Given a lattice Λ we have constructed a polynomial $z^3+g_2z+g_3$ and a variety $\tilde{R}_\Lambda$ which is the completion of $\{(z,w): w^2=z^3+g_2z+g_3\}$. We have constructed functions $\wp$ and $\wp'$ so that $u \overset{\Phi}{\mapsto} (\wp(u), \wp'(u))$ is a conformal isomorphism.

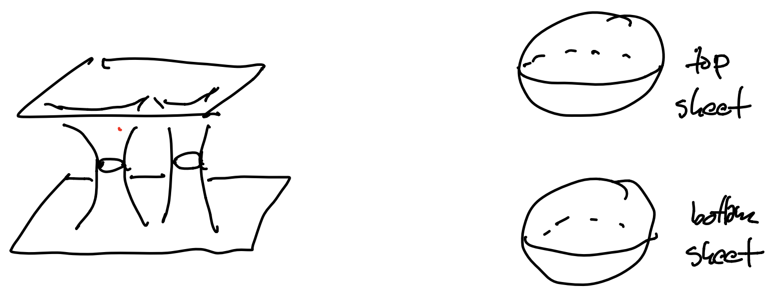
Which polynomials Q arise from this construction? (We know that Q must have distinct roots.)

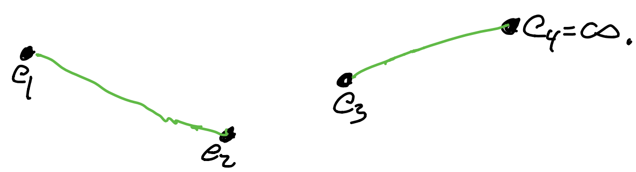
Simpler question: Given a polynomial $Q(z)=z^3+az+b$ with distinct roots. If we know that Q does come from this construction how do we find Λ?

Answer: We can find Λ by integrating $\int_\gamma \frac{dz}{\sqrt{Q(z)}}$ over loops in $\mathbb{C}$.

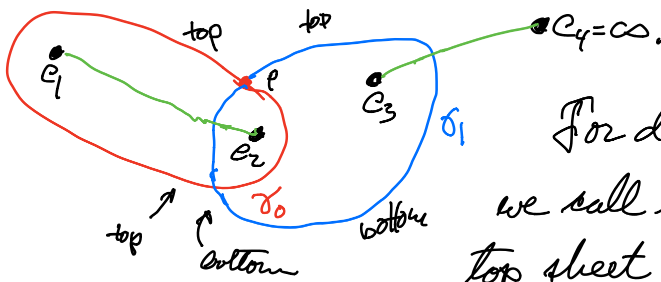
Recall that R_Λ can be built by gluing together two sheets along slits.

The function $\pi_z : R_\Lambda \to \mathbb{C}$ extends to $\pi_z : \hat{R}_\Lambda \to \mathbb{C}_\infty$ and π_z is a degree 2 function with 4 branch points at $e_1, e_2, e_3, e_4 = \infty$ where we identify e_1, e_2, e_3 with the roots of Q.





Now we construct two curves γ_0 and γ_1 in $\mathbb{C}_\infty - \{e_1, e_2, e_3, e_4\}$ and their lifts to $\hat{R}$.



For definiteness we call these the top sheet and bottom sheet

Pick a base point p on the top sheet. Note that the lifts intersect at only 1 point.

Each sheet corresponds to $\mathbb{C}_\infty$ minus 2 slits.

Consider the loop γ_0.

Since γ_0 does not cross a slit we can lift γ_0 to the (say) top sheet. Get $\tilde{\gamma}_0$.

Consider γ_1. Choose a lift that starts on the top sheet at p. This goes to the bottom sheet

Get $\tilde{\gamma}_1$.

So we can lift γ_0 and γ_1 to curves on R_Λ.

