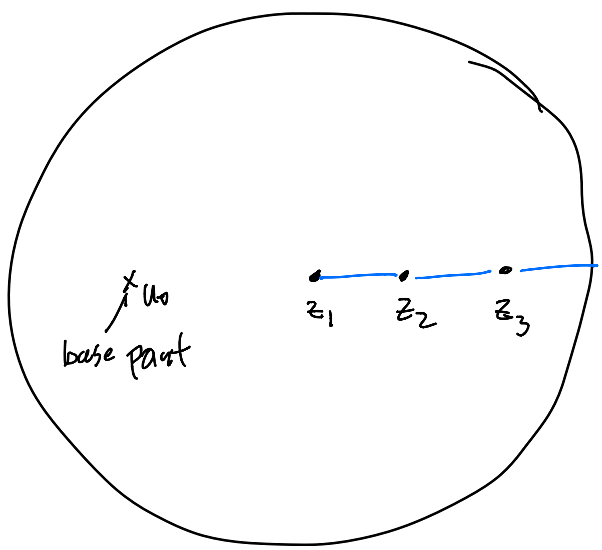
Recall from last time that we are constructing a topological picture of a hyper-elliptic surface

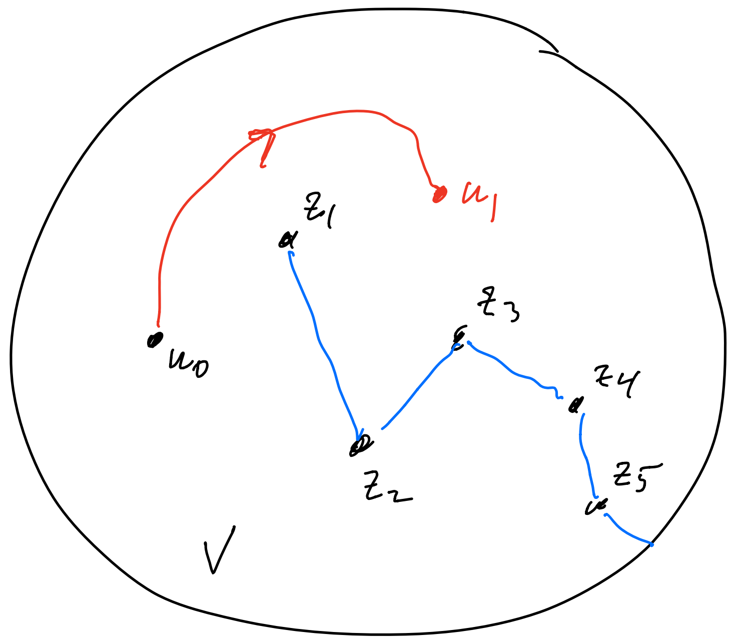
$R = \{(z,w) : w^2 = P(z)\}$ P has m simple zeros $z_1 \dots z_m$.



Draw a large disk D containing $z_1 \dots z_m$ and arcs between the z_j.

We also choose an arc connecting z_m to ∂D. We do this in order to make the complementary region simply connected.

Picture could be:



Write V for $D - \{\text{arcs}\}$. We defined $\phi_0, \phi_1 : V \to R$ by lifting paths.

u_0 plays the role of a basepoint in V.

w_0 and w_1 satisfy $w_j^2 = P(u_0)$ and $w_1 = -w_0$

$\phi_j(u_1) = w_j \cdot \exp\left(\frac{1}{2}\int_\gamma \frac{P'(z)}{P(z)}\,dz\right)$. | $(u_1, \phi_j(u_1)) \in R$, $\phi_j^2(u_1) = P(u_1)$.

where γ is a path in V from u_0 to u_1.

Note that for any point u_1 in D $\phi_0(u_1) = -\phi_1(u_1)$ $\neq 0$.

so the sets $(u, \phi_j(u))$ are disjoint

for $u \in D$.

