Example 3 : Algebraic varieties.

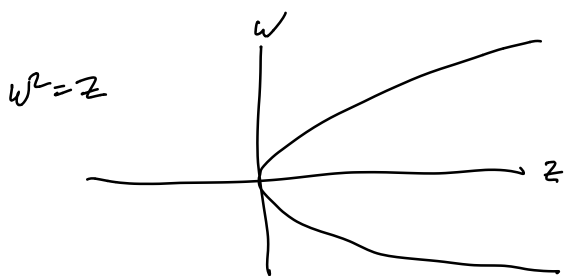
We will not give a coherent development of these we will discuss some examples. On the other hand the examples we discuss are historically interesting and perhaps give us some feel for this branch of the field.

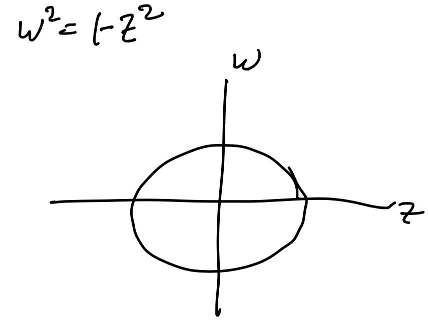
Let $P(z)$ be a polynomial $P(z) = a_1 + a_2 z + \dots a_m z^m$.

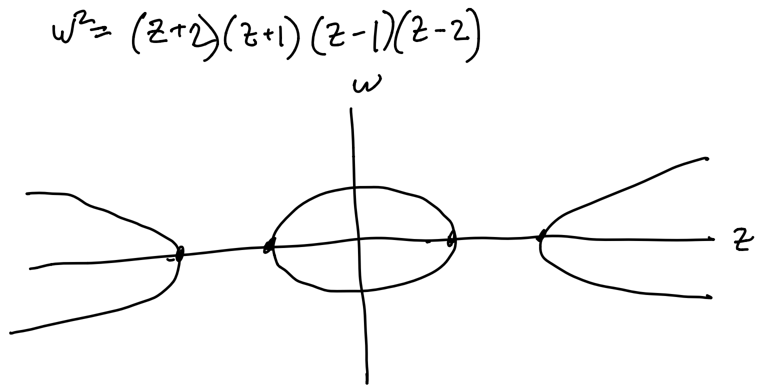
A hyperelliptic variety has the form

$$R = \{(z, w) : w^2 = P(z)\}.$$

If P has real coefficients we can draw the corresponding real locus:







To give these varieties Riemann surface structures we construct charts (in the real or complex setting) by projecting onto the z or w axes and using the implicit function theorem to write our variety locally as the graph of a function, writing either z as a function of w or w as a function of z.

We will not describe the general case we only look at the hyper-elliptic case.

Prop. If P has simple zeros then R has a Riemann surface structure.

Say $(z_0, w_0) \in R$

When $P(z_0) \neq 0$ we can get an inverse chart for R in a nbd. of (z_0, w_0) by choosing a branch $\phi : U \to V$ with $\phi(P(z_0)) = w_0$ of the square root function in a nbd. of $P(z_0)$ and defining $w = \sqrt{P(z)}$ for z near z_0 hence $P(z)$ near $P(z_0)$. A second branch is given by $w = -\sqrt{P(z)}$.

Get a local inverse chart of the form $z \mapsto (z, \phi(z))$.

$z \mapsto (z, \phi(z))$ inverse chart

$w \mapsto (\psi(w), w)$

When $P(z)=0$ we have that $P'(z) \neq 0$ since the roots are simple. This implies that P is locally invertible.