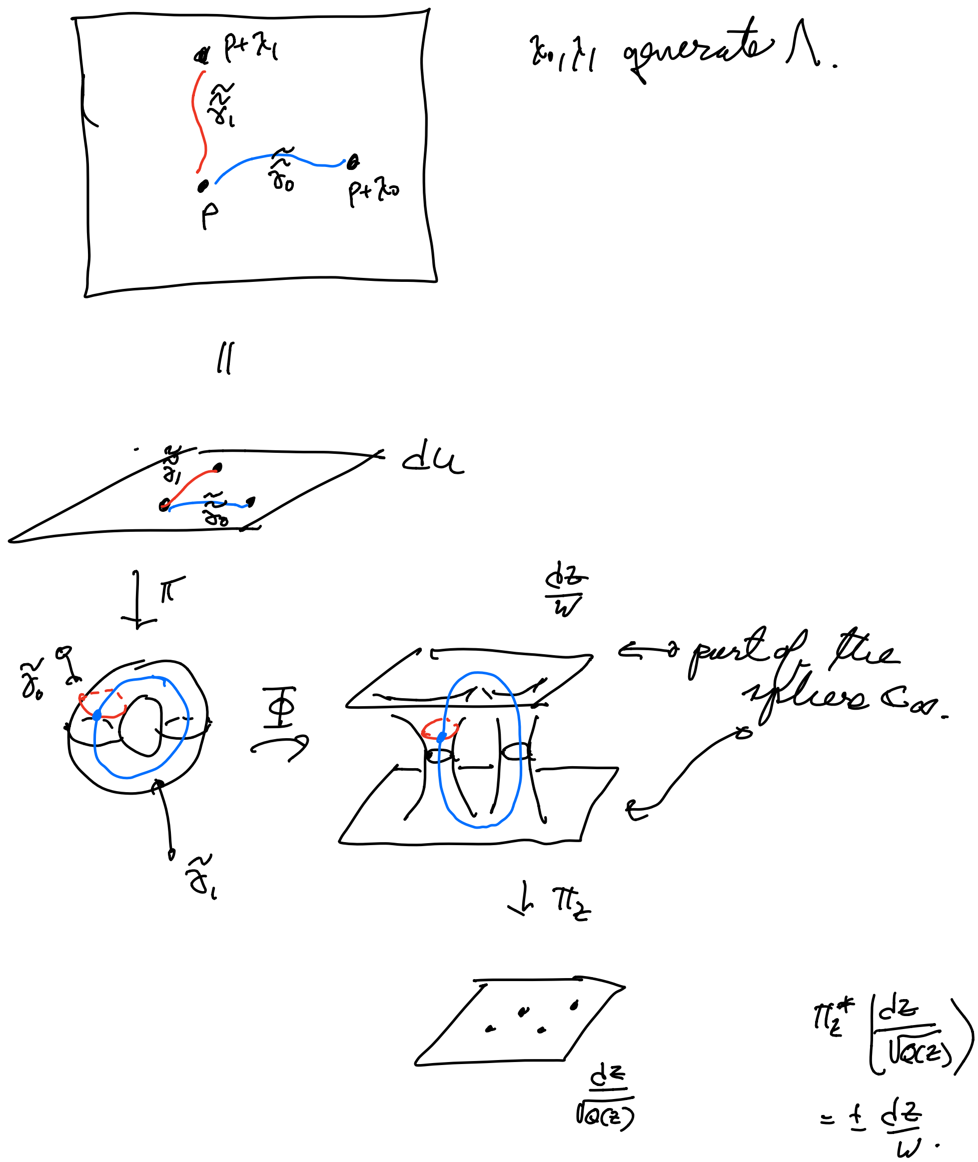
In particular along γ_0 and γ_1 the expression $\sqrt{Q(z)}$ has a consistent interpretation (as $\pi_W(\tilde{\gamma}_j)$).

Proposition,

$$\int_{\gamma_0} \frac{dz}{\sqrt{z^3+g_2z+g_3}} \quad \text{and} \quad \int_{\gamma_1} \frac{dz}{\sqrt{z^3+g_2z+g_3}}$$

generate the lattice Λ.

Proof.



Consider the covering map $\pi: \mathbb{C} \to \mathbb{C}/\Lambda = T^2$. Recall that we can identify the deck group (Λ) with

$\pi_1(T^2)$ as follows. If α is a loop in T^2 we lift α to $\tilde{\alpha}$ in $\mathbb{C}$ then $\tilde{\alpha}(1)$ and $\tilde{\alpha}(0)$ differ by the action of the deck group. In our case $\tilde{\alpha}(1) = \tilde{\alpha}(0) + \lambda_\alpha$.

In particular if α and β generate $\pi_1(T^2)$ then λ_α and λ_β generate Λ.

$$\int_{\gamma_0} \frac{dz}{\sqrt{Q(z)}} = \pm \int_{\tilde{\gamma}_0} \frac{dz}{w}$$

$$= \pm \int_{\tilde{\tilde{\gamma}}_0} du$$

$$= \pm \hat{\gamma}_0(t) \Big|_{t=0}^{t=1}$$

$$= \pm \tilde{\tilde{\gamma}}_0(1) - \tilde{\tilde{\gamma}}_0(0)$$

$$= \pm \big((p + \lambda_0) - p\big) = \pm \lambda_0$$

where λ_0 is the deck group element corresponding to $\tilde{\gamma}_0$.

