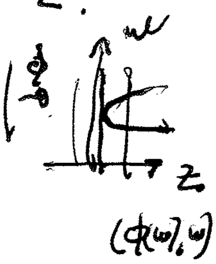
Proposition. Let $C = \{P(z,w)=0\}$. Let $\pi_\infty(z,w) = z$.

Assume that $(z_0,w_0)\in C$ and $\frac{\partial P}{\partial w}(z_0,w_0) = 0$ but that C is a non-singular Riemann surface. Let $\phi(w)$ be such that $C = \{(\phi(w), w)\ w \text{ close to } w_0\}$ in a nbd. of (z_0,w_0). Then the order of vanishing of $(V_\phi(w_0))$ ϕ at w_0 is equal to the order of vanishing of ~~$\frac{\partial P}{\partial w}$ at (z_0,w_0). (plus 1)~~ the polynomial $P(z_0,w)$ at $w=w_0$. (order of the zero)



~~(order of vanishing~~ (note $\phi(w_0)$ can take any value. $P(z_0,w_0)=0$)

Proof. Non-singularity implies $\frac{\partial P}{\partial z}(z_0,w_0)\neq 0$. The fact that $(\phi(w),w)$ parametrises C means that $P(\phi(w),w)=0$. Implicit differentiation gives

$$\frac{\partial}{\partial w} P(\phi(w),w) = \frac{\partial P}{\partial z}\cdot\frac{\partial \phi}{\partial w} + \frac{\partial P}{\partial w} = 0.$$

Showed that:

for $\boxed{\frac{\partial P}{\partial z}\cdot\frac{\partial \phi}{\partial w} = -\frac{\partial P}{\partial w}}$ ← order of vanishing of both sides is the same since $\frac{\partial P}{\partial z}(z_0,w_0)\neq 0$.

Since $\frac{\partial P}{\partial z}\neq 0$ at (z_0,w_0) either both $\frac{\partial \phi}{\partial w}$ and $\frac{\partial P}{\partial w}$ vanish, or neither does.

Assume both do. Differentiate again w.r.t. $\frac{\partial}{\partial w}$

$$(**)\quad \frac{\partial^2 P}{\partial z\,\partial w}\cdot\frac{\partial \phi}{\partial w} + \frac{\partial P}{\partial z}\,\frac{\partial^2\phi}{\partial w^2} + \frac{\partial^2 P}{\partial w^2} = 0$$

Evaluate at (z_0,w_0) to get $\frac{\partial P}{\partial z}\,\frac{\partial^2 \phi}{\partial w^2} = -\frac{\partial^2 P}{\partial w^2}$.

Condition 1 holds. Proof by induction. Inductive hypothesis is.

Say $\frac{\partial \phi}{\partial w} = \frac{\partial^2 \phi}{\partial w^2} = \dots \frac{\partial^{k-1}\phi}{\partial w^{k-1}} = 0$ and $\frac{\partial P}{\partial z}\frac{\partial^k \phi}{\partial w^k} = -\frac{\partial^k P}{\partial w^k}$

(condition k)

$\frac{\partial P}{\partial w} =$ neither $\frac{\partial^{k-1} p}{\partial w} = 0$.

Then either ~~both~~ $\frac{\partial^k \phi}{\partial w^k}$ and $\frac{\partial^k p}{\partial w^k}$ vanish in which case the proposition holds (order ϕ = $\operatorname{ord}_z(P(z_0,w))$) are both vanish. In this case condition $(k+1)$ holds. Differentiate formula k: w r t $\frac{\partial}{\partial z}$

$$\underbrace{\frac{\partial^2 P}{\partial z^2}\cdot\frac{\partial \phi^k}{\partial w^k}}_{\substack{\text{vanishes}\\ \text{by assumption.}}} + \frac{\partial P}{\partial z}\frac{\partial^{k+1}\phi}{\partial w^{k+1}} = -\frac{\partial^{k+1} p}{\partial w^{k+1}}$$

$V_\phi(w_0) \geq k$

$$\begin{array}{ccc} & \boxed{\text{condition } k} & \\ \swarrow{\scriptstyle \frac{\partial^k P}{\partial w} = 0} & & \searrow{\scriptstyle \frac{\partial^k P}{\partial w} \neq 0} \\ \boxed{\text{condition } k+1} & & \boxed{\text{prop is true}} \end{array}$$

order $V_\phi(w_0)$ = order of vanishing of $P(z_0, w)$ = k.