$w^2 = P(z)$

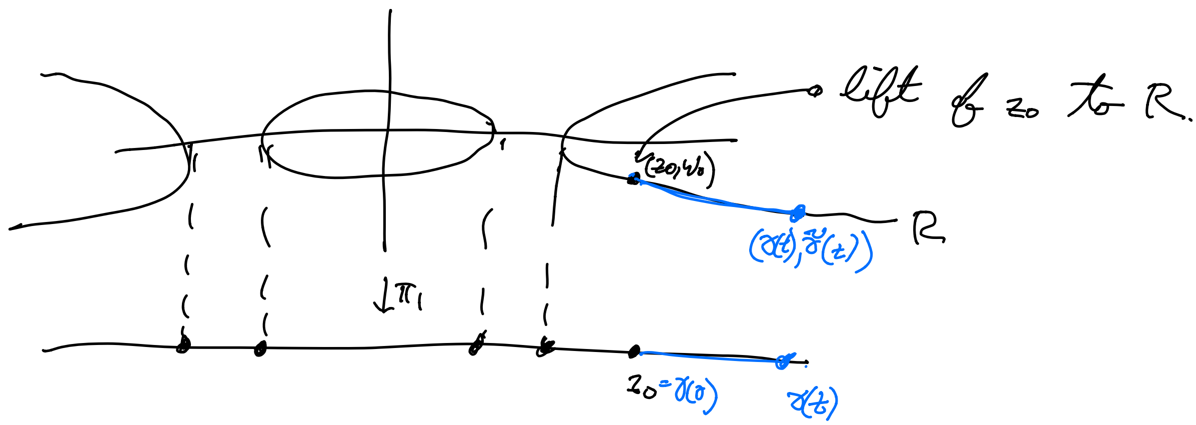
In a nbd. of $(z_j, 0)$ we can solve for z as a function of w. Since $P'(z_j) \neq 0$ P is locally invertible by the inv. fcn. thm. Write $\underbrace{z = P^{-1}(w^2)}_{\text{to } P(z)=w^2}$. $= \psi(w^2)$ for w near 0.

$\psi = P^{-1}$

Our next objective is to get a "global" picture of these surfaces as topological objects. In the real case this would involve counting components and determining which are circles and which are open intervals. For surfaces we want to look at the genus.

Traditionally done with too little detail. Balance this by giving you too much detail.

We will proceed as in our last lecture to find explicit parametrization of the surface. That is find a formula for lifting paths in $\mathbb{C} - \{z_1 \ldots z_m\}$ to R.



Formula for $\tilde{\gamma}(t)$ in terms of w_0 and γ.

Recall that $\displaystyle\int_\gamma \underbrace{\frac{f'(z)}{f(z)}\,dz}_{f^*(\theta)} = \int_{f(\gamma)} \underbrace{\frac{dz}{z}}_{\theta} \quad \left(= 2\pi i \cdot \mathrm{wind}(f(\gamma), 0)\right).$

Naturality of integration. Change of variables formula.

Observe: $\displaystyle \exp\left(\int_\gamma \frac{P'(z)}{P(z)}\,dz\right) = \exp\int_{P\circ\gamma} \frac{dz}{z} = \frac{P(\gamma(1))}{P(\gamma(0))}.$

In other words when we apply this to paths rather than loops this formula gives us the value of P. $\displaystyle\int \frac{P'(z)}{P(z)}dz = \text{"}\log P\text{"}.$

We can easily modify this formula to get the value of $\sqrt{P}$ by multiplying the integral by $1/2$. using $\sqrt{z} = \text{"}\exp\left(\frac{1}{2}\log z\right)\text{"}.$

Say γ is a path in $\mathbb{C} - \{z_j\}$.

$\gamma : [0,1] \longrightarrow \mathbb{C} - \{z_j\}.$

Let $(\gamma(0), w_0) \in R$ (so $w_0^2 = P(\gamma(0))$)

then define

$$\tilde{\gamma}(t) = w_0 \cdot \exp\left(\frac{1}{2}\int_0^t \frac{P'(\gamma(s))\gamma'(s)}{P(\gamma(s))}\, ds\right)$$

Claim that $t \mapsto (\gamma(t), \tilde{\gamma}(t))$ solves the lifting problem for

$\pi_1 : R \longrightarrow \mathbb{C} - \{z_i\}$.

Proof. By construction $\pi_1((\gamma(t), \tilde{\gamma}(t))) = \gamma(t)$ and $(\gamma(0), \tilde{\gamma}(0)) = (\gamma_0, w_0)$.

