We continue with our discussion of forms.

This is an example of modern mathematics

There is a formalism and there is a geometric interpretation.

The formalism is set up so that certain proofs will be trivial. In particular the motivation for the formalism is not transparent, it reveals itself as we go along.

Hopefully the geometric connections reveal themselves as you go along. Example sheets and recitation sections can help with this.

Additional reference for forms:

Hubbard & Hubbard.

Vector calculus, linear algebra and differential forms.

(i.e. a function)

Exterior derivative. If f is a 0-form

$$d(f) = \sum \frac{\partial f}{\partial x_j} \cdot dx_j \quad \text{is a 1-form.}$$

$$\boxed{\text{derivative of } f \text{ in direction } v}$$

Note $d(f)[v] = Df_v$ so the exterior derivative for functions can be identified with the derivative of a function.

Exterior derivative of a k-form θ is the $k+1$ form $d\theta$ defined by

say $\theta = f dx_I$

$$d\theta = d(f\, dx_I) = \overset{1\text{-form}}{df} \wedge \underset{k\text{-form}}{dx_I}$$

$$d\left(\sum_I f_I\, dx_I\right) = \sum_I \underbrace{df_I \wedge dx_I}_{k+1 \text{ form}}.$$

Prop. $d^2\theta = 0$. $\theta = \sum_I f_I \, dx_I$

$$d(d\theta) = d\left(\sum_I \left(\sum \frac{\partial f_I}{\partial x_j} dx_j\right) dx_I\right)$$

$$= \sum_{i,j,I} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I$$

$$= \sum_{i \neq j, I} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I$$

$$= \sum_{i<j,I} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I + \sum_{i>j,I} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I$$

$$- \sum_{i<j,I} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_i \wedge dx_j \wedge dx_I$$

$$= \sum_{i<j,I} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j \wedge dx_I$$

$$= 0.$$

Product formula: θ a l-form ψ an m-form

$$d(\theta\wedge\psi) = d\theta\wedge\psi + (-1)^{l}\,\theta\wedge d\psi.$$

Proof.

Case 1.
2 functions $d(f\cdot g) = \sum \frac{\partial f\cdot g}{\partial x_i}\,dx_i$

$$= \sum\left[\left(\frac{\partial f}{\partial x_i}\right)g + f\left(\frac{\partial g}{\partial x_i}\right)\right]dx_i$$

$$= g\cdot\left(\sum \frac{\partial f}{\partial x_i}\,dx_i\right) + f\cdot\left(\sum \frac{\partial g}{\partial x_i}\,dx_i\right)$$

$$\left(\sum \frac{\partial}{\partial x_i} \cdots\right) + \left(\sum \frac{\partial}{\partial x_i} \cdots\right)$$

$$= (df) \cdot g + f \cdot (dg).$$

Case 2.

$$\theta = \sum_I f_I \, dx_I \qquad \psi = \sum_J g_J \, dx_J$$

Now say

$$\theta \wedge \psi = \sum_{I,J} f_I \cdot g_J \; dx_I \wedge dx_J$$

regular product formula applied to $f_I g_J$.

then

$$d(\theta \wedge \psi) = \sum_{I,J} \Big((df_I) g_J + f_I (dg_J) \Big) dx_I \wedge dx_J$$

gather terms

$$= \left(\sum_{I,J} df_I \wedge g_J \wedge dx_I \wedge dx_J + \sum_{I,J} f_I \, dg_J \wedge dx_I \wedge dx_J \right)$$

deg 1 deg l deg 0 deg m deg 1 deg l

$$= \sum_{I,J} df_I \, dx_I \wedge g_J \, dx_J + (-1)^l \sum_{I,J} f_I \, dx_I \wedge dg_J \, dx_J$$

$$= \left(\sum_I df_I \, dx_I \right) \wedge \left(\sum_J g_J \, dx_J \right)$$

$$+ (-1) \left(\sum_I f_I \, dx_I \right) \wedge \left(\sum_J dg_J \, dx_J \right)$$

$$= d\theta \wedge \psi + (-1)^l \theta \wedge d\psi.$$

Demonstrates that pullback does not on the

Pullbacks
(change of variables)

system?