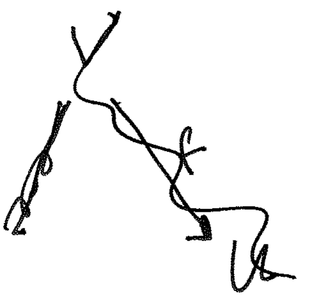
Can take left branch only finitely many times since order of a zero of P ~~at~~ (z_0, w) is bounded by deg P in w.

Proposition. Every finite ~~degree~~ sheeted covering space of a surface of finite type is a surface of finite type.

Proof. Say $f: R \to S$ is a ~~finite covering space of~~ finite sheeted covering space.

We can write S as a compact piece X together with open ~~disc~~ sets $U_1 \dots U_k$ each equivalent to a punctured disk. ~~Since f has finite is a finite cover~~ Can describe X as a finite union of compact simply connected piece. Inverse image of each of these is homeo. ~~to~~ $\{1 \dots d\} \times Y_j$ so is compact.

Remains to show that each component of the inverse image of U_j is a punctured disk. Say V is such a component



$$\begin{tikzcd} V \arrow[r, "H"] \arrow[dr, "f"'] & \Delta - \{0\} \arrow[d, "z \mapsto z^j \quad g"] \\ & \Delta - \{0\} \end{tikzcd}$$

$f_*(\pi_1(V))$ is a subgroup of $\pi_1(\Delta - \{0\})$ of finite index say j. Let $g: \Delta - \{0\} \to \Delta - \{0\}$ be defined $g(z) = z^j$, $g_*(\pi_1(\Delta - \{0\})) = f_*(\pi_1(V))$ So covers are ...

Topological picture of Riemann surfaces.

All compact Riemann surfaces with ~~non-zero~~ non-constant meromorphic functions ~~are~~ are determined by their monodromy representation and their branch locus.
(Fund. Thm. of covering spaces.) Gives equiv. as Riemann surfaces.

Conversely given a set $\{q_1 \ldots q_k\} \subset S^2$ and a representation $h: \pi_1(\mathbb{C}^2 - \{q_1 \ldots q_k\}) \to \mathrm{Perm}(\{1 \ldots d\})$ there is a Riemann surface which realizes it. ~~(together with a~~

The representation h tells us how to build a covering space C of $S^2 - \{q_1 \ldots q_k\}$. This covering space is a Riemann surface of finite type. We can complete it to a compact Riemann surface $\bar{C}$. $\bar{C}$ is unique.

Given an abstract compact Riemann R surface it is not clear that R has meromorphic functions on it.

If ~~Q~~ R is a compact Riemann surface which is conformally equivalent to a ~~proces~~ projective curve $C \subset \mathbb{CP}^2$ then R has many meromorphic functions as we will now see.

(0)

~~Points outside the chart $(x, 1, z)$ correspond to $y=0$. $X^4+Z^4=0$~~

Meromorphic functions on $\mathbb{CP}^2$ correspond to ratios of homogeneous polynomials of the same degree. $\dfrac{f(x,y,z)}{g(x,y,z)}$.

$$\frac{f(\lambda x,\lambda y,\lambda z)}{g(\lambda x,\lambda y,\lambda z)} = \frac{\lambda^d}{\lambda^d}\,\frac{f(x,y,z)}{g(x,y,z)} = \frac{f(x,y,z)}{g(x,y,z)}$$

so the values are well defined away from the set where $f=0$ and $g=0$ simultaneously. Indeterminacy locus.

The restriction of a meromorphic function on $\mathbb{CP}^2$ to a curve C is ~~usually~~ a meromorphic function on C.

Let C* be the projective curve determined by the homogeneous polynomial $X^4+Y^4+Z^4=0$ (in $\mathbb{CP}^2$). (Fermat curve of degree 4.)

Consider the meromorphic function

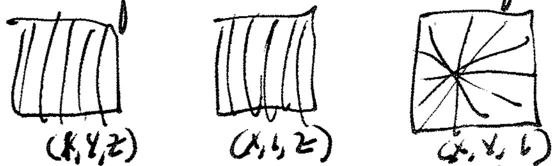
$f(X,Y,Z) = \frac{X}{Y}$. Well defined away

f has degree 4.