Need to check that

$$(\overset{\parallel\, z}{\gamma(t)}, \overset{\parallel\, w}{\tilde{\gamma}(t)}) \in R \quad \text{or} \qquad w^2 = P(z)$$

equivalently that

$$\left(\tilde{\gamma}(t)\right)^2 = P(\gamma(t)).$$

So

$$\left(\tilde{\gamma}(t)\right)^2 = \left(w_0 \cdot \exp\left(\frac{1}{2}\int_0^t \frac{P'(\gamma(s))\cdot\gamma'(s)}{P(\gamma(s))}\,ds\right)\right)^2$$

$$= w_0^2 \cdot \exp\left(\int_0^t \frac{P'(\gamma(s))\cdot\gamma'(s)}{P(\gamma(s))}\,ds\right)$$

$$= w_0^2 \cdot \frac{P(\gamma(t))}{P(\gamma(0))}$$

$$= P(\gamma(t)). \qquad \text{QED.}$$

$\gamma : [0,1] \to \mathbb{C} - \{z_j\}$.

$$\tilde{\gamma}(t) = w_0 \cdot \exp\left(\frac{1}{2}\int_0^t \frac{P'(\gamma(s))\gamma'(s)}{P(\gamma(s))}\,ds\right)$$

$$\tilde{\gamma}(1) = w_0 \cdot \exp\left(\frac{1}{2}\int_\gamma \frac{P'}{P}\,dz\right).$$

Use π_z for π_1.

Remark. $\pi_z : R' \to \mathbb{C} - \{z_j\}$ is a normal cover and we can identify the deck group with ± 1. $\quad (z, w) \mapsto (z, \pm w)$

We can use the solution to the lifting problem to compute the map from $\pi_1(S - \{z_i\}) \longrightarrow$ (deck group vs. monodromy group?).

What do you get when you lift a loop?
Say γ is a loop.

$P(z) = a_m \prod (z - z_j)$.

$$\int_\gamma \frac{P'(z)}{P(z)}\,dz = \int_\gamma \frac{\left(a_m \prod_j (z-z_j)\right)'}{a_m \prod_j (z-z_j)}\,dz$$

$$= \int_\gamma \sum_j \frac{a_m (z-z_1) \cdots \widehat{(z-z_j)} \cdots (z-z_m)}{a_m (z-z_1) \cdot (z-z_j) \cdots (z-z_m)}\,dz$$

$$= \sum_j \int_\gamma \frac{dz}{z-z_j}\,dz$$

$$= \sum_j 2\pi i \,\mathrm{wind}(\gamma, z_j).$$

When γ is a loop:

$$\frac{\tilde\gamma(1)}{\tilde\gamma(0)} = \frac{w_0}{w_0} \cdot \exp\left(\frac{1}{2} \cdot 2\pi i \sum_j \mathrm{wind}(\gamma, z_j)\right)$$

$$= \exp\left(\pi i \sum_j \mathrm{wind}(\gamma, z_j)\right)$$

$$= (-1)^{\sum_j \mathrm{wind}(\gamma, z_j)}$$

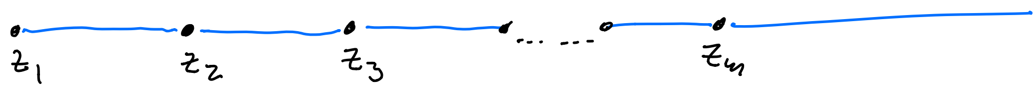
Procedure is to cut R into pieces that we can understand and then calculate how to glue the pieces back together.

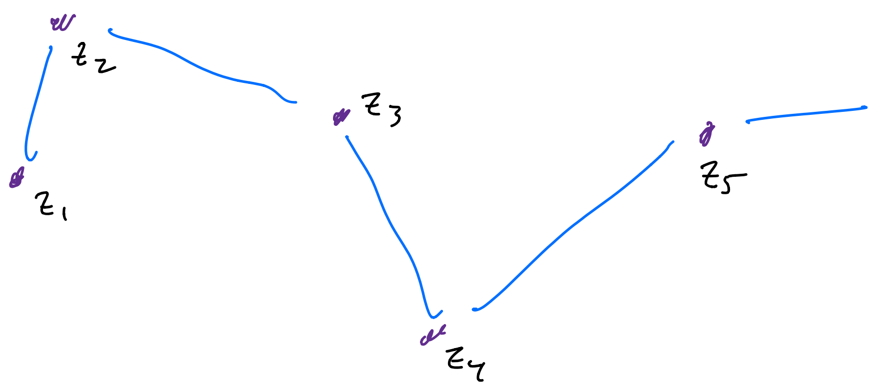
$f(z) = \sqrt{(z-z_1)(z-z_2)\cdots(z-z_n)}$

