Prerequisites Complex analysis MA 3B8

MA 3F1 Intro to topology

Monday 11:00-12:00 H5.45 Humanities 5th floor.

Wednesday 11:00-12:00 MS.05

Friday 12:00-1:00 MA_B1.01

surfaces fundamental groups, covering spaces.

Revision? today.

MA3H5

Manifolds. Prof. Bowditch

Not a pre-req. but ~~might~~ who has taken it?

Who not?

Example sheets will appear on my personal website.

Will post sketchy, incomplete notes. Guide to topics. Not a substitute for your own class notes.

The subject of Riemann surfaces has been taught many times.

Guide to the literature. Also some internet sources.

Focus on terminology and work which is not so standard in Riemann surfaces courses.

Problem 1. There are natural algebraic expressions which have ambiguities in their solutions.

For example $f(z) = \sqrt{z}$.

Want to be able to deal with more complicated algebraic expressions in a coherent way.

Problem 2. There are natural constructions which lead to multivalued functions

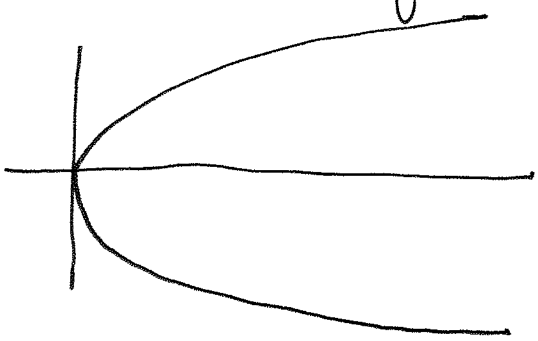
Example: $\oint_\gamma \frac{dz}{z}$. $\int_1^w \frac{dz}{z}$ We get a holomorphic function but the value of this function depends not just on w but on the path we chose from ~~our starting point~~ 1 to w.

$\boxed{\frac{df}{dz} = g}$

This is an example of a differential equation $f'(z) = \frac{1}{z}$. Similar phenomena arise with the issue of piecing together local solutions of more complicated differential equations.

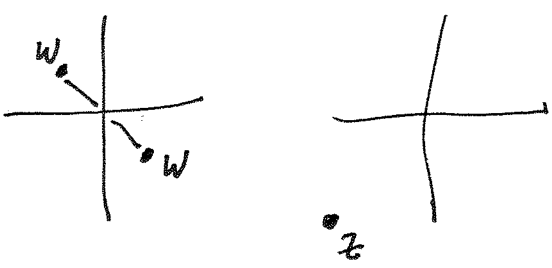
Let's return to problem 1.

In the case of a real variable we can



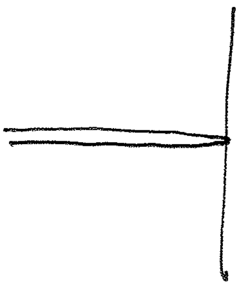
deal with the ambiguity of $f(x)=\sqrt{x}$ once and for all by choosing a branch of the function over the positive reals and by convention we choose the positive branch.

In the complex case we are forced to make a somewhat unnatural choice.
For a given z we are choosing a w that satisfies $w^2=z$. How do we make a choice? We can choose the value of w with a positive real part. This gives



a unique choice except ~~on the~~ when z is on the negative real axis when both choices have zero real part.

So we can make a slit in $\mathbb{C}$ along the



negative real axis then over the slit plane ~~on the~~ our function has two branches each of which is continuous and holomorphic.

Rather than making slits in our domain ($\mathbb{C}$), we want to make the conceptual shift (following Riemann) to replacing the domain of our function by the graph of the function:

$$S = \{(z,w) \in \mathbb{C}^2 : w^2 = z\}.$$

This is some type of surface sitting inside a 4 real dimensional space.

On the domain S the function $w = \sqrt{z}$ ~~is the single va~~ is single valued.

S is an example of a Riemann surface.

We can do this more generally by starting with a polynomial in 2 complex variables

$$P(z,w) = \sum_{n,m=0}^{k} a_{n,m} z^n w^m$$

and considering the graph

$$S = \{ (z,w) \in \mathbb{C}^2 : P(z,w) = 0 \}$$

We want to know:

① What kind of objects are Riemann surfaces?

