Corollary. A meromorphic function on a compact Riemann surface has the same number of zeros as poles (counted up to multiplicity.)

Proof. A meromorphic function on R is a hol. function $f: R \to \mathbb{C}_\infty$.

We have $\delta_f(0) = \delta_f(\infty)$ in such a case.

If $f: R \to S$ is a map between compact Riemann surfaces then

Definition. we call $d_f \equiv \delta_f(q)$ the <u>degree</u> of f. Note $d_f \geq 1$. (We saw before that a non-constant hol. map is surjective, This is a refinement of that.).

Corollary. A holomorphic map between compact Riemann surfaces of degree 1 is a conformal equivalence.

Cor. If a merom. function on R has 1 zero of order 1 then R is conformally equivalent to S^2.

Proof. If $d_f = 1$ then every point has 1 inverse image and non-zero derivative. So f is invertible. Inverse is holomorphic by the inverse function theorem.

Comments on Euler Characteristic.

Recall the Euler characteristic of a surface R.

$\chi(R) = \# v(R) - \# e(R) + \# f(R)$ $\qquad \chi(R) = \dim H_0 - \dim H_1 + \dim H_2.$

Recall $\chi(R) = 2-2g$.
Typically $\chi < 0$.

$\chi = 2$, $\chi = 0$, $\chi = -2$ ... $\chi = -4$ ...

$g=0$ $\quad g=1$ $\quad g=2$ $\quad g=3$

Gauss-Bonnet. $2\pi\chi(R) = \int_R \text{curvature } d\text{vol}$.

Cor. If R is a compact Riem. surface with a metric of constant curvature then the curvature is positive if the surface is a sphere, 0 if the surface is a torus and negative if the surface has higher genus.

Remark. What about annuli?

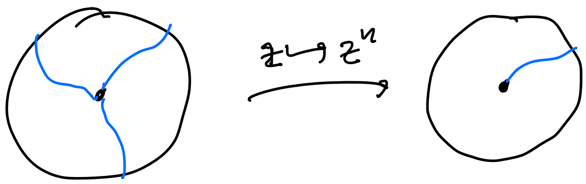
Comment. In the following proof we use the existence of triangulations of surfaces. These do exist but we are not proving that they exist.

Riemann-Hurwitz Theorem. Let $f: R \to S$ be a non-constant holomorphic map between compact Riemann surfaces. Then

$$(1) \quad \chi(R) = d \cdot \chi(S) - \underbrace{\sum_{\substack{p \in R \\ v_f(p) > 1}} (v_f(p) - 1)}_{\text{sign is determined}}.$$

Proof. There are only finitely many points $p \in R$ with $v_f(p) > 1$.

Construct a triangulation of S where the set of vertices $q_1 \dots q_k$ contains the images under f of $p_1 \dots p_j$. Away from the inverse images of the q's the map is a covering map of degree d so each open edge lifts to d open edges in R.

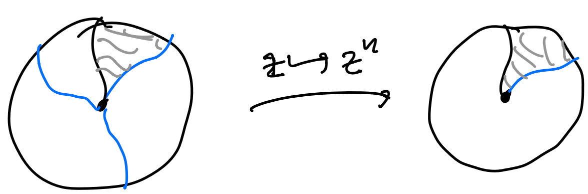


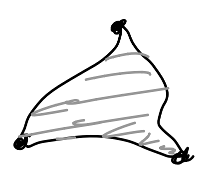
Can use the local model of the map to show that we can extend the lift of the path to its endpoints:

$$\begin{tikzcd} & R - \{p_1 \dots p_j\} \arrow[d, "f"] \\ \sigma : [0,1] \arrow[r] \arrow[ur, "\tilde{\sigma}"] & S - \{q_1 \dots q_k\} \end{tikzcd}$$

If $\lim_{t\to 0} \sigma(t) = 0$ then $\lim_{t \to 0} \sigma^{1/n}(t) = 0$.

Each triangle downstairs lifts to d topological triangles upstairs since triangles are simply connected.





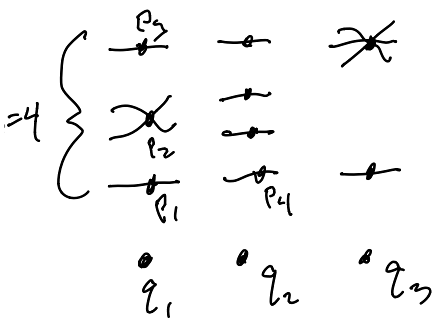
The Euler characteristic can be calculated as the alternating sum of #'s of simplices in a triangulation.

$$\chi(S) = \# v(S) - \# e(S) + \# f(S)$$

$$\chi(R) = \# v(R) - \# e(R) + \# f(R)$$

Now $\# e(R) = d \, \# e(S)$ and $\# f(S) = d \cdot \# f(R)$

$$\chi(R) - d\,\chi(S) = \# v(R) - d \cdot \# v(S).$$



$$= \sum_{q \in V(S)} \# f^{-1}(q) - d \sum_{q \in V(S)} 1$$

write the sum in terms of images of pts

$$= \sum_{q \in V(S)} \left(\# f^{-1}(q) - d \right)$$

counting actual cardinality here. not with multiplicity.

$$= \sum_{q \in V(S)} \left(\# f^{-1}(q) - \sum_{p : f(p) = q} \nu_f(p) \right)$$

Now recall that

$$\sum_{p : f(p) = q} \nu_f(p) = d$$

$$= \sum_{q \in V(S)} \left(\sum_{p : f(p) = q} 1 - \sum_{p : f(p) = q} \nu_f(p) \right)$$

$$= \sum_{q \in V(S)} \sum_{p : f(p) = q} \left(1 - \nu_f(p) \right)$$

this quantity vanishes at regular pts. Only a finite # of non-reg. pts

$$= \sum_{\substack{p \in V(R) \\ f'(p) = 0.}} \left(1 - \nu_f(p) \right)$$

finite sum

Note that points p with $V_f(p)=1$ make no contribution.

$$= \sum_{p \in R} 1 - V_f(p).$$

Cor. If $f: R \to S$ is hol. and non constant then $g(S) \le g(R)$.

Proof. It is useful here to rewrite the Riemann-Hurwitz equation in terms of the genus.

$1 - g = \frac{\chi}{2}$ $\quad g = \frac{\chi}{2} + 1$

$-g = \frac{\chi}{2} - 1$

$$2 - 2g(R) = d(2 - 2g(S)) - \sum V_f(p) - 1.$$

$$2g(R) - 2 = \underset{\ge 1}{d}(2g(S) - 2) + \underbrace{\sum V_f(p) - 1}_{\ge 0}$$

If $S = S^2$ then the assertion is true.

If $S \ne S^2$ then $2g(S) - 2 \ge 0$ so

$$2g(R) - 2 \ge d(2g(S) - 2) \ge 2g(S) - 2$$

$$g(R) \ge g(S)$$

$$\chi(R) = d\cdot\chi(S) - \underbrace{\sum_{\substack{p\in R\\ V_f(p)>1}} (V_f(p)-1)}_{\text{sign is determined}}.$$

Claim $\chi(R) < \chi(S)$. Assume $\chi(R), \chi(S) < 0$

$\chi(R) \le d\cdot\chi(S)$ If $\chi(S)=2$ then $\chi(R)\le 2$.

If $\chi(S)=0$ then $\chi(R)\le 0$

If $f: R\to R$ f not const $\chi(R)<0$ then $d=1$.

$\chi(R) - d\chi(S) \le 0.$

$\chi(R) \le d\chi(S)$