Similarly $\displaystyle\int_{\gamma_1} \frac{dz}{\sqrt{Q(z)}} = \pm\lambda_1$ where

λ_1 corresponds to $\tilde{\gamma}_1$.

Since $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ generate $\pi_1(T^2)$

$\{\pm\lambda_0, \pm\lambda_1\}$ generate Λ.

What happens if we don't know that R_Q has the form R_Λ for some Λ?

It is still true that $\tilde{R}_Q$ is a compact Riemann surface of genus 1 with a non-vanishing 1-form.

The above construction gives a subgroup of $\mathbb{C}$. Is this subgroup discrete? If so then we can start from this subgroup and produce a conformally equivalent surface $\tilde{R}_\Lambda$.

Answer. It follows from the Main Theorem that any compact Riemann surface of genus 1 has the form $\mathbb{C}/\Lambda$

Why? A Riemann surface R has the

form S^2, $\mathbb{C}/\Gamma$ or Δ/Γ. In each case it has a metric of constant curvature preserved by Γ. If the surface is compact then by the Gauss-Bonnet formula $\chi(R)>0$ in case 1, $\chi(R)=0$ in case 2 and $\chi(R)<0$ in case 3.

For a torus $\chi(T^2)=0$.

We can also see that T^2 has a non-vanishing holomorphic 1-form dz.

It is also true that any 2 non-vanishing 1-forms differ by a constant.

Prop. If a compact Riemann surface has a non-vanishing holomorphic 1-form then any 2 non-vanishing 1-forms differ by mult. by a constant.

Proof. If α and β are non-vanishing 1-forms then in a chart $\alpha = f(z)dz$, $\beta = g(z)dz$.
Claim that the ratio "$\frac{\alpha}{\beta}$" $\stackrel{\text{def}}{=} \frac{f(z)}{g(z)}$ gives a holomorphic function independent of the chart. If u is the variable coming from a different chart then

$$\alpha = f(u)\frac{dz}{du} \cdot du$$

$$\beta = g(u)\frac{dz}{du} \cdot du$$

so $\frac{\alpha}{\beta} = \frac{f}{g}$. Compactness implies this function is constant.

Given a lattice Λ we have constructed a polynomial P and an elliptic surface $R = R_P$.

Given an elliptic variety how do we recover the lattice Λ?

Compute the image of the homomorphism $\pi_1(R) \to \mathbb{C}$ given by integrating $\frac{dz}{w}$.

In fact we can define a holomorphic map

$$z \mapsto \int_{z_0}^{z} \frac{dz}{w} \to \mathbb{C}/\Lambda.$$

In particular there is a conformal isomorphism

$$\Psi : \mathbb{C}/\Lambda \longrightarrow R_Q \quad \text{and} \quad \Psi^*\left(\frac{dz}{w}\right) = c \cdot du.$$

In particular the above construction will produce the lattice $c\Lambda$. Note that $\mathbb{C}/\Lambda$ and $\mathbb{C}/c\Lambda$ are conformally equivalent.

What we get are maps in two directions:

$$\begin{tikzcd}
\text{lattices } \Lambda \arrow[r, bend left, "\text{constructing } \wp \text{ function}"] & \begin{array}{c}\text{cubic}\\ \text{polynomials}\\ \text{with distinct}\\ \text{roots}\end{array} \arrow[l, bend left, "\text{integrating}"]
\end{tikzcd}$$

The compositions need not be the identity but the compositions will have the property that they preserve conformal equivalence.

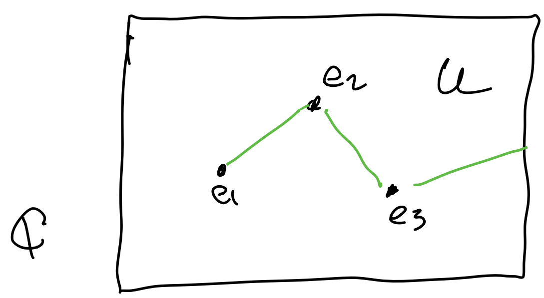
Key question: When do two lattices Λ and Λ' give conformally equivalent surfaces?

When do two polynomials Q, Q' give conformally equivalent surfaces?

Integration discussion (in class this was part of the previous lecture.)

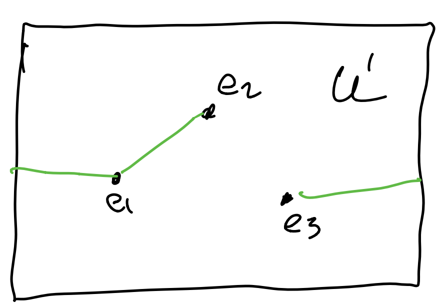
Branches of $\wp^{-1}$

Think of $\wp$ as a doubly periodic function from $\mathbb{C}$ to $\mathbb{C}_\infty$.