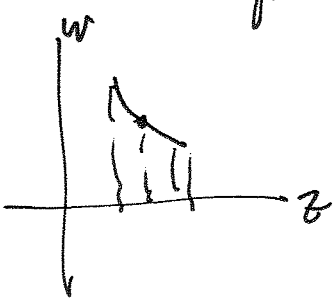
② How can we do complex analysis on them?

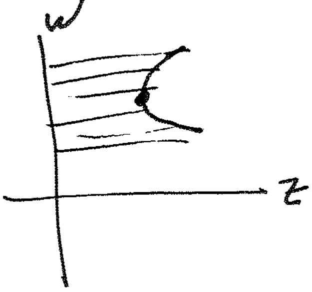
Let's return to our Riemann surface S. For this particular S $= \{(z,w) : z = w^2\}$ we can answer both these questions. Note that there is a bijection $\psi : \mathbb{C} \longrightarrow S$ given by $\psi(w) = (w^2, w)$ since S is really a ~~graph~~ single valued graph if we think of z as a function of w.

In particular S is homeomorphic to $\mathbb{C}$ and we can define the class of holomorphic functions on S to be the $g: S \to \mathbb{C}$ so that the composition $\mathbb{C} \xrightarrow{\psi} S \xrightarrow{g} \mathbb{C}$ is holomorphic.

In other words we are using the function ψ (which we will call a chart) to determine the class of holomorphic functions on S.

~~Let's consider a second example which~~
More generally we may have an S which doesn't have a single chart but that each point in S has a nbd that projects bijectively onto the w or z plane. (or both).





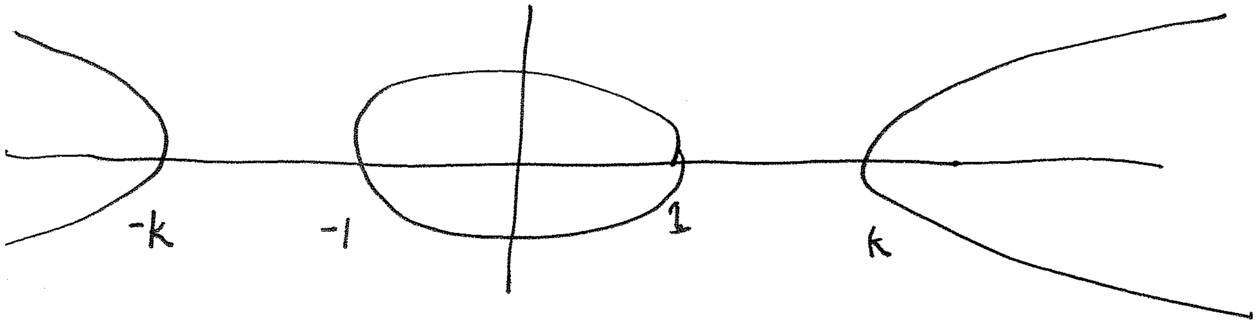
Before we define this rigorously let's look at a more complicated (and more typical) example.

Consider $w = \sqrt{(z^2-1)(z^2-k^2)}$ where $k > 1$.

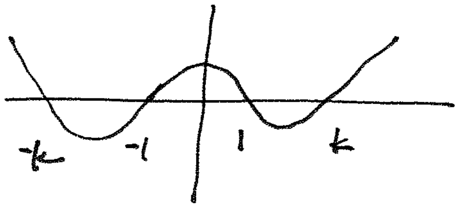
Following the method of the previous example we look at

$$T = \{(z,w) \in \mathbb{C}^2 : w^2 = (z^2-1)(z^2-k^2)\}$$

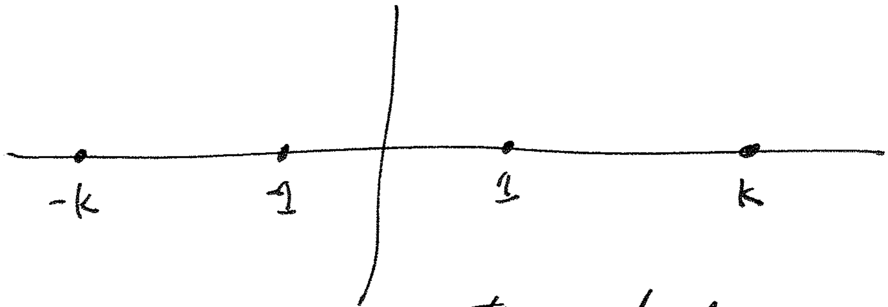
If we sketch the real locus of ~~the~~ T ie $T \cap \mathbb{R}^2$ we see



(since $(z^2-1)(z^2-k^2)$



In $\mathbb{C}^2$ general there are two values of w for each value of z other than $z = \pm 1, \pm k$



For any path in the z-plane that avoids $\pm 1, \pm k$ we can continue our 2 solutions along that path.

