Now we want to make a connection between meromorphic functions and algebraic curves.

This is essentially an algebraic fact. Field of meromorphic functions is a finite extension of $\mathbb{C}(z)$. Transcendence degree 1. Any 2 elts satisfy some poly. relation.

Prop. $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$ for certain constants g_2 and g_3 that depend on the lattice Λ.

Proof. $\wp(z) - \frac{1}{z^2} = \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$

vanishes at 0 since each term vanishes at 0. We also see that it is an even function (since $\wp$ and $\frac{1}{z^2}$ are even functions).

Thus $\quad \wp(z) = \quad z^{-2} + \quad \lambda z^2 + \quad \mu z^4 + O(z^6)$

$$\wp'(z) = -2z^{-3} + 2\lambda z + 4\mu z^3 + O(z^5)$$

$$(\wp'(z))^2 = 4z^{-6} - 8\lambda z^{-2} - 16\mu + O(z^4)$$

$$\wp^3(z) = z^{-6} + 3\lambda z^{-2} + 3\mu + 3\lambda^2 z^2 + O(z^4)$$

$$(\wp'(z))^2 - 4\wp^3(z) = \underbrace{-20\lambda z^{-2} - 28\mu}_{-20\lambda \wp(z) - 28\mu} + O(z^2)$$

$$-20\lambda \wp(z) - 28\mu + O(z^2)$$

Thus $(\wp'(z))^2 - 4\wp^3(z) + 20\lambda \wp(z) + 28\mu$ has no pole at 0 and has value 0 at 0.

Since the only poles of $\wp$ and $\wp'$ occur at lattice points this function has no other poles.

Thus we have a hol. fcn on a compact Riemann surface $(\mathbb{C}/\Lambda)$ so it is constant.

Evaluating at 0 we see that the value of the constant is 0.

Now set $g_2 = -20\lambda$ and $g_3 = -28\mu$.

Write $Q(z)$ for $4z^3+g_2z+g_3$.

Cor. The functions $u \overset{\Phi}{\mapsto} (\wp(u), \wp'(u))$ from $\mathbb{C}/\Lambda - \{0\}$ to $\mathbb{C}^2$ takes values in the curve

$$V_Q = \{(z,w) : w^2 = 4z^3+g_2z+g_3\}.$$

Cor. The polynomial $\overbrace{4z^3+g_2z+g_3}^{Q(z)}$ has distinct 0's.

Proof.

We have a map to the surface $w^2=Q(z)$ given by $z=\wp(u)$, $w=\wp'(u)$.

The polynomial $w^2=Q(z)$

$w^2=\underbrace{4z^3+g_2z+g_3}_{Q(z)}$ is satisfied for $w=\wp'(u)$, $z=\wp(u)$.

$$(\wp'(u))^2 = 4\wp^3(u)+g_2\wp(u)+g_3$$

The zeros of Q occur at the points $\wp(u)$ for which $\wp'(u)=0$. These are the 3 half-lattice points other than 0.

We have proved that $\wp$ takes distinct values at these 3 points.

Recall that $\bar{R} \subset \mathbb{C}_\infty \times \mathbb{C}_\infty$ is $R \cup \{(\infty,\infty)\}$ and we have constructed a Riemann surface structure for $\bar{R}$, $\tilde{R}$. (Odd # of roots.

We have shown that $\bar{R}$ has a Riem. surface structure. Do not have to separate pts. at ∞.)

Remark. The condition that the zeros of $4z^3 - g_2 z - g_3$ be distinct was required so that $R \subset \mathbb{C}^2$ have no singular points and yield a Riemann surface.

Prop.

This parametrization Φ extends to a conformal isomorphism from $\mathbb{C}/\Lambda$ to $\tilde{R}$ where $\Phi(0) = (\infty, \infty)$.

Proof.

We have hol. maps.

We are using the non-singular property here.

① $\mathbb{C}/\Lambda - \{0\} \xrightarrow{u \overset{\Phi}{\mapsto} (\wp(u), \wp'(u))} \mathbb{C}^2$

$u \longmapsto (\wp(u), \wp'(u))$

②
$$\begin{tikzcd} & \tilde{R} - \{0\} \arrow[dr] & \\ \mathbb{C}/\Lambda - \{0\} \arrow[ur] \arrow[rr, "\Phi", "u \mapsto (\wp(u), \wp'(u))"'] & & \mathbb{C}^2 \end{tikzcd}$$

By the cor.

③ The function Φ extends to

$$\mathbb{C}/\Lambda \longrightarrow \mathbb{C}_\infty \times \mathbb{C}_\infty$$

taking 0 to (∞, ∞)

since each coordinate is a meromorphic function and 0 is a pole for $\wp$ and $\wp'$.

(Both coord. fcns. are holomorphic.)

(4) So we have

$$\begin{tikzcd} & \tilde{R} \arrow[dr] & \\ \mathbb{C}/\Lambda \arrow[ur, "\varphi"] \arrow[rr, "\Phi"] & & \mathbb{C}_\infty \times \mathbb{C}_\infty \end{tikzcd}$$

φ is continuous and holomorphic on $\mathbb{C}/\Lambda - \{0\}$ so φ is actually holomorphic on $\mathbb{C}/\Lambda$

(Problem.)

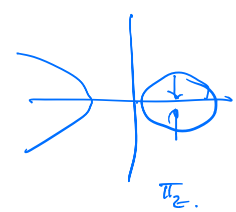
(5) Want to show that φ is a hol. isomorphism.

Suffices to show $\deg(\varphi) = 1$ since range and domain are compact.

$$\begin{tikzcd} & \tilde{R} \arrow[dr] \arrow[dd, dashed] & \\ \mathbb{C}/\Lambda \arrow[ur, "\varphi"] \arrow[rr, "\Phi"] \arrow[dr, "\wp"'] & & \mathbb{C}_\infty \times \mathbb{C}_\infty \arrow[dl, "\pi_z"] \\ & \mathbb{C}_\infty & \end{tikzcd}$$

or

$$\begin{tikzcd} & \tilde{R} \arrow[dd, "\pi_z \circ \iota \quad \deg = 2"] \\ \mathbb{C}/\Lambda \arrow[ur, "\varphi"] \arrow[dr, "\wp"', "\deg = 2"] & \\ & \mathbb{C}_\infty \end{tikzcd}$$



$\deg \varphi \cdot \deg \pi_z \circ L = \deg P.$ (Problem)

Now $\pi_z|\tilde{R}$ is a covering map of degree 2 (away from zeros of w) so its degree is 2.

We have shown $\deg P = 2$.

$\deg \varphi \cdot 2 = 2.$

So $\deg \varphi = 1.$

$$\begin{array}{c} R \\ \big\downarrow \pi_z \circ L \leftarrow \deg = 2 \\ \mathbb{C} \end{array}$$

For a generic $z \in \mathbb{C}$ there are 2 values of w so that $w^2 = P(z)$ $(w, -w)$.

Conclude that we have established a map from lattices to non-singular hyper-elliptic surfaces.