$z^{1/2}$.

$|f(z)|$ is "single valued". Does not depend on the branch chosen.

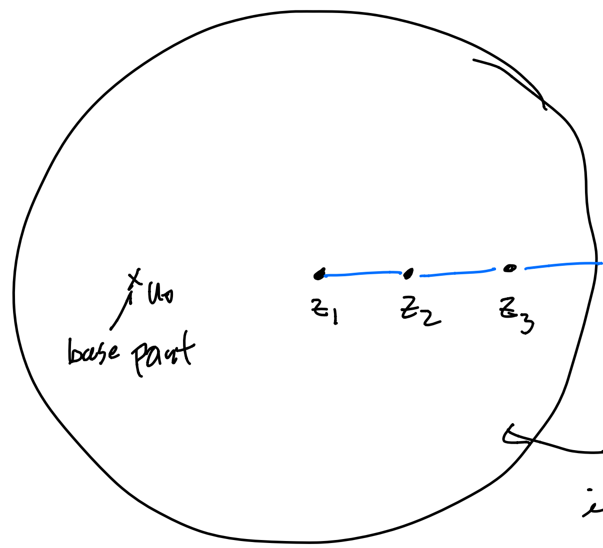
Let's assume that the z_j are real for convenience.





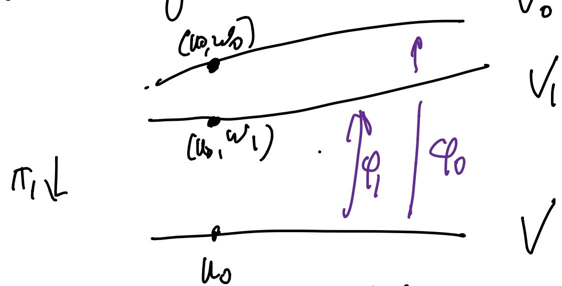
Choose a large disk D. Let's analyze $\pi_1^{-1}(D)\cap R$.

Assume for now that $\pi_1\big|_R \to \mathbb{C}-\{z_j\}$ is a covering.



$D-\{\text{arcs}\}$ is simply connected.

z_0 has 2 lifts



V is simply connected so $\pi^{-1}(V)$ consists of two sheets.

φ gives w coordinate.

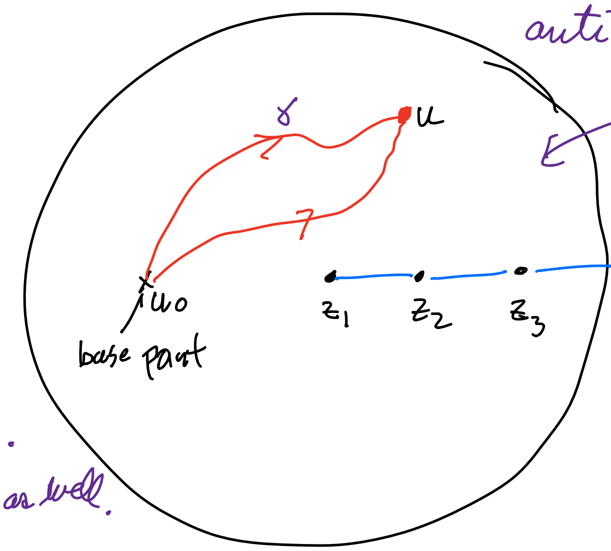
$z \mapsto (z, \varphi_0(z))$.

$z \mapsto (z, \varphi_1(z))$

Pick two inverse images of z_0. (u_0,w_0) and (u_0,w_1).