If the path is a loop then when we come back to our starting point the 2 solutions either come back to themselves or are interchanged.

We saw something similar for $\sqrt{z}$: If we take a path going around the origin once then the two solutions get interchanged. ~~If we do not go~~ If the path does not go around the origin the solutions do not get interchanged.

What about a small loop around $z=1$?

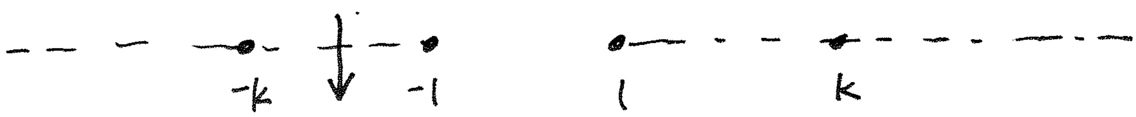
Let's write $z = 1+\varepsilon$ then

$$
\begin{aligned}
w &= \sqrt{(z^2-1)(z^2-k^2)} \\
&= \sqrt{\left((1+\varepsilon)^2-1\right)\left((1+\varepsilon)^2-k^2\right)} \\
&= \sqrt{(2\varepsilon+\varepsilon^2)\left((1+\varepsilon)^2-k^2\right)} \\
&= \sqrt{\varepsilon}\cdot\sqrt{(2+\varepsilon)\left((1+\varepsilon)^2-k^2\right)}
\end{aligned}
$$

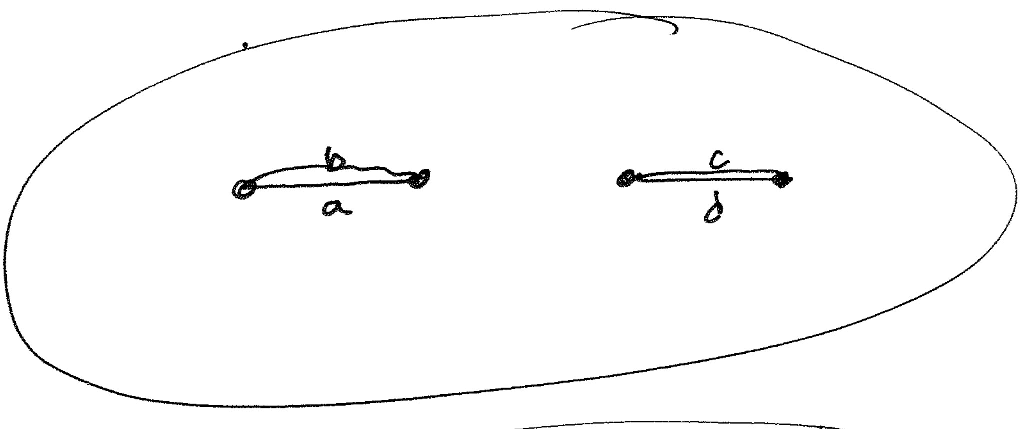
non-zero ~~holo~~ holomorphic function

On the r.h. side there are two distinct branches ~~of~~ $\sqrt{\varepsilon}$ ~~changes~~ over $\{|\varepsilon| \ll 1\}$

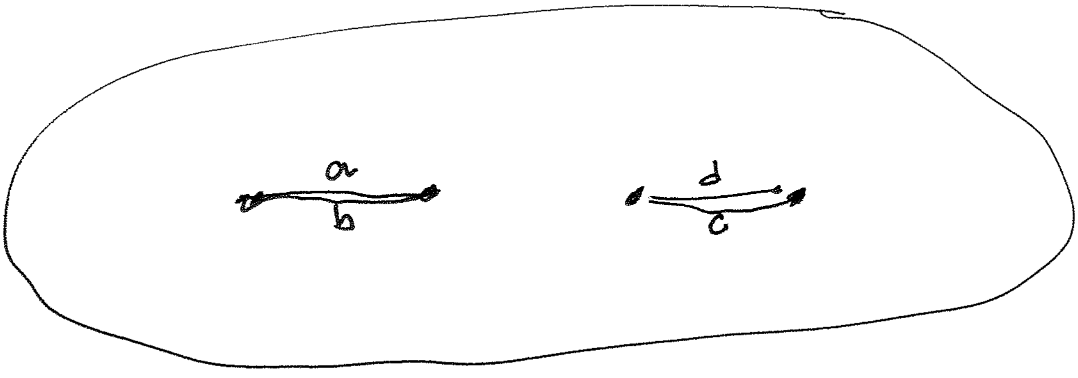
As we traverse our loop once the function $\sqrt{\varepsilon}$ changes ~~sign~~ sign. Same analysis applies at $-k, -1, 1$



