Theorem. A function f is meromorphic on $\mathbb{C}_\infty$ if and only if it is a rational function.

Completion of proof

If $z_j \in \mathbb{C}$ is a finite pole we can write

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-z_j)^n \text{ for } z \text{ near } z_j.$$

The function $f_j(z) = \sum_{n=-N}^{-1} a_n (z-z_j)^n$ is rational and tends to 0 as $z \to \infty$ since it only contains negative powers of $(z-z_j)$.

Let $R(z) = \sum_j f_j(z)$.

Now $f - R$ is holomorphic at each z_j since we have subtracted off the negative powers of $(z-z_j)$ so it is holomorphic in $\mathbb{C}$.

Furthermore $f-R$ is continuous and finite valued at ∞ so it is bounded.

By Liouville theorem $f-R$ is constant so

$$f(z) = R(z) + c = \sum f_j(z) + c$$

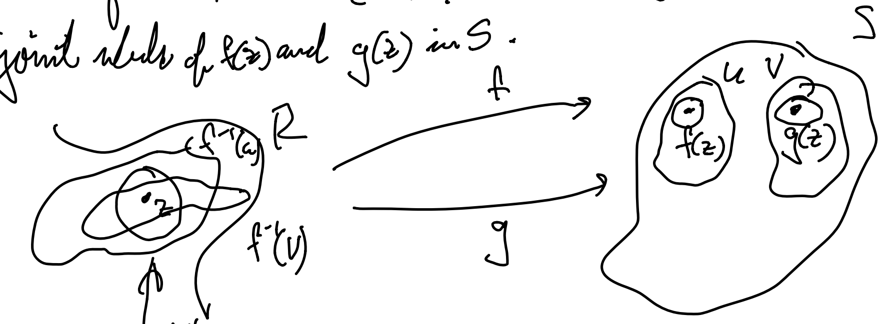
showing that f is rational.

(R connected)

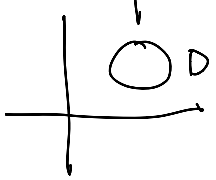
Theorem. Let R and S be Riemann surfaces and suppose that f and g are holomorphic maps of R to S. Then either $f = g$ at every point of R or $f = g$ at isolated points.

Remarks. We proved this already when R and S are domains in $\mathbb{C}$. Made use of $f - g$. Need to modify the proof in general still making use of the previously proved fact.

Proof. Pick $z \in R$. If $f(z) \neq g(z)$ then we can find disjoint nbds of $f(z)$ and $g(z)$ in S.

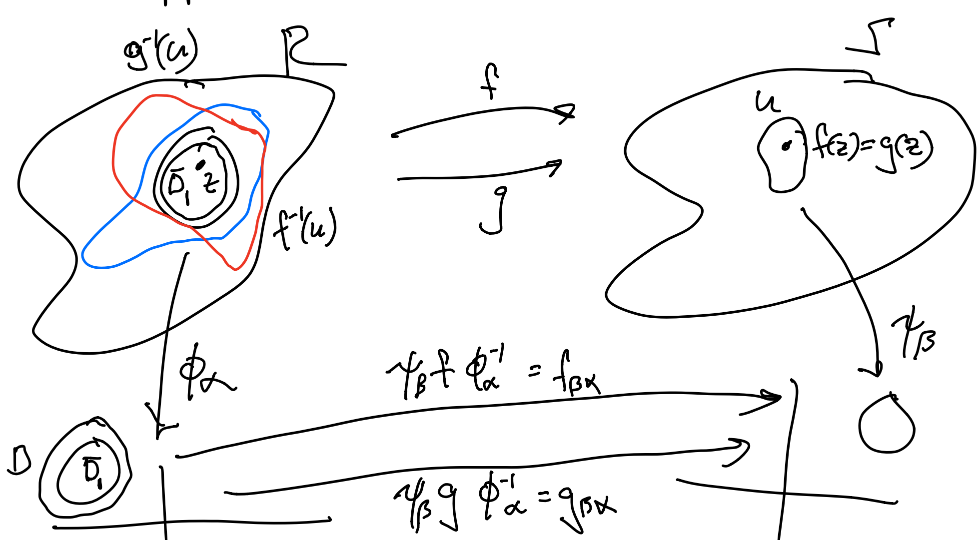


$z \quad f^{-1}(U) \cap f^{-1}(V)$.



So there is a coordinate chart around z and a $D \in \mathbb{C}$ so that $\phi(D) \subset f^{-1}(U) \cap f^{-1}(V)$ thus $f(z) \neq g(z)$ for $z \in \phi(D)$.

Now suppose $f(z) = g(z)$. Let U be a nbd. of $f(z) = g(z)$.



There is a compact disk $\bar{D}$ so that $\phi_\alpha^{-1}(\bar{D}) \subset g^{-1}(U) \cap f^{-1}(U)$.