We can use our path lifting function to get explicit formulas for u_0 and u_1.

Analogous to the construction of the anti-derivative.



$\mathcal{U}$

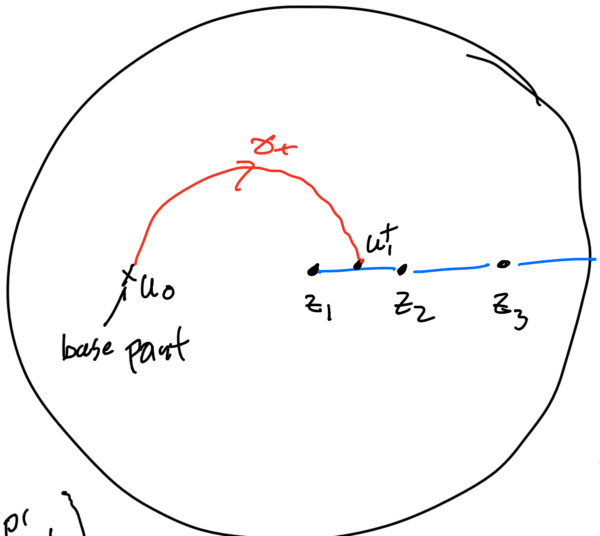
Say $(u_0, w_0) \in R$.
Then $(u_0, -w_0) \in R$ as well.

$(u, \phi_0(u)) \in R$ $\quad \phi_0(u) = w_0 \cdot \exp\left(\frac{1}{2}\int_\gamma \frac{p'}{p}\,dz\right)$.

$(u, \phi_1(u)) \in R$ $\quad \phi_1(u) = -w_0 \cdot \exp\left(\frac{1}{2}\int_\gamma \frac{p'}{p}\,dz\right)$.

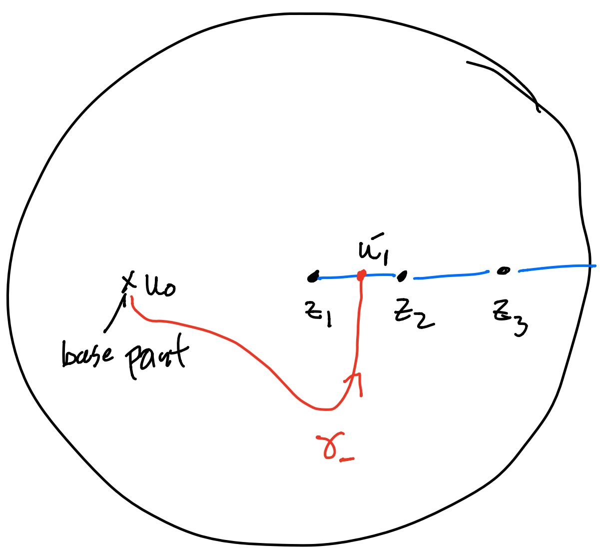
The lift is independent of the path since $\mathcal{U}$ is simply connected

Now we can extend the functions ϕ_0 and ϕ_1 to the slits but we have to make a choice. Do we approach the slit from the top or from the bottom?

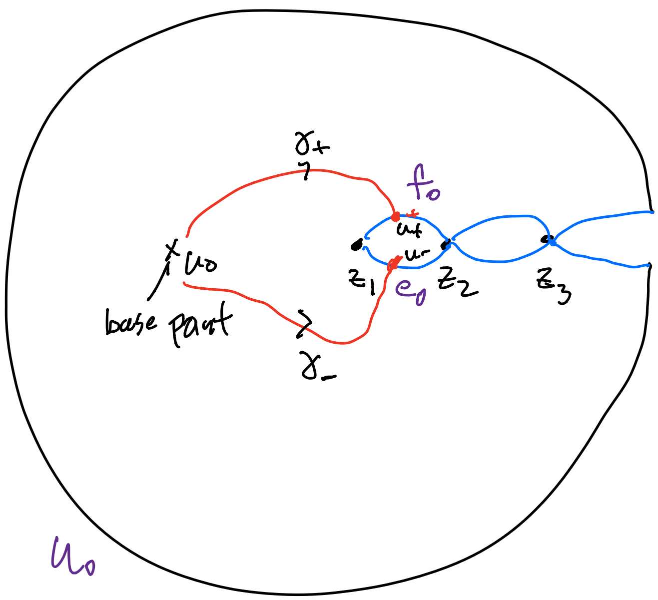


$$\phi_0(u^+) = w_0 \exp\left(\frac{1}{2}\int_{\gamma_+} \frac{p'}{p}\,dz\right)$$

