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In the last class we discussed maps from the set of lattices $\{\mathbb{C}/\Lambda\}$ to the set of Riemann surfaces $\{R_Q\}$ where Q is a cubic poly. with distinct roots and back again.

It is useful to turn such a sets into a topological spaces. This would be a a space X where every $p \in X$ corresponds to a Riemann surface.

Issue arose that we might start with an object apply the maps in both directions and get back to a conformally equiv. object.

We say X has the additional property of being a <u>moduli</u> space if distinct points correspond to conformally inequivalent surfaces.

We start with the set $\{\mathbb{C}/\Lambda : \Lambda \text{ a lattice in } \mathbb{C}\}$.
Recall that <u>every</u> compact genus 1 Riemann surface is equivalent to (at least) one element of this set.

In order to construct coordinates it is useful to introduce general Riemann surfaces with an extra piece of structure called a marking. A marking on T^2 is a choice of an ordered basis (γ_0, γ_1) for $\pi_1(T^2) = \mathbb{Z}^2$.

We have a homomorphism $h : \pi_1(\mathbb{C}/\Lambda) \to \mathbb{C}$ which takes a loop γ to $\int_\gamma dz$. This map identifies $\pi_1(\mathbb{C}/\Lambda)$ with Λ. An ordered basis for $\pi_1(\mathbb{C}/\Lambda)$ determines an ordered basis for Λ.

We say that two marked conformal tori T and T' are equivalent if there is a hol. isomorphism which takes T to T' and takes the marking for T to the marking for T'.

Prop. Two marked tori (λ_0, λ_1) and (λ_0', λ_1') are conformally equivalent iff $\frac{\lambda_1}{\lambda_0} = \frac{\lambda_1'}{\lambda_0'}$.

Proof. Say $f: \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda'$ is holomorphic.

By covering space theory we have a lift to $\tilde{f}: \mathbb{C} \longrightarrow \mathbb{C}$ which is holomorphic so that $f(z+\lambda) = f(z) + \lambda'$ (*).

A holomorphic automorphism of $\mathbb{C}$ has the form $f(z) = az + b$. If we compose

f with a translation we still have property $(*)$ but we can assume $b=0$.

Now $f(\Lambda)=\Lambda'$ so $f(\lambda_0)=\lambda_0'$, $f(\lambda_1)=\lambda_1'$

Thus $\dfrac{\lambda_1'}{\lambda_0'} = \dfrac{a\lambda_1}{a\lambda_0} = \dfrac{\lambda_1}{\lambda_0}$.

Conversely if the ratios are the same then define a to be $\dfrac{\lambda_0'}{\lambda_0} = \dfrac{\lambda_1'}{\lambda_1}$. Then $f(z)=az$ induces a conformal isomorphism from $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$.

$$\frac{\lambda_1'}{\lambda_0'} = \frac{\lambda_1}{\lambda_0} \Rightarrow \lambda_1' = \frac{\lambda_1\lambda_0'}{\lambda_0} \Rightarrow \frac{\lambda_1'}{\lambda_0'} = \frac{\lambda_1}{\lambda_0}.$$

The set of markings falls into two classes. Recall that λ_0, λ_1 are linearly independent over $\mathbb{R}$ so

$$\det \begin{bmatrix} \operatorname{Re}\lambda_0 & \operatorname{Re}\lambda_1 \\ \operatorname{Im}\lambda_0 & \operatorname{Im}\lambda_1 \end{bmatrix} \neq 0.$$

matrix of the map $h: \mathbb{R}^2 \to \mathbb{C}$ expressed in the basis $1, i$.

This quantity can be positive or negative.

If it is negative we can make it positive by switching λ_0 and λ_1. Let us assume that we only deal with positive markings. ⑤

Prop. We can identify the space of conformal classes of <u>marked</u> tori with the upper half plane $(\lambda_0\ \lambda_1)^{\in\Lambda} \rightsquigarrow \frac{\lambda_1}{\lambda_0}$.

Proof.

$$\mathrm{Im}\left(\frac{\lambda_1}{\lambda_0}\right) = \mathrm{Im}\left(\frac{\lambda_1\bar{\lambda}_0}{|\lambda_0|^2}\right) = \frac{1}{|\lambda_0|^2}\,\mathrm{Im}\,(\lambda_1\bar{\lambda}_0) = \frac{\det}{|\lambda_0|^2}$$

$$\mathrm{Im}\Big(\big(\mathrm{Re}\,\lambda_1 + i\,\mathrm{Im}\,\lambda_1\big)\big(\mathrm{Re}\,\lambda_0 - i\,\mathrm{Im}\,\lambda_0\big)\Big)$$

$$= \left(\mathrm{Im}\,\lambda_1 \cdot \mathrm{Re}\,\lambda_0 - \mathrm{Re}\,\lambda_1\,\mathrm{Im}\,\lambda_0\right)$$

Remark. The space of marked tori is not only a topological space it is actually a simply connected Riemann surface! With a metric of curvature -1!

The next step in the construction of a moduli space is to determine when two marked Riemann surfaces correspond to the same (unmarked) Riemann surface.

This is equivalent to asking when two pairs λ_0, λ_1 and μ_0, μ_1 generate the same lattice.

Prop. λ_0, λ_1 and μ_0, μ_1 generate the same lattice iff there is an integral matrix of det 1 with

$$\begin{bmatrix} \lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \mu_0 & \mu_1 \end{bmatrix}$$

Proof. Say that $\Lambda = \{ m\lambda_0 + n\lambda_1 : m, n \in \mathbb{Z} \}$.

If $\mu_0 \in \Lambda$ then $\mu_0 = a\lambda_0 + b\lambda_1$

$$\mu_1 = c\lambda_0 + d\lambda_1.$$

This says that

$$\begin{bmatrix} \lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \mu_0 & \mu_1 \end{bmatrix}.$$

Conversely

$$\begin{bmatrix} \mu_0 & \mu_1 \end{bmatrix} \begin{bmatrix} a' & c' \\ b' & d' \end{bmatrix} = \begin{bmatrix} \lambda_0 & \lambda_1 \end{bmatrix}$$

Implies A and A' are inverses so $\det A = \pm 1$.

But $\det \begin{bmatrix} \lambda_0 & \lambda_1 \end{bmatrix} > 0$, $\det \begin{bmatrix} \mu_0 & \mu_1 \end{bmatrix} > 0$ so $\det A = I$.

Now we can identify the upper half space with the coset space

$$\mathbb{C}^* \backslash GL(2,\mathbb{R}) \approx \mathbb{H}$$

$$z \backslash \begin{bmatrix} \lambda_0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} z\lambda_0 & z\lambda_1 \end{bmatrix}$$

$$\frac{z\lambda_1}{z\lambda_0} = \frac{\lambda_1}{\lambda_0}.$$

We can identify the moduli space of genus 1 Riemann surfaces with the double coset space:

$$\mathbb{C}^* \backslash \{ \begin{bmatrix} \lambda_0 & \lambda_1 \end{bmatrix} \} / SL(2,\mathbb{Z}).$$