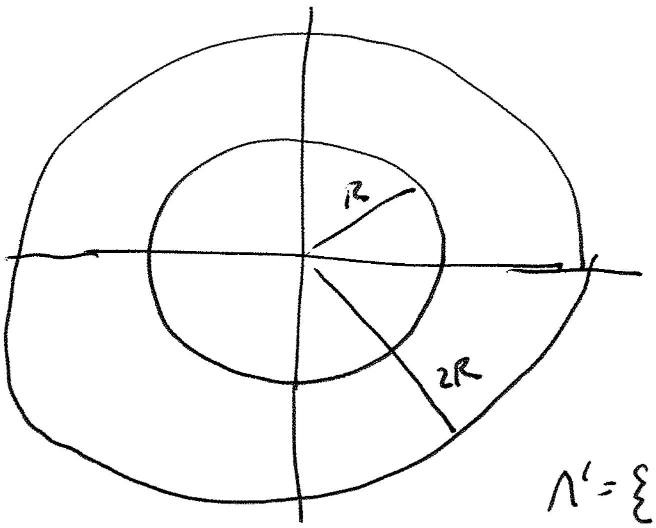

Write

$$\sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^{\ell}}$$

as $$\sum_{\omega \in \Lambda'} \frac{1}{(z-\omega)^{\ell}} + \sum_{\omega \in \Lambda - \Lambda'} \frac{1}{(z-\omega)^{\ell}}$$

$$\Lambda' = \{\omega \in \Lambda : |\omega| < 2R\}.$$

Use Weierstrass M-test to show second sum converges in $\{|z| < R\}$.

Estimate the size of a term where $|z| < R$ $|\omega| > 2R$.

$\cancel{|z-\omega|^{-\ell} = |\omega|^{-\ell} \cdot |\tfrac{z}{\omega} - 1|^{-\ell}}$ $\quad |\tfrac{z}{\omega}| \leq \tfrac{1}{2} \quad |\tfrac{z}{\omega} - 1| \geq \tfrac{1}{2}$

$|\cdot| < \tfrac{1}{2}$

$|\omega| \geq \sqrt{2} \cdot k$

$|\omega| \geq \text{const.}\, k$

$$|z-\omega| = |\omega| \cdot |\tfrac{z}{\omega} - 1| \geq |\omega| \cdot (\tfrac{1}{2})$$

$$|z-\omega|^{-\ell} \leq |\omega|^{-\ell} \cdot 2^{\ell}. = \text{const.}\, |\omega|^{-\ell} \leq \text{const.}\, k^{-\ell}$$

use M-test $\cancel{\sum_k |z-\omega|^{-\ell}}$ $\sum_k \sum_{\max\{m,n\} = k} |z-\omega|^{-\ell} \leq \text{const} \cdot k \cdot k^{-\ell}$

$\leq M$

$\leq \frac{1}{k^{\ell+1}}$

$\leq \frac{1}{k^{\ell-1}}$

Case $\ell = 2$.

Replace $\cancel{\frac{1}{z^2} + \sum_{\ell}}$ $\sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^2}$ by

$$\frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

No reason to think this function is invariant under $z \mapsto z+\omega$.

Estimate the ~~gener~~ size of the general term.

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{\omega^2 - (z-\omega)^2}{(z-\omega)^2 \cdot \omega^2} \right| = \left| \frac{z^2 + 2z\omega}{(z-\omega)^2 \cdot \omega^2} \right|$$

$$\leq \frac{|z| \cdot |z+2\omega|}{|z-\omega|^2 \cdot \omega^2} \leq \frac{R \cdot \frac{5}{2} |\omega|}{\left(\frac{3}{2}\right)^2 |\omega|^2 \cdot |\omega|^2} \leq \text{const.} \frac{1}{|\omega|^3}.$$

Note $|z+2\omega| = \cancel{|2\omega} \; 2|\omega| \cdot \left| \frac{z}{2\omega} + 1 \right| \leq 2|\omega| \cdot \frac{5}{4}.$

$$\left| \frac{z}{2\omega} \right| \leq \frac{1}{4}$$

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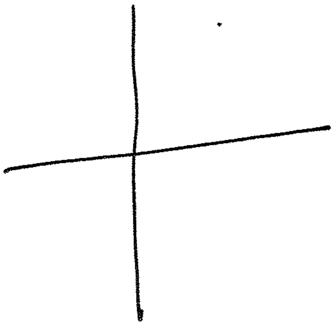


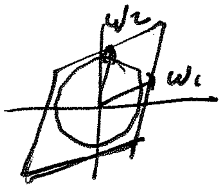
For arbitrary lattice we write

$\Lambda = \{ m\omega_1 + n\omega_2 : m,n \in \mathbb{Z} \}$

Sum over ~~parallelo~~ parallelograms

$m\omega_1 + n\omega_2 : \max\{m,n\} = k.$

Need to estimate the size of $(m\omega_1+n\omega_2)^{-l}$ from above in terms of



Let γ be the radius of the maximal disk contained in the parallelogram. Set

$|m\omega_1 + n\omega_2| \geq \gamma \cdot \max\{m,n\}$

Def. The Weierstrass $\wp$-function is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

Lemma,
~~Prop.~~ ~~Ell~~ $f(z) = \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^\ell}$ is a Λ-invariant

for $\ell \geq 3$. $\wp(z)$ is invariant under $z \mapsto -z$.