Define $V_0 = \{(u, \phi_0(u)) : u \in D\}$ $V_1 = \{(u, \phi_1(u)) : u \in D\}$

V_0, V_1 are disjoint. How do their closures intersect?

Claim. ϕ_j extends continuously to

the z_j and takes the value 0 at z_j.

Proof. The equation $\phi_j^2(u_1) = P(u_1)$ gives

us $|\phi_j(u)|^2 = |P(u)|$ so $|\phi_j(u)| = \sqrt{|P(u)|}$.

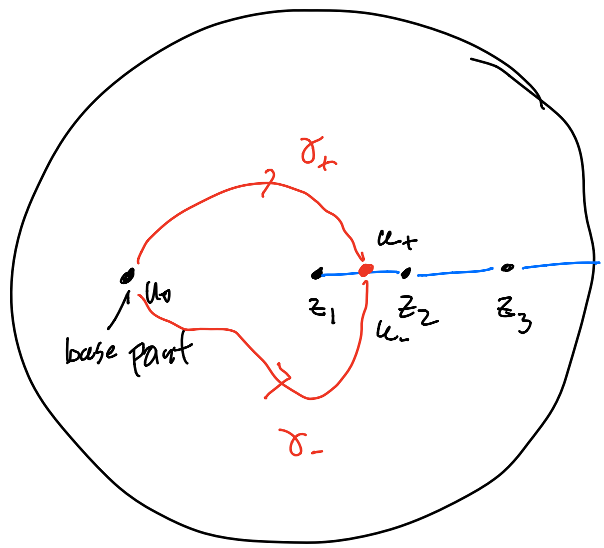
If $u \to z_j$ then $|P(u)| \to 0$ so $|\phi_j(u)| \to 0$.

The next step is to extend ϕ_j to

the slits. This will tell us how

the sets $\phi_j(V)$ are glued together

in R.



$$\phi_j(u_\pm) = w_j \cdot \exp\left(\frac{1}{2}\int_{\gamma_\pm} \frac{p'}{p}\,dz\right)$$