Let me revert to putting e_4 at ∞. Construct a collection of curves through e_1, e_2, e_3 so that the complement $\mathcal{U}$ is simply connected. The restriction of $\wp \circ \pi$ to $\wp^{-1}(\mathcal{U})$ is a covering map. Let $\wp^{-1}$ be a branch of this map.

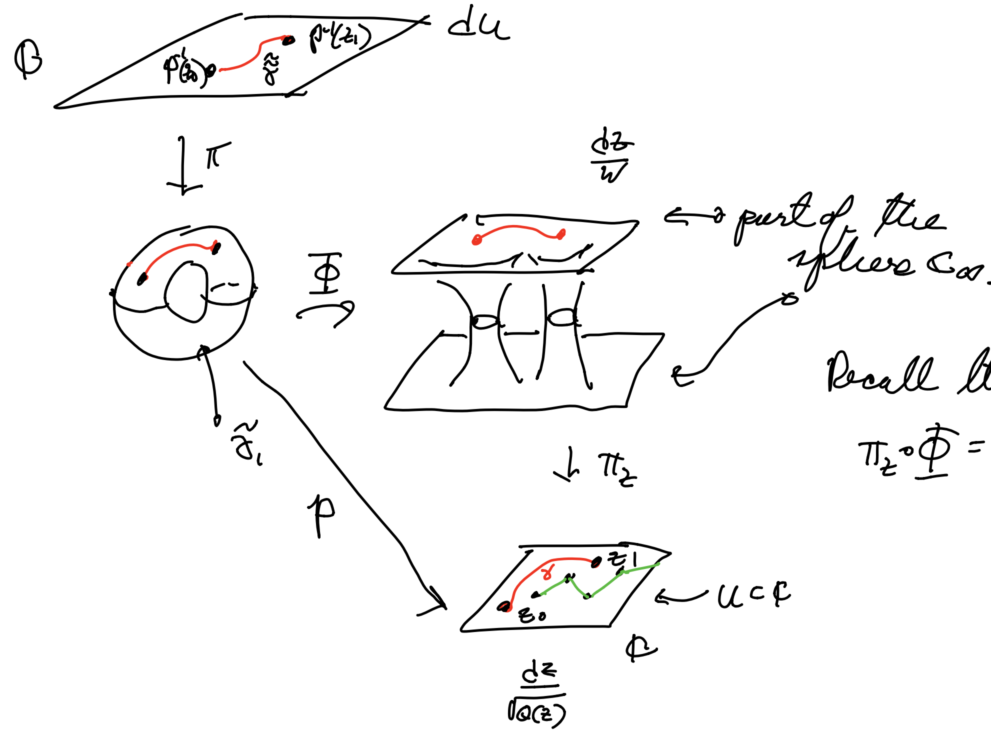
Alternate choice:



Claim. Let P^{-1} be a branch of the inverse function over U. Let $z_0, z_1 \in U$. Then:

$$\int_{z_0}^{z_1} \frac{dz}{\sqrt{Q(z)}} = P^{-1}(z_1) - P^{-1}(z_0).$$

Proof.



Recall that

$\pi_z \circ \Phi = P.$

Consider the map

$$\pi_z \circ \Phi \circ \pi = P.$$

If we remove the green curves from $\mathbb{C}$ then the result is a simply connected region $U \subset \mathbb{C}$

so we can define a branch of P^{-1}: $P^{-1}: U \to \mathbb{C}$.

Say that z_0 and z_1 are points in U and $\gamma: [0,1] \to \mathbb{C}$

is a path between them. $\gamma(0) = z_0$, $\gamma(1) = z_1$.

Since U is simply connected any two such

curves are homotopic.

If γ is a path in U then $P^{-1}(\gamma)$ is a lift of γ to

a path $\tilde{\tilde{\gamma}}$ in $\mathbb{C}$. Set $\tilde{\tilde{\gamma}}(t) = P^{-1}(\gamma(t))$.

$$\int_\gamma \frac{dz}{\sqrt{Q(z)}} = \int_{P(\tilde{\tilde{\gamma}})} \frac{dz}{\sqrt{Q(z)}}$$

$$= \int_{\tilde{\tilde{\gamma}}} P^*\left(\frac{dz}{\sqrt{Q(z)}}\right)$$

$$= \int_{\tilde{\tilde{\gamma}}} du$$

$$= \tilde{\tilde{\gamma}}(1) - \tilde{\tilde{\gamma}}(0)$$

$$= P^{-1}(z_1) - P^{-1}(z_0).$$