Proof. $f(z)$ converges absolutely

The series for $f(z+\omega_0)$ and $f(z)$ have the same terms presented in different orders

"ω term" for $f(z+\omega_0)$ is $\frac{1}{(z-\omega+\omega_0)^\ell}$.

"$\omega-\omega_0$" term for $f(z)$ is $\frac{1}{(z-(\omega-\omega_0))^\ell}$.

~~Series for $\wp(z)$ and~~

By ~~absolute~~ M-test $\Rightarrow$ absolute convergence
$\Rightarrow$ invariance under rearranging terms

ω- term for $\wp(-z) = \frac{1}{(-z)^2} + \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(-z+\omega)^2} - \frac{1}{\omega^2}$

$-\omega$- term for $\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \ldots} \frac{1}{(z-\omega)^2} - \frac{1}{(-\omega)^2}$

Again we are rearranging terms.

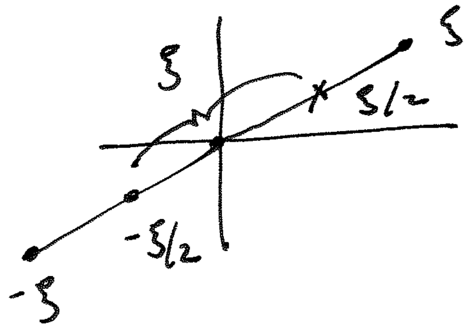
Proposition. $\wp(z+\omega) = \wp(z)$.

Proof. Note that by the M-test we can compute the derivative of $\wp$ term by term. We see that $\wp'(z) = -2\sum \frac{1}{(z-\omega)^3}$ is invariant under Λ.

So $(\wp(z+\omega) - \wp(z))' = 0$ and

$$\wp(z+\omega) = \wp(z) + c(\omega) \text{ for some constant.}$$

Now we use the fact that $\Gamma = \{z \mapsto \pm z + \omega : \omega \in \Lambda\}$ acts with fixed points on $\mathbb{C}$. Fix $\xi \in \Lambda$.



$$-\xi/2 \xrightarrow{z \mapsto z+\xi} \xi/2 \xrightarrow{z \mapsto -z} -\xi/2.$$

$$\wp(\xi/2) = \wp(-\xi/2) + c(\xi)$$

$$\wp(\xi/2) = \wp(-\xi/2) \text{ so } c(\xi) = 0.$$

Conclude that $\wp$ is Γ invariant.

$\iota(z) = -z$

$z \mapsto -z$ induces an involution on $\mathbb{C}/\Lambda$.

If $z \sim z'$ then $z - z' \in \Lambda$.

$\iota(z) - \iota(z') = \iota(z - z') \in \cancel{-\Lambda \subset \Lambda}$.

$= z' - z \in -\Lambda = \Lambda$.

Quotient is the same as $\mathbb{C}/\Gamma$ where $\Gamma = \{z \mapsto \pm z + \omega : \omega \in \Lambda\}$

The action of ι is not free.

$\iota(z) = z$ means $\iota(z) - z \in \Lambda$

or $-z - z \in \Lambda$ $\quad -2z \in \Lambda$ $\quad z \in \Lambda/2$.

Two elements of $\Lambda/2$ are equivalent if they differ by an element of Λ.

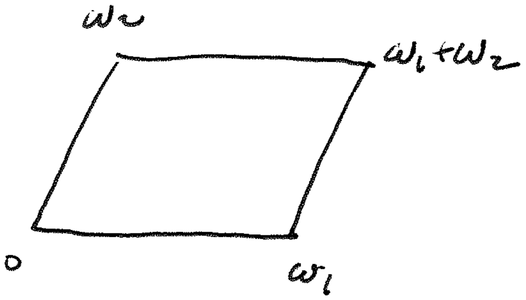
Any ω_1, ω_2 generate Λ then $\frac{\omega_1}{2}, \frac{\omega_2}{2}$ generate $\Lambda/2$. $\frac{n\omega_1}{2} + \frac{m\omega_2}{2}$ mod Λ is equivalent to

$\delta = 0$ if n, m are even

$\alpha = \frac{\omega_1}{2}$ $\quad n$ odd m even

$\beta = \frac{\omega_2}{2}$ $\quad n$ even m odd

$\gamma = \frac{\omega_1 + \omega_2}{2}$ $\quad n, m$ odd.