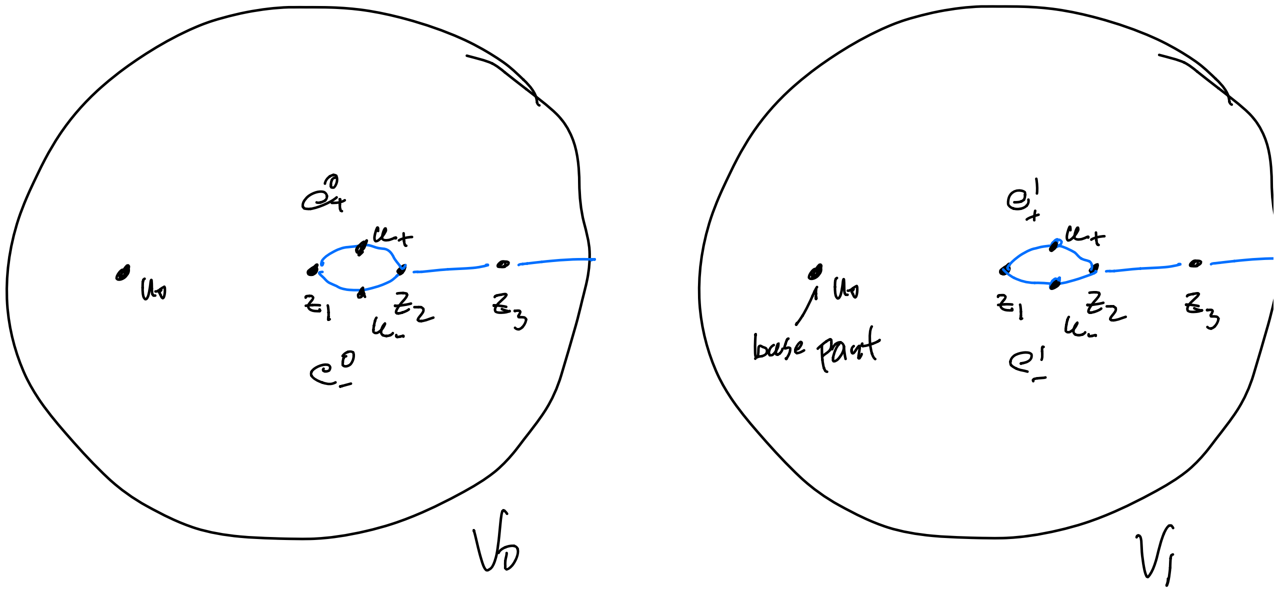
Note that

$$\phi_0(u_+) = w_0 \cdot \exp(\quad) = -w_1 \cdot \exp(\quad) = -\phi_1(u_+).$$

Similarly

$$\phi_0(u_-) = -\phi_1(u_-).$$

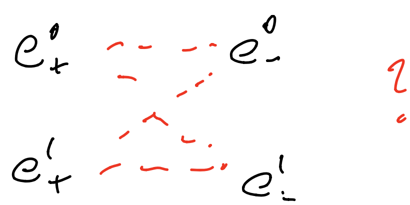
The bottom slit on V_0 is disjoint from the bottom slit on V_1. Similarly the top two slits are disjoint. How do the bottom slits glue to the top slits?



How do these edges pair up in R?

e^0_+ and e^1_+ are disjoint.

e^0_- and e^1_- are disjoint?



Look at $\phi_0(u_+)$ and $\phi_0(u_-)$.

$$\frac{\phi_0(u_+)}{\phi_0(u_-)} = \frac{w_0 \cdot \exp\left(\frac{1}{2}\int_{\gamma^+} \frac{p'}{p}\, dz\right)}{w_0 \cdot \exp\left(\frac{1}{2}\int_{\gamma^-} \frac{p'}{p}\, dz\right)}$$

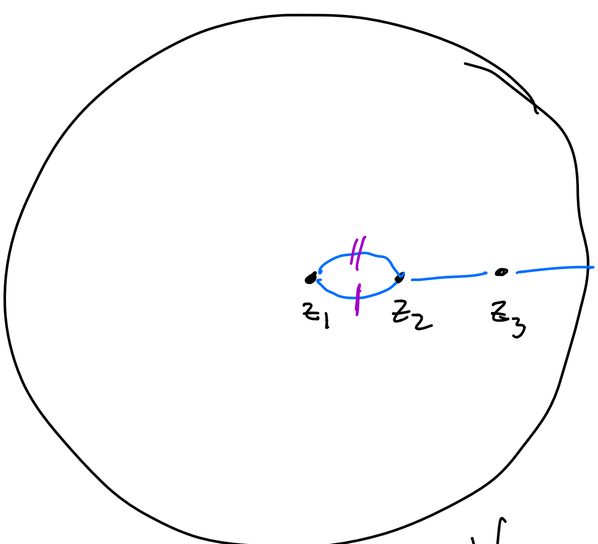
$$= \exp\left(\frac{1}{2}\left(\int_{\gamma^+}\frac{p'}{p}\,dz - \int_{\gamma^-}\frac{p'}{p}\,dz\right)\right)$$

$$= \exp\left(\frac{1}{2}\left(\int_{(\gamma^+)\cdot(\gamma^-)^{-1}}\frac{p'}{p}\,dz\right)\right)$$