If we make these cuts then there are 2 distinct branches over the remaining region. On the interval $-k, -1$ we interchange branches. over $(-\infty, -k)$ we do not interchange branches

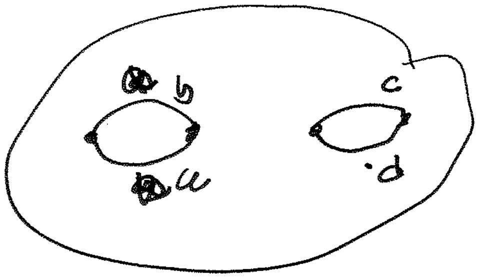


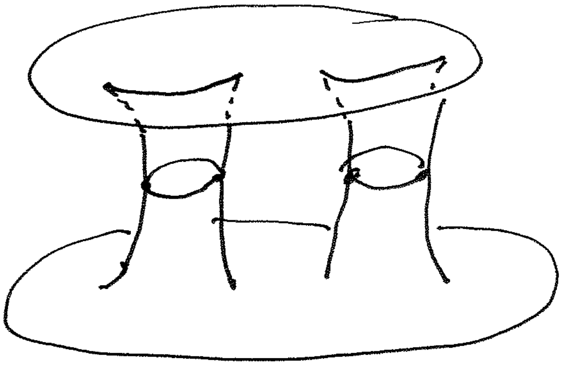
branch 1

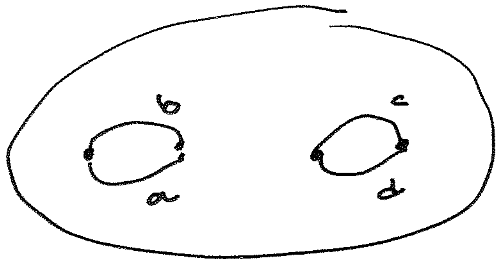


branch 2

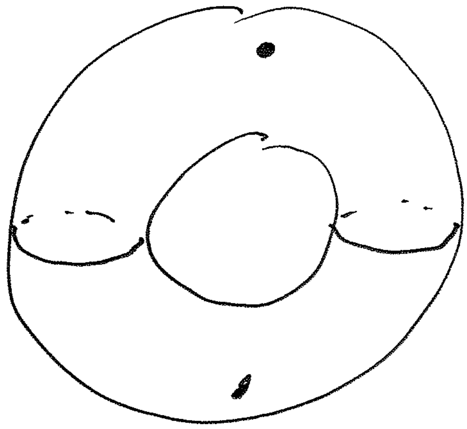
Flip the bottom picture







Implicit fcn. thm.?



T is a torus with two punctures

A Riemann surface is given by the following collection of data:

(1) A Hausdorff topological space X

(2) A collection of open sets $U_\alpha \subset X$ ~~whose~~ where α ranges over some index set. We assume that the U_α cover X. $(X = \bigcup_\alpha U_\alpha)$