$$\phi_0(z_1) = w_0 \cdot \exp\left(\frac{1}{2}\int_{\gamma_+} \frac{p'}{p}\,dz\right)$$



$$\int_{\gamma_+ \cdot \gamma_-^{-1}} \frac{p'}{p}\,dz = \int_{\gamma_+} \frac{p'}{p}\,dz + \int_{\gamma_-^{-1}} \frac{p'}{p}\,dz = \int_{\gamma_+} \frac{p'}{p}\,dz - \int_{\gamma_-} \frac{p'}{p}\,dz$$



$$\phi_0(u_+) = w_0 \cdot \exp\left(\frac{1}{2}\int_{\gamma_+} \frac{p'}{p} dz\right)$$

$$\phi_0(u_-) = w_0 \cdot \exp\left(\frac{1}{2}\int_{\gamma_-} \frac{p' dz}{p}\right)$$

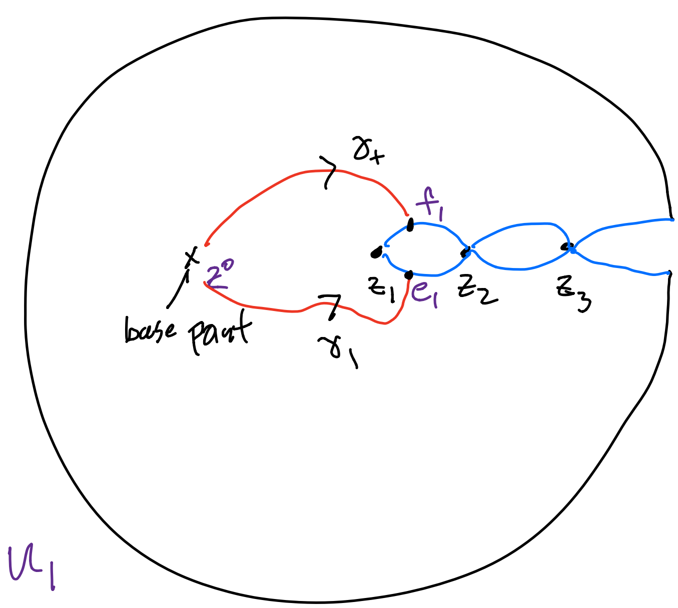
$$\phi_1(u_+) = w_1 \cdot \exp\left(\frac{1}{2}\int_{\gamma_+} \frac{p'}{p} dz\right)$$
$$= -w_0 \cdots = -\phi_0(u_+)$$

$$\phi_1(u_-) = -\phi_0(u_-).$$

$\phi_0(u_+) \in f_0$ $\quad\quad$ $\phi_1(u_+) \in f_1$

$\phi_0(u_-) \in e_0$ $\quad\quad$ $\phi_1(u_-) \in e_1$

~~Is~~ $\phi_0(u_+) = \phi_0(u_-)$ or $\underset{-\phi_0(u_-)}{\underset{\shortparallel}{\phi_1(u_-)}}$?



$$\frac{\phi_0(u_+)}{\phi_0(u_-)} =$$
$$\exp\left(\frac{1}{2}\left(\int_{\gamma_+} \frac{p'}{p} dz - \int_{\gamma_-} \frac{p'}{p} dz\right)\right)$$
$$= \exp\left(\frac{1}{2}\int_{\gamma_+ \cdot \gamma_-^{-1}} \frac{p'}{p} dz\right)$$
$$= (-1)^{\mathrm{wind}(\gamma_+ \cdot \gamma_-^{-1},\, z_1)}$$
$$= -1.$$