Applying our previous result to $f_{\beta\alpha}$ and $g_{\beta\alpha}$ we see that the set where

Set of points where $f=g$ in D is consist of isolated points. So there are only finitely many points where $f(z)=g(z)$ in $\overline{D}$.

Now let A be the set of points p in R which have a nbd N_p where $f=g$ at finitely many points in N_p. A is open

Let B be the set of points with nbds where $f=g$ throughout N. B is open.

A and B are open and disjoint so $R=A$ or $R=B$ by connectivity.

Cor. If R is a Riemann surface then the collection of meromorphic functions on R (other than $f(R)=\infty$) is a field iff R is connected.

Proof. If f, g are meromorphic functions $p \in R$ and $f(p), g(p) \neq \infty$ we can add multiply functions so the set of functions forms a ring. If g has a discrete set of zeros we can form $\frac{1}{g}$. At a pole or zero of g, $\frac{1}{g}$ has a zero or pole so $\frac{1}{g}$ is meromorphic.

If g does not have a discrete set of zeros and R is connected then $g \equiv 0$.

If $R = R_1 \dot{\cup} R_2$, $f_1 \equiv 1$ on R_1, $f_1 \equiv 0$ on R_2 $\quad f_2 \equiv 1$ on R_2, $f_2 \equiv 0$ R_1
then $f_1 \cdot f_2 = 0$ while $f_1, f_2 \neq 0$ so f_1, f_2 are zero divisors.

Remark. There is an interesting parallel between fields of meromorphic functions on Riemann surfaces and number fields: finite degree extensions of $\mathbb{Q}$.

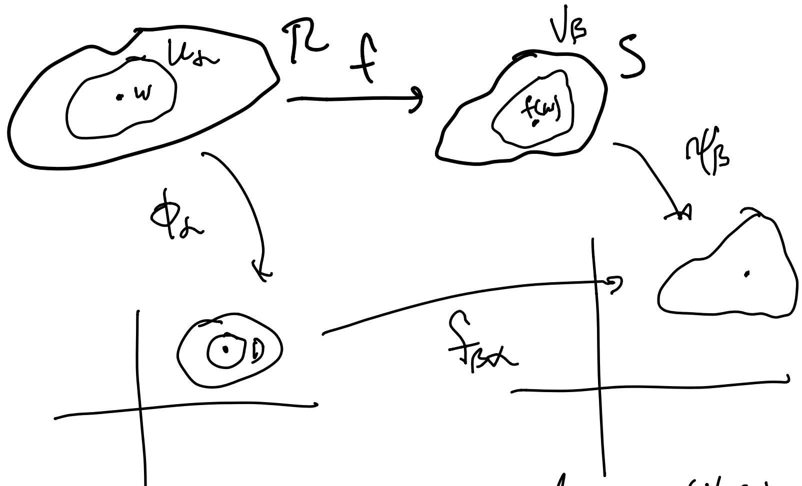
Remark. When $R = \mathbb{C}_\infty$ we can identify this field with the field of fractions $\mathbb{C}(z)$.

Theorem. Let R, S be Riemann surfaces and suppose that $f: R \to S$ is holomorphic but not constant. If $A \subset R$ is open then $f(A)$ is open.

Proof. Say $A \subset R$ is open. Assume A is not empty. Let $w \in A$. We have a coordinate chart φ mapping a nbd. of w, U to $\mathbb{C}$.

Let D be a disk around $\varphi(w)$.

φ^{-1}



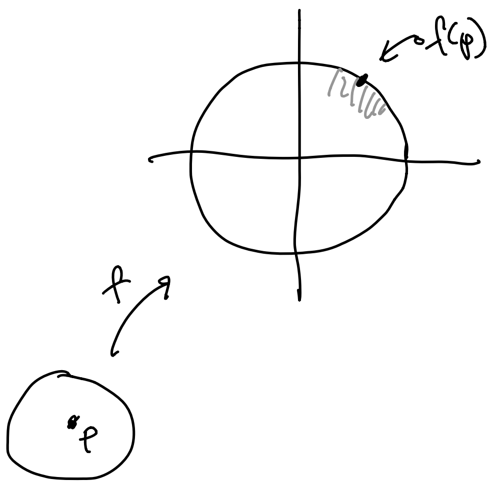
Choose a disk D around $\phi_\alpha(w)$ so that $\phi_\alpha^{-1}(D) \subset f^{-1}(V_\beta)$.

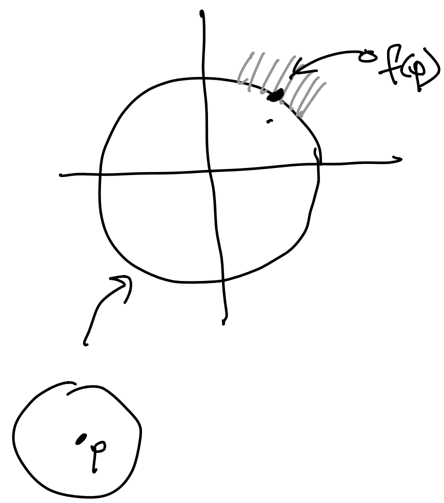
By the previous thm. since f is not constant it is not locally constant so $f_{\beta\alpha}$ is not constant. Thus $f_{\beta\alpha}(D)$ is open by the open mapping thm for holomorphic functions from $\mathbb{C}$ to $\mathbb{C}$. So $\psi_\beta^{-1}(f_{\beta\alpha}(D))$ is open since ψ_β is continuous.