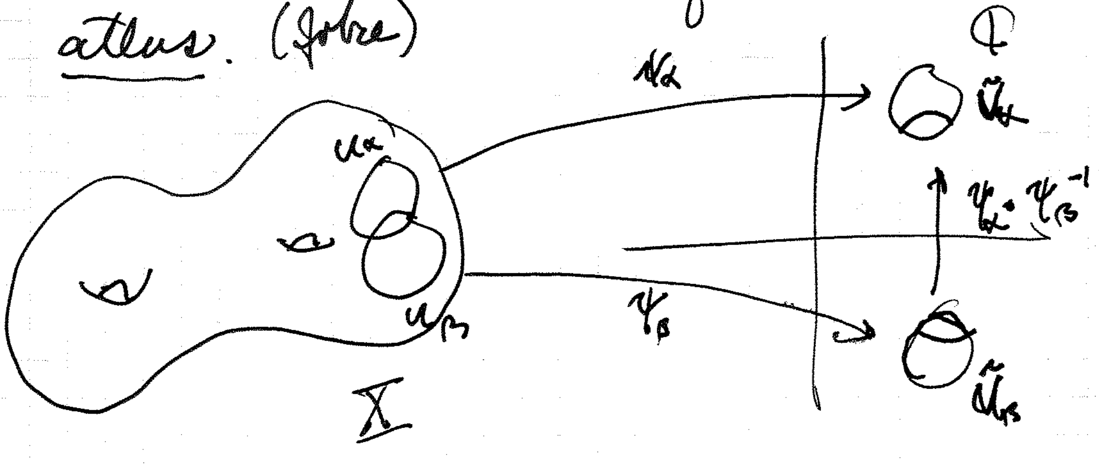
(3) For each α a homeomorphism

$$\psi_\alpha : U_\alpha \to \tilde{U}_\alpha \subset \mathbb{C}$$

with the property that for all α, β the map $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic on its domain of definition.

The maps ψ_α are referred to as "charts", "coordinate charts" or "local coordinates".

The entire collection of charts is called an atlas. (Jobra)



We assume $\tilde{U}_\beta$ open in $\mathbb{C}$.

Remarks:

ψ_α^{-1} is defined to have domain $\tilde{U}_\alpha$ and range U_α

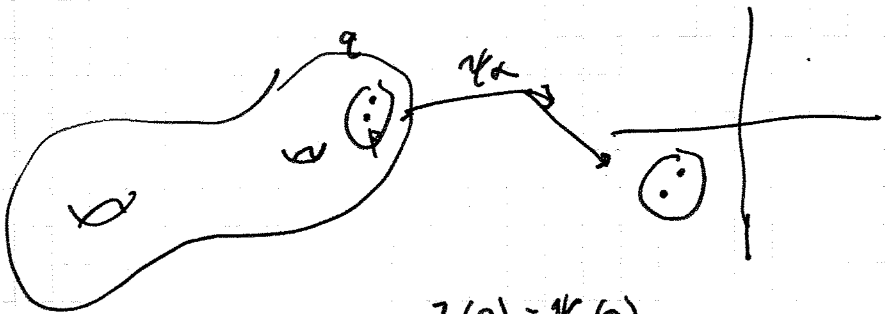
The domain of definition of $\psi_\alpha \circ \psi_\beta^{-1}$ is $\psi_\beta(U_\alpha \cap U_\beta)$ and the range is $\psi_\alpha(U_\alpha \cap U_\beta)$.
(Could be empty in which case there is nothing to check.)

Thus $\psi_\alpha \circ \psi_\beta^{-1}$ maps an open set in $\mathbb{C}$ to an open set in $\mathbb{C}$ so the notion of what it means to be holomorphic is well defined.

The way we have set things up $\psi_\alpha \circ \psi_\beta^{-1}$ is the inverse to $\psi_\beta \circ \psi_\alpha^{-1}$ so by interchanging indices we see that it is ~~assum~~ a consequence of the definition that $\psi_\alpha \circ \psi_\beta^{-1}$ has a holomorphic inverse.
(Such a map is often called "conformal".)
(For maximal confusion we sometimes say a holomorphic atlas gives a "conformal structure".) Conformal atlas
(Other terminology: Complex structure)

This atlas determines the topology on X.
Defn topology on each U_α. Set is open in X iff its intersection with each U_α is open.

This is a fairly complicated definition but in working with Riemann surfaces we rarely see this bulky collection of data explicitly. Suppose we have a point $p \in X$. p lies in some U_α so we choose one. ψ_α is just a complex valued function defined on a neighborhood of p and we normally denote ψ_α by a symbol such as z. Then, in making calculations near to p we label points by the corresponding value of the variable z.



