11:00 am
Mon B2.01 Science Concourse
11:00 am
Wed MB0.07 (New building
12:00 noon
Fri MA_B1.01

Shengxuan Liu Monday.

There are notes and example sheets from last year on my personal website. These will be updated with current notes and example sheets as the course progresses.

Useful books.

A Primer on Riemann Surfaces

Beardon - Follow this book except we don't prove the classification of simply connected Riemann surfaces. Instead we supplement with material from:

Riemann Surfaces

Donaldson - Includes "further developments"
Connects with other directions.

Algebraic Curves

Kirwan - Info on algebraic curves.

Assuming basics of topology including fundamental groups and covering spaces.

Assuming complex analysis course but will review some of the high lights.

Not assuming that you have had a course in manifolds or differential forms.

We can think of this course as providing interesting (but limited) examples of both things.

Start with motivation for studying Riemann surfaces from 2017 notes.

Algebraic expressions define "multi-valued" functions.

E.G. $f(z) = \sqrt{z}$

$f(z) = \sqrt{z^3 - z}$

Equations in 2 complex variables $\{(x,y) \in \mathbb{C}^2 : y = P(x)\}$.

$\pi_x : (x, y) \mapsto x$.

Generalize $\{(x,y) \in \mathbb{R}^2 : y = P(x)\}$

Integration leads to "multi-valued functions".

$$\int_1^y \frac{dz}{z} \rightarrow \text{"}\log(z)\text{"}$$

Branch cuts.

Branches of the logarithm.

Combination $\int \frac{dz}{z(z-1)}$ or $\int_\gamma \frac{dz}{\sqrt{z^3 - z}}$

elliptic integrals

Solutions of differential equations.

$\frac{d}{dz} = f(z)$
simplest example.

Modern viewpoint is to ~~think~~ of
the "graphs" of these objects as
geometric objects in their own rite
by introducing the notion of Riemann surfaces.

Special case of manifolds.
Ubiquitous in modern mathematics.

Review of complex analysis. Let $f : U^{\subset \mathbb{C}} \to \mathbb{C}$ be a (real) differentiable function.

If we write vectors in $\mathbb{R}^2$ as column vectors.

Geometric picture

$U^{\subset \mathbb{C}}$

$f : U \longrightarrow \mathbb{C}$ $\qquad f = u + iv$

f satisfies the Cauchy-Riemann equations if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in U.

$$Df = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = m \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Geometrically:

If $Df \neq 0$ then Df preserves angles between vectors. Df has <u>positive</u> determinant a^2+b^2

Df <u>need not preserve lengths</u>.

We say that f is <u>holomorphic</u> if Df satisfies the CR equations. f is <u>conformal</u> if it is holomorphic and $Df \neq 0$ in its domain.

Algebraic picture. The Cauchy-Riemann equations are conditions on a real linear map in order for it to actually be complex linear.

In terms of the standard basis of $\mathbb{R}^2$ $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

multiplication by i is a real linear map with matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Call this matrix $J: \mathbb{R}^2 \to \mathbb{R}^2$.

A real linear map A from $\mathbb{R}^2$ to $\mathbb{R}^2$ is complex linear if $AJ = JA$. This condition is equivalent to the Cauchy-Riemann equations.

The algebraic discussion extends to higher dimensions whereas the geometric discussion is strictly 2-real or 1-complex dimensional.

Remark: To put a complex structure on a 2 dim real vector space we need a metric + orientation.

Basic tool for analyzing consequences of the Cauchy-Riemann equation is path integration.

The path integral is used to find anti-derivatives of holomorphic functions

Def.

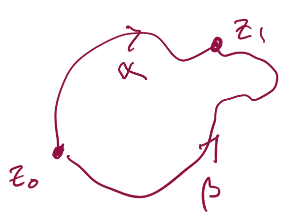
Path integral. $\gamma : [a,b] \longrightarrow U$

$$\int_\gamma f \, dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt.$$

Let U be a domain in $\mathbb{C}$ (open connected set).

Say that $\int_\gamma f dz = 0$ for all <u>loops</u> in U then given f on U and $z_0 \in U$ we can define a function F on U by

$$F(z_1) = \int_\gamma f \, dz \quad \text{where } \gamma(0) = z_0, \ \gamma(1) = z_1$$



Then $F(z_1)$ is independent of γ since if we have 2 paths α, β from z_0 to z_1 then

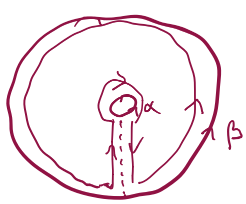
$\int_\alpha f\,dz - \int_\beta f\,dz = \int_{\alpha\cdot\beta^{-1}} f\,dz = 0$ since $\alpha\cdot\beta^{-1}$ is a loop. Furthermore the function F that we produce this way satisfies $F' = f$. It is an <u>anti-derivative</u> for f.

Cauchy's theorem. If f is <u>holomorphic</u>, U is a disk and γ is a loop then in U

$$\int_\gamma f\,dz = 0.$$

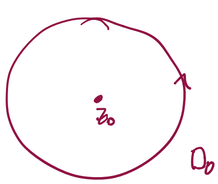
Corollary. f has a local anti-derivative (unique up to the addition of a constant).

If U is an annulus and f is holomorphic in U then



$$\int_\alpha f\,dz = \int_\beta f\,dz.$$

We can use this to prove Cauchy's integral formula. If f is holomorphic in a disk



$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D_0} \frac{f(z)}{z - z_0}\,dz.$$

(The value of f at z_0 is determined by its value on ∂D.)

Or $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

If f is holomorphic in a domain D then f has a Taylor expansion
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$
valid in a disc $\{z : |z-z_0| < r\}$ contained in D.

Consequence:

Morera's Theorem. If f is continuous in a domain U and if $\int_\gamma f = 0$ for every closed path in U then f is holomorphic in U.

Proof. By the previous discussion f has an anti-derivative F which is holomorphic hence infinitely differentiable. Follows that f is infinitely differentiable.

Theorem. If $f_1, f_2 \ldots$ are holomorphic in D and converge locally uniformly to f in D then f is holomorphic in D and for each $k \geq 1$ the sequence of k-th derivatives converges locally uniformly in D.

Proof. Derivatives are not controlled by uniform convergence but integrals are. Apply Morera's thm. and Cauchy integral formula.