If f is holomorphic in some domain

$$D = \{z : 0 < |z-w| < r\}$$

then we say f has an isolated singularity at w. D is called a punctured disk.

We can expand f in a Laurent series around w:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-w)^n.$$

Let $N = \inf\{n : a_n \neq 0\}$ then f has a removable singularity if $N \geq 0$, a pole if $-\infty < N < 0$ and an essential singularity if $N = -\infty$.

If f has a removable singularity then we can extend f to a holomorphic function defined at w given by $\sum_{n \geq 0} a_n (z-w)^n$.

If f has a pole then set $m = n - N$ $\quad n = m + N$

$$f(z) = \sum_{n=N}^{\infty} a_n (z-w)^n = \sum_{m=0}^{\infty} a_{m+N} (z-w)^{m+N}$$

$$= (z-w)^N \underbrace{\sum_{m=0}^{\infty} a_{m+N} (z-w)^m}_{\text{Call this } g.}$$

g is holomorphic and

$g(w) = a_N \neq 0$.

So $f(z) = \dfrac{g(z)}{(z-w)^n}$ $\quad n \geq 1$.

It follows that $\lim_{z \to w} |f(z)| = \infty$.

Also follows that <u>poles are isolated</u>.

If f has an essential singularity at w then according to the Casorati-Weierstrass Thm for any ζ in $\mathbb{C}$ there exists a sequence $z_n \to w$ with $f(z_n) \to \zeta$.

So we can distinguish these 3 cases by the limiting behaviour.

Proof of Casorati-Weierstrass:

Say $f(D-\{0\})$ avoids a nbd. of w.
Note that $z \mapsto \frac{1}{z-w}$ takes w to ∞.

Let $g(z) = \frac{1}{f(z)-w}$.
Now g avoids a nbd. of ∞ so it is bounded.

If a bounded holomorphic function has an isolated singular pt. then it is a removable singularity.

So $\frac{1}{f(z)-w} = g(z)$ where g is holomorphic.

$$f(z)-w = \frac{1}{g(z)}$$

$f(z) = w + \frac{1}{g(z)}$ is mero-morphic or has a remov. sing.

We say
a function is meromorphic in a domain $U \subset \mathbb{C}$ if f is holomorphic in $U - \{z_j\}$, each z_j is isolated in U and f has at worst a pole at each z_j.

If f is meromorphic in U then we can define an extension of $f^+ : U \to \mathbb{C}_\infty$ by setting $f^+(z) = f(z)$ when z not a pole of f and $f^+(z) = \infty$ when z is a pole of f.
Clear from the prev. discussion that f^+ is continuous.

Remark. $U \subset \mathbb{C}$ can be viewed as a Riemann surface where its atlas consists of the single chart $\iota : U \to \mathbb{C}$ corresponding to the inclusion.

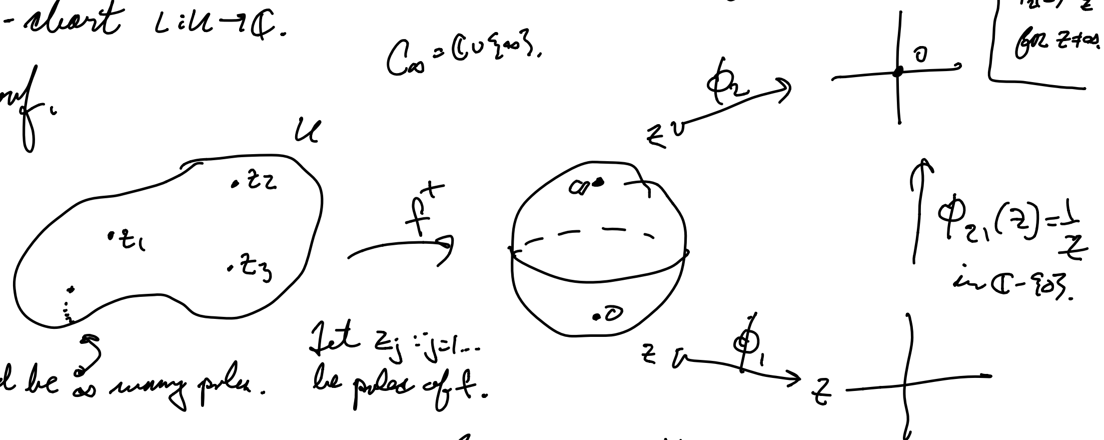
Prop. Viewed as a map between Riemann surfaces f^+ is a holomorphic map.

Recall what it means to be a holomorphic map between Riemann surfaces. It means the map is holomorphic when expressed in charts. Now U is a Riemann surface with 1-chart $\iota : U \to \mathbb{C}$.

Recall
$\phi_2(\infty) = 0$
$\phi_2(z) = \frac{1}{z}$ for $z \neq \infty$

$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Proof.



Could be as many poles.

Let $z_j : j = 1 \ldots$ be poles of f.

Let $z_0 \in U$. If $z_0 \neq z_j$ then $\phi_1 \circ f^+(z) = f(z)$ is holomorphic.

If $z_0 = z_j$ then choose a nbd of z_j where $f \neq 0$ (now N is positive.)

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_j)^n = \frac{g(z)}{(z - z_j)^N} \quad g(z_j) \neq 0$$

$$\phi_2 \circ f(z) = \frac{1}{f(z)} = (z - z_j)^N \cdot \frac{1}{g(z)} \text{ is holomorphic}$$

since $g(z_j) \neq 0$.

Theorem. A function f is meromorphic on $\mathbb{C}_\infty$ if and only if it is a rational function.

Note: Meromorphicity is a local assumption. Rationality is a global conclusion.

"Locally nice + compactness $\Rightarrow$ algebraic"

Proof. Let f be a rational function.

$f = \frac{P}{Q}$. $\quad Q = \prod (z - z_j)^{m_j}$. At each zero z_{j_0} of Q

$$f(z) = \frac{1}{(z - z_{j_0})^{m_{j_0}}} \underbrace{\frac{P(z)}{\prod_{j \neq j_0} (z - z_j)^{m_j}}}_{\text{holomorphic near } z_j.}$$

so f is meromorphic in $\mathbb{C}$.

To check that f is meromorphic at ∞.

Say $f(\infty)$ is finite.

$$\begin{tikzcd}
\mathbb{C}_\infty \arrow[r, "f"] & \mathbb{C}_\infty \arrow[d, "\varphi_1"] \\
\mathbb{C} \arrow[u, "z \mapsto \frac{1}{z}", "\varphi_2^{-1}"'] \arrow[r, "F"] & \mathbb{C}
\end{tikzcd}$$

$a_0 + \frac{a_1}{z} + \dots \frac{a_n}{z^n}$

$\parallel$

Let $F(z) = f(\frac{1}{z})$. Now $f(1/z) = \frac{P(1/z)}{Q(1/z)} = \frac{z^m \cdot P(\frac{1}{z})}{z^m \cdot Q(\frac{1}{z})}$.

For n suff. large numerator and denominator are polynomials so f is rational hence meromorphic.

End here.

If $f(\infty)=\infty$ we $1/z$ on RHS and look at $\frac{1}{f(1/z)} = \frac{Q(1/z)}{P(1/z)}$. Still rational.