It follows that $f(A)$ contains a nbd. of each of its points.

Theorem. Let $f: R \to \mathbb{C}$ be holomorphic but not constant on a Riemann surface R. Then $|f|$ has no local maximum and no positive local minimum.

This follows from the open mapping theorem.





R, S connected

Theorem. If $f : R \to S$ is analytic but not constant and if R is compact then $f(R) = S$ and so S is compact.

In particular a holomorphic function on a compact surface is constant.

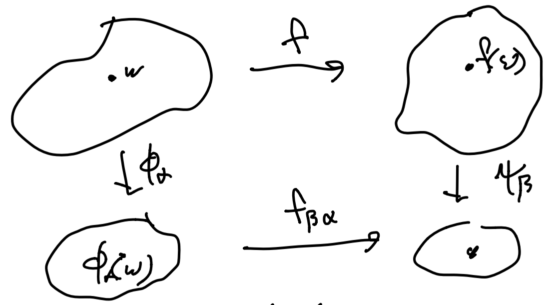
If f is meromorphic on a compact surface then $f(R) = \mathbb{C}_\infty$ unless f is constant.

Local behaviour of analytic functions.

Recall that if $w \in U \subset \mathbb{C}$ and $f : U \to \mathbb{C}$ then

$$\nu_f(w) = \min\{n \geq 1 : f^{(n)}(w) \neq 0\}.$$

Can we define v_f when f maps between Riemann surfaces?



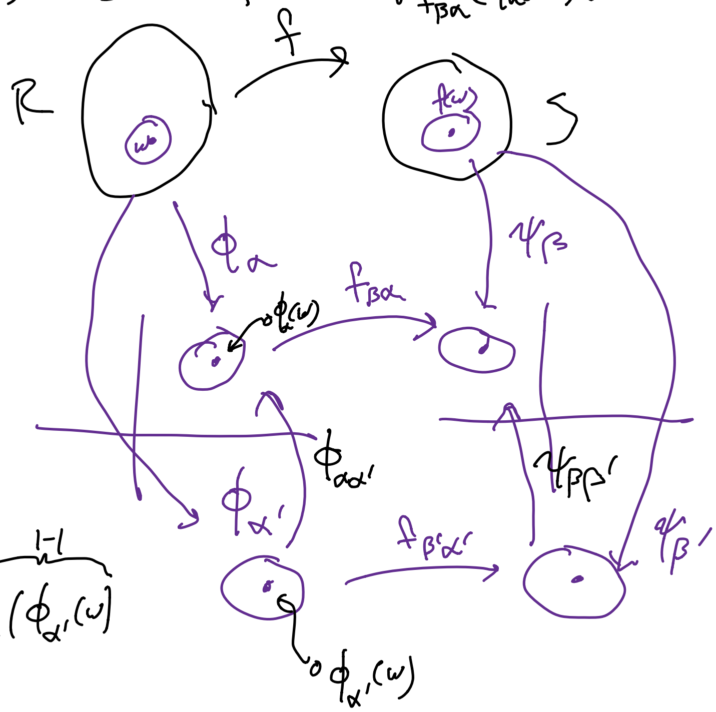
Now the derivatives of $f_{\beta\alpha}$ <u>do</u> depend on ϕ_α and ψ_β.

Problem. If h and g are 1-1 and holomorphic

Then $\quad V_{g\circ f\circ h}(w) = V_g(f h(w))\cdot V_f(h(w))\cdot V_h(w)$

$\qquad = V_f(h(w))$

Now if $f: R \to S$ is non-constant we can define $V_f(w)$ as $\quad V_f(w) \stackrel{def}{=} V_{f_{\beta\alpha}}(\phi_\alpha(w))$.



$V_{f_{\beta'\alpha'}}(\phi_{\alpha'}(w)) \qquad V$

$= \underbrace{V_{\psi^{-1}_{\beta\beta'}}(f_{\beta\alpha}(\phi_\alpha(w)))}_{1-1} V_{f_{\beta\alpha}}(\phi_\alpha(w)) \underbrace{V_{\phi_{\alpha\alpha'}}(\phi_{\alpha'}(w))}_{1-1}$

If we chose different coordinate charts

the derivatives of $f_{\beta'\alpha'}$ need not be the same as those of $f_{\beta\alpha}$ but the valence will be the same.

Model for transferring a definition from holomorphic maps $f: U \subset \mathbb{C} \to \mathbb{C}$ to Riemann surfaces.