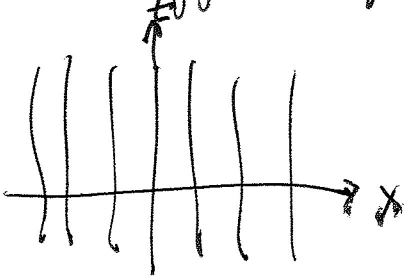
from the point $(0,0,1)$ which does not lie on C. (alt. notation: $(X:1:Z)$)

f is a "projection from a point".

Consider f in the charts $(X,1,Z)$ and $(1,Y,Z)$. Complement of the union of these charts is $(0,0,1)$.



In the chart $(X,1,Z)$ C corresponds to the affine polynomial $P(X,Z)=X^4+1+Z^4$.



$f((X,1,Z)) = X$.

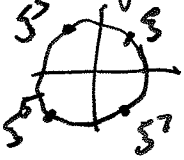
f projects onto $\mathbb{CP}^1$ "at ∞".

Critical points of f are zeros of $\frac{\partial P}{\partial Z} = 4Z^3$ on C.

Crit. pts. are soln of $Z=0$ and $X^4+1+Z^4=0$. $\Rightarrow X^4=-1$.

... points are solutions of $X^4+1=0$. $X^4=-1$

If ξ is a primitive 8-th root of 1 then the critical pts are $(\xi, 1, 0)$ $(\xi^3, 1, 0)$ $(\xi^5, 1, 0)$ $(\xi^7, 1, 0)$



The order of the critical point at $(x_0, 1, z_0)$ is the order of vanishing of $z \mapsto P(x_0, 1, z)$ at z_0.

Remark.

$$z \mapsto P(\xi^j, 1, z) = \xi^{4j} + 1 + z^4 = z^4.$$

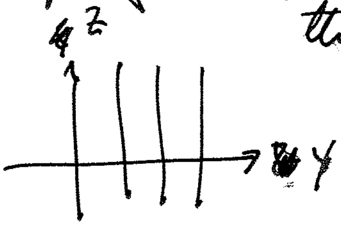
No other critical points.
Need to check points in chart $(1, Y, Z)$ where $\frac{\partial P}{\partial Z} = 0$ and $Y = 0$ (already counted solns with $X \neq 0$ and $Y \neq 0$ in chart $(x : 1 : z)$).

$P(1 : Y : Z) = 1 + Y^4 + Z^4$, $Y = 0$

$\frac{\partial P}{\partial Z} = 0 \Rightarrow 4Z^3 = 0 \Rightarrow Z = 0$. No common solutions.

onto the coordinate

$(1, Y, Z)$ proj. to ~~Y coordinate~~ the coordinate



$P(Y, Z) = 1 + Y^4 + Z^4$

$\frac{\partial P}{\partial Z} = 4Z^3$ $Z = 0$ in tangencies

Points on the curve with $Z = 0$

$1 + Y^4 = 0$ $Y^4 = -1$ $Y = \xi, \xi^3, \xi^5, \xi^7$

$(\xi, 1, 0)$

ξ^3

$(1, \xi, 0)$

$(1, \xi^3, 0)$

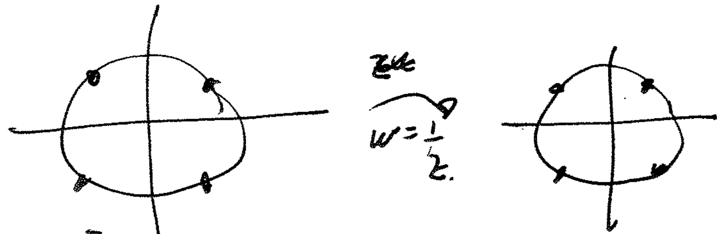
$(1, \xi^5, 0)$

$(1, \xi^7, 0)$

4 pts. each of order 4

Lemma. C is connected.
This is equivalent to $h: \pi_1(\mathbb{C} - 4\text{pts}) - \text{Perm}$ being transitive (since the base is connected. Consider a loop around a puncture. Since $\nu_f(p) = 4$ it corresponds to a cycle of length 4.

Since a cycle acts transitively.



Covering of degree $= 4$.

$\chi(R) - d\,\chi(S^2) = -\sum_{i=1-4} (4-1)$

$\chi(R) - 4 \cdot 2 = -12$

~~$\chi(R) = -8 = 2-2g$~~ $g = 5?$

$\chi(R) = 8 - 12 = -4$ $g = 3$

To get this representation $f: R \to S$

$\pi_1(S - \text{pts}) \to \text{Perm}$ we only need to assume that S is connected. R will be connected exactly when the representation is transitive.