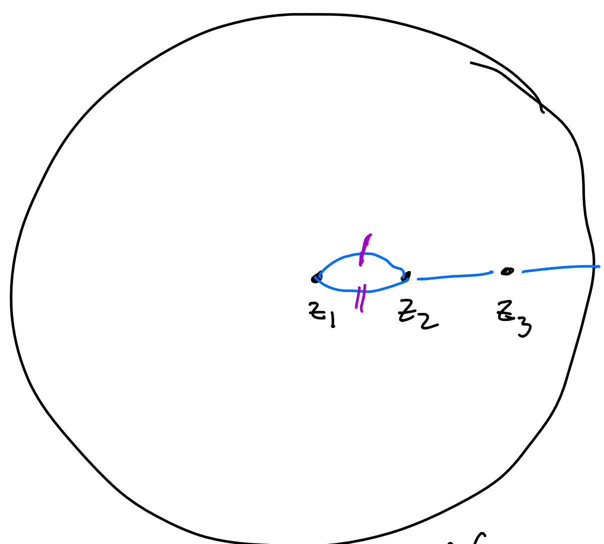
$$= \exp\left(\frac{1}{2}\left(2\pi i\sum_j \mathrm{wind}\left((\gamma^+)\cdot(\gamma^-)^{-1}, z_j\right)\right)\right)$$

$$= \exp\Big(\pi i \cdot \underbrace{\mathrm{wind}\left(\gamma^+\cdot(\gamma^-)^{-1}, z_1\right)}_{=\,1}\Big)$$