Definition. Let $G: U^{\subset \mathbb{R}^m} \longrightarrow V^{\subset \mathbb{R}^n}$ be a smooth function. Let θ be a k-form on V. We define the pullback $G^*\theta$ by:

$$(G^*\theta)[V_1 \ldots V_k] = \theta\left[DG(V_1) \ldots DG(V_k)\right]$$

Fact behind the definition: k-forms give all alternating k-linear functions. It is clear that the pullback of an alternating k-linear function is an alternating k-linear function so the pullback of a k-form is a k-form.

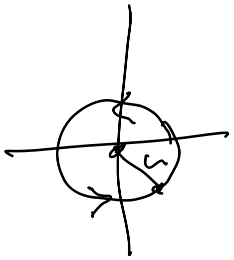
Definition adapted to calculation:

If f is a function on V then we pull it back by composing with G: $G^*(f) = f \circ G$.

Example of a pullback. .

$$\omega = \frac{-y\,dx + x\,dy}{x^2+y^2} \in \mathcal{C}^1(\mathbb{R}^2 - \{0\})$$

$$\overset{(t)}{G: \mathbb{R}} \longrightarrow \overset{\omega \quad (x,y)}{\mathbb{R}^2 - \{0\}}$$



$$G(t) = \begin{bmatrix} r\cdot\cos(t) \\ r\cdot\sin(t) \end{bmatrix} \begin{array}{l} \leftarrow x \\ \leftarrow y \end{array}$$

$$d\cos(t) = -\sin(t)\,dt \qquad d(r\cdot\cos(t)) = -r\sin t\,dt$$

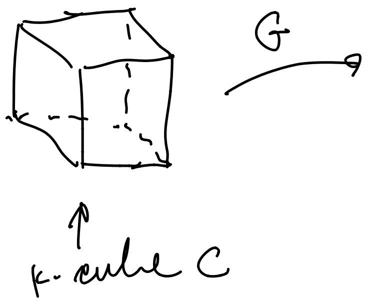
$$G^*(\omega) = \frac{-r\cdot\sin(t)\,d(r\cdot\cos(t)) + r\cdot\cos(t)\,d\,r\cdot\sin(t)}{r^2(\cos^2(t)+\sin^2(t))}$$

$$= \frac{r^2\sin^2(t)\,dt + r^2\cos^2(t)\,dt}{r^2(\sin^2(t)+\cos^2(t))}$$

$$= dt.$$

Wedge and exterior derivative pullback naturally. (One forms transform <u>differently</u> from vector fields.)

Integration:



θ a k-form.

This order of coordinates corresponds to a certain orientation on C.

Write $\int_{G(C)} \theta = \int_C G^*(\theta) = \int_C f \, \underbrace{dx_1 \wedge \dots \wedge dx_k}$

for forms the order of the 1-forms affects the sign

$\overset{\text{def.}}{=} \int_C f \, \underbrace{dx_1 \dots dx_k}$

in integration typically the order of coords plays no role.

Step 1. On a k-cube C we have

$\omega = f \cdot dx_1 \wedge \dots dx_k.$

We define $\int_C \omega = \int_C f \, dx_1 \dots dx_k.$

On the image of a k-cube in $\mathbb{R}^n$

$G: C \longrightarrow \mathbb{R}^n$ ω a k-form on $\mathbb{R}^n$.

then $\displaystyle\int_{G(C)} \omega = \int_C G^*\omega$.

Def.
Day 29 35:04.

Change of variables formula 28:30.

$\Omega \subset \mathbb{R}^n$

$$\int_{\vec{g}(\Omega)} f \, dV_{\vec{x}} = \int (f\circ\vec{g}) \left|\det D\vec{g}\right| dV$$

Change of variables

$$\vec{g}^*\omega = (f\circ\vec{g})\, dg_1 \wedge \dots dg_k$$

$$= (f\circ\vec{g})(\det D\vec{g})\, dx_1 \wedge \dots dx_k.$$

Compare with change of variables for an n-form. Only difference is the sign corresponds to orientation.