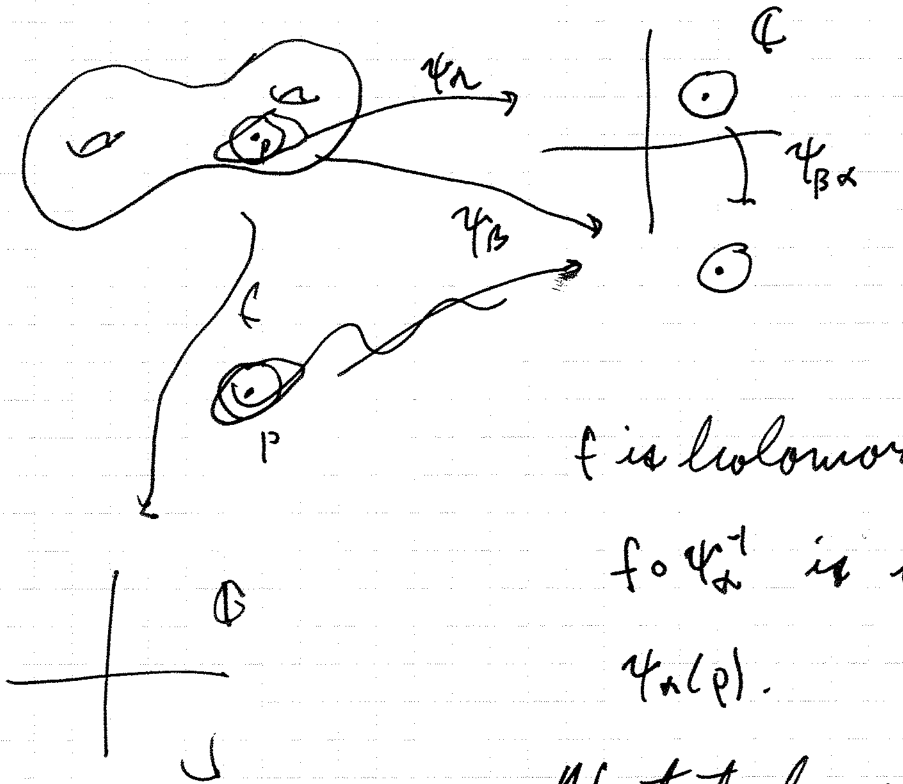
$$z(p) = \psi_\alpha(p)$$

$$z(q) = \psi_\alpha(q).$$

Now we are effectively working with the standard notation of complex analysis.

To insure that our calculations and constructions are valid we need to analyse how they change under a holomorphic coordinate change.

Example: It makes sense to say that a function $f : X \to \mathbb{C}$ is holomorphic at p since if f is holomorphic with respect to one collection of coordinate charts ~~covering X~~ near p it is holomorphic (i.e. $f \circ \psi_\alpha^{-1}$ is holomorphic)



f is holomorphic near p if $f \circ \psi_\alpha^{-1}$ is holomorphic near $\psi_\alpha(p)$.

Want to know that this is independent of our choice of chart.

Is $f \circ \psi_\beta^{-1}$ holomorphic near $\psi_\beta(p)$? Yes since

$$f \circ \psi_\beta^{-1} = \underbrace{f \circ \psi_\alpha^{-1}}_{\text{holomorphic}} \circ \underbrace{\psi_{\alpha\beta}}_{\substack{\psi_\alpha \circ \beta_\beta^{-1} \\ \text{holomorphic}}}$$

Uses the fact that the composition of holomorphic functions is holomorphic

Example: Given 2 ~~ff~~ holomorphic functions f and g it does not make sense to say that $f' = g$, the derivative of f is g.

This relationship between functions f and g <u>depends</u> on the choice of the local variable.

This example suggests the advantage of having a more coordinate independent way of talking about differentiation of functions. The language of 1-forms provides this.

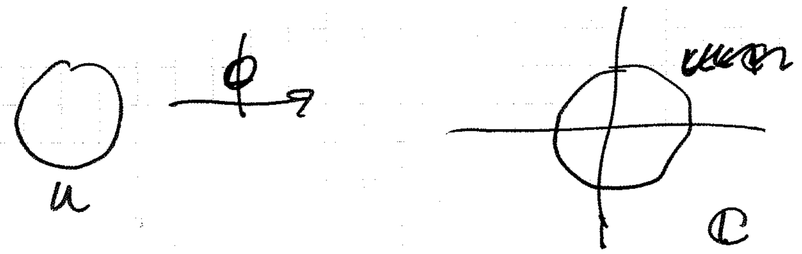
Language of tangent vectors.

We will introduce ~~this~~ ~~be~~ these notions at a later point.

There are two ways of dealing with this issue.

Example I: Any open set U in $\mathbb{C}$ is a Riemann surface. ~~and~~ It has an atlas consisting of a single chart. (U, U, I).

$$\phi : U \longrightarrow \cancel{U}\ \mathbb{C}$$



Only change of coordinate map is the identity which is holomorphic.