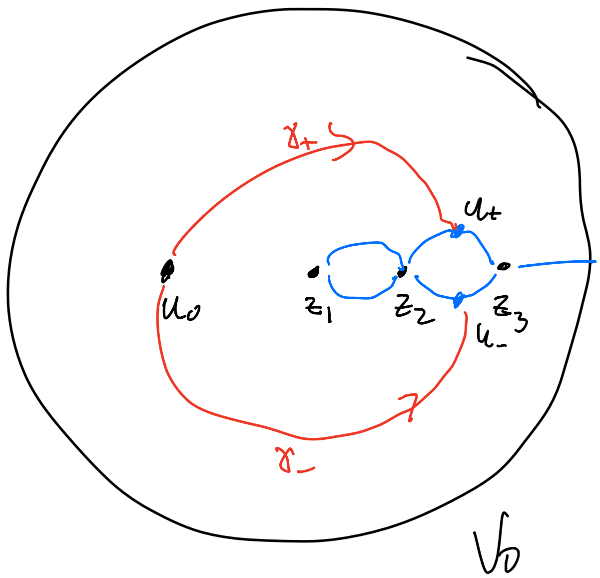
$$= e^{\pi i} = -1.$$

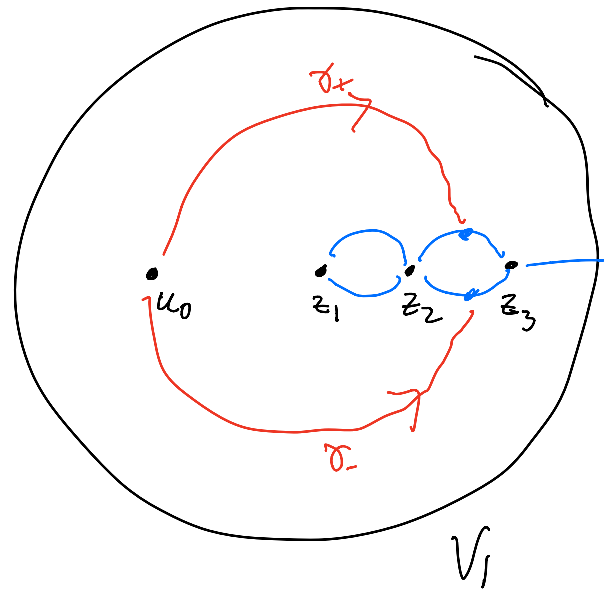


V_0



V_1





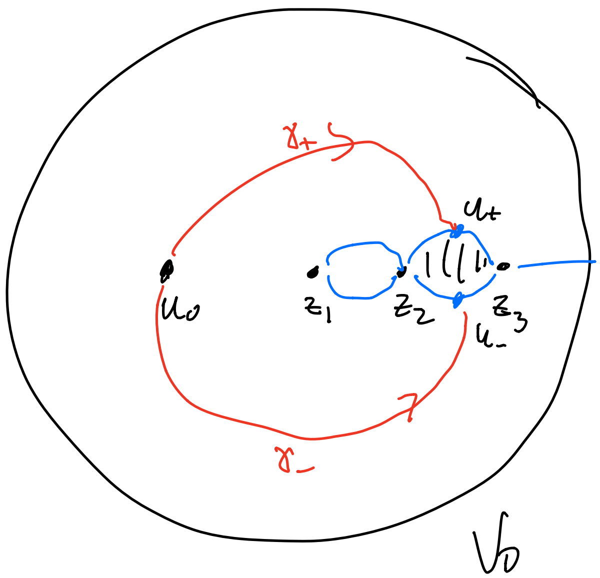
Do these two edges get identified?

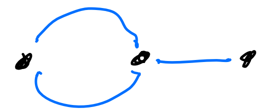
Calculate:

$$\frac{\phi_0(u_+)}{\phi_0(u_-)} = \exp\Big(\pi i \cdot \sum_j \mathrm{wind}(\gamma_+ \cdot \gamma_-^{-1}, z_j)\Big)$$

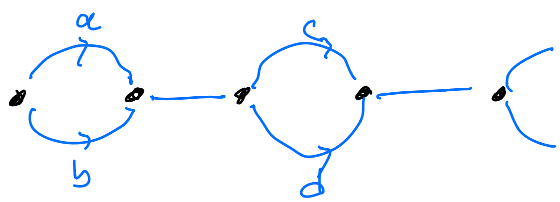
$$= e^{\pi i \cdot 2} = +1,$$

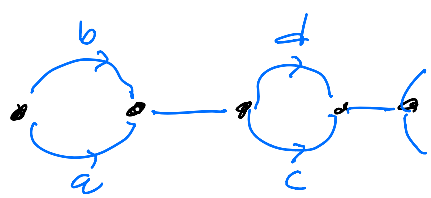
Opposite sides in V_0 are identified:



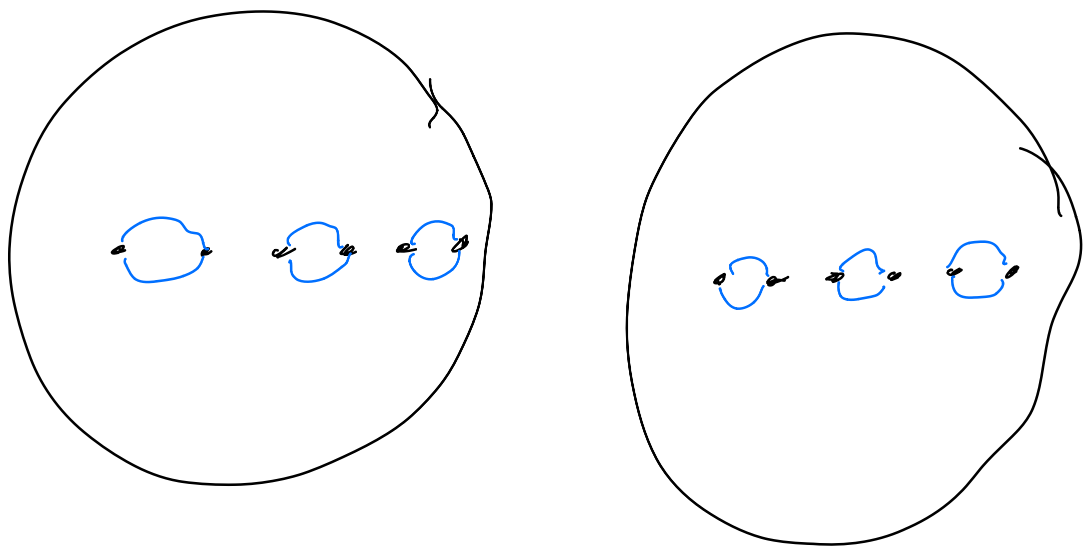


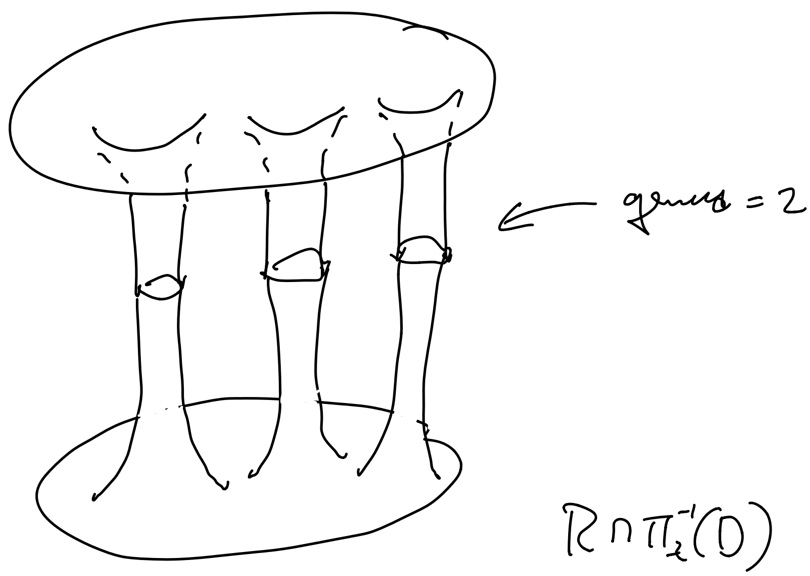
See the pattern: In V_0 every other slit has the property that the two edges in V_0 are identified.





Case 1 n is even



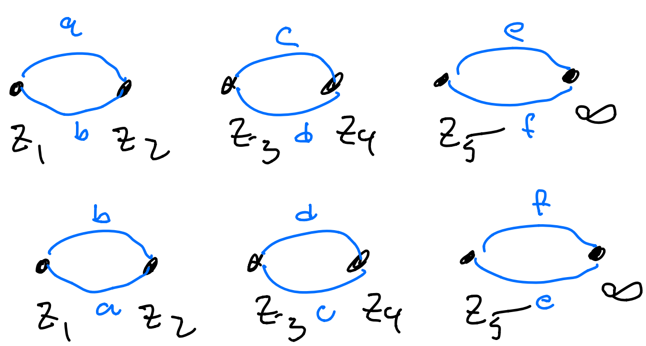


Our surface is homeomorphic to a surface of genus $\frac{n}{2}$ with 2 boundary components.

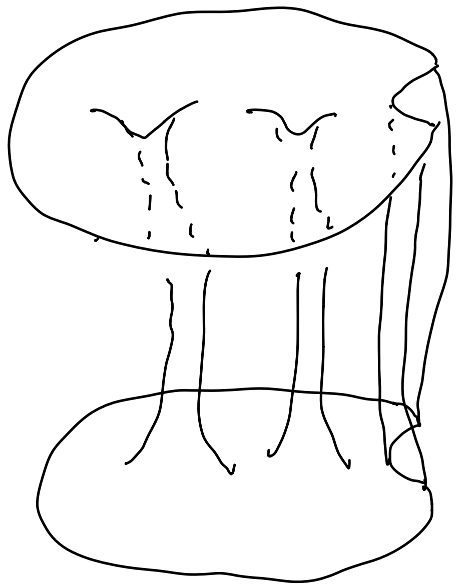
R is homeomorphic to a closed surface of genus $\frac{m}{2}-1$ with 2 pts. removed.

If m is odd:

Think of our base surface as $\mathbb{C}_\infty$ instead of $\mathbb{C}$.



R is homeomorphic to a surface of genus $\frac{m-1}{2}$ with one point removed.



genus = 2.