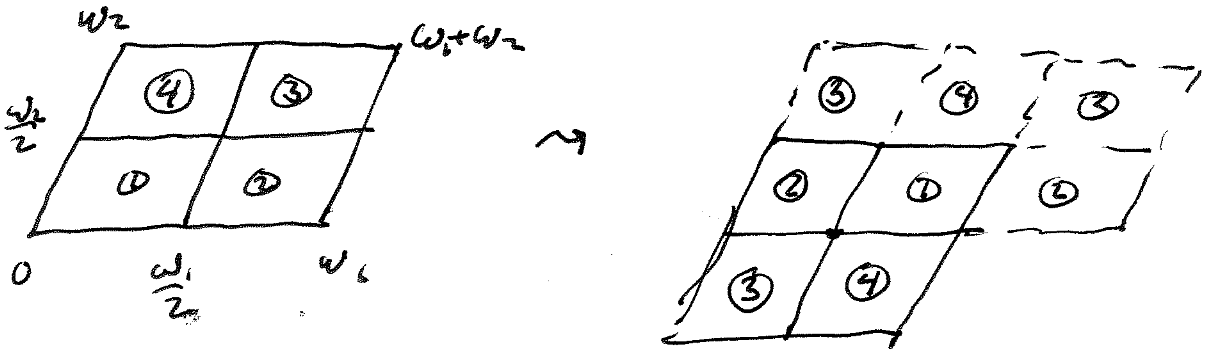


Fundamental domain for Λ.

Construction of the quotient.

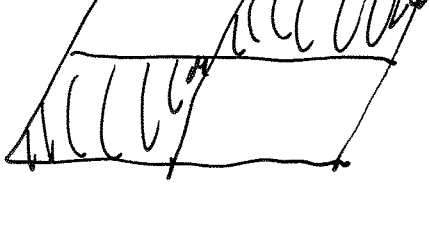
$z = L(w)$ if $z - L(w) \in \Lambda$ $\quad z + w \in \Lambda$.

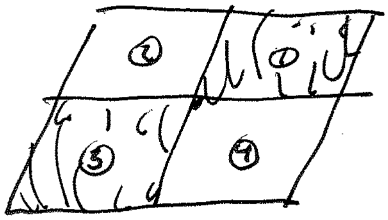
Consider a fundamental domain for $\mathbb{C}/2\Lambda$. ~~Choose~~ For convenience Choose a fundamental domain invariant under $L(z) = -z$,



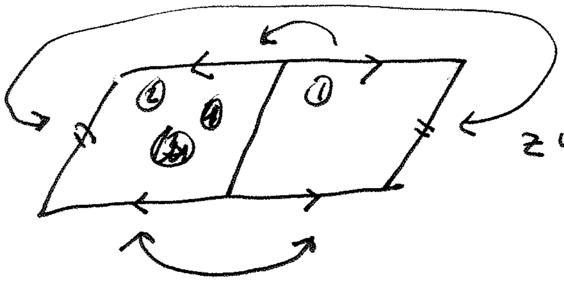
We see that ~~1~~ $L(1) = 3$. $L(2) = 4$.

To get a fundamental domain for Γ action use one black and one white parallelogram.





Gluing pattern on the boundary.

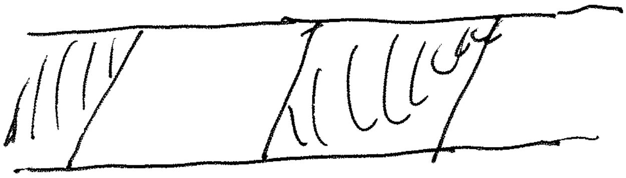


top of 1 ~ bot of 4 ~ ~~top~~ top of 2

$z \mapsto z-\omega_2 \quad \mapsto \quad -z+\omega_2$.

~~Note that $\mathbb{C}/\Gamma$ has~~

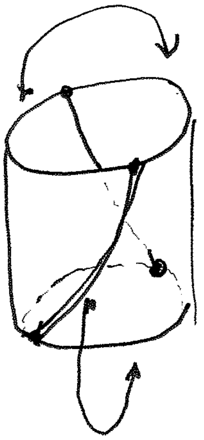
geometric picture.



Gaussian integers

$m+ni$ lattice: pillow case

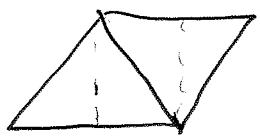
Built from 2 squares

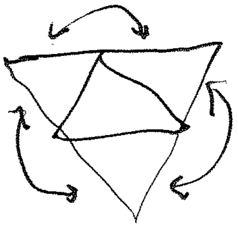


$m+n\omega$

6th root of 1.

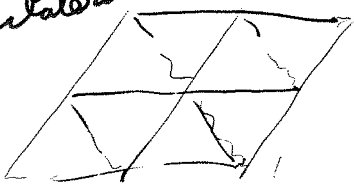
Eisenstein integers





$\omega = \left(\frac{1+\sqrt{3}i}{2}\right)$

Built from 4 equilateral triangles.



~~Recall~~

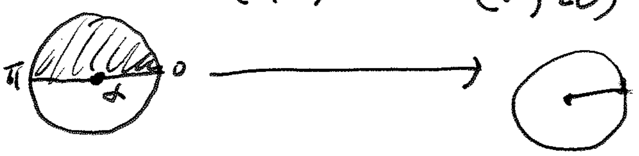
~~Q $\mathbb{C}/\Gamma$ has a ha~~

$\mathbb{C}/\Lambda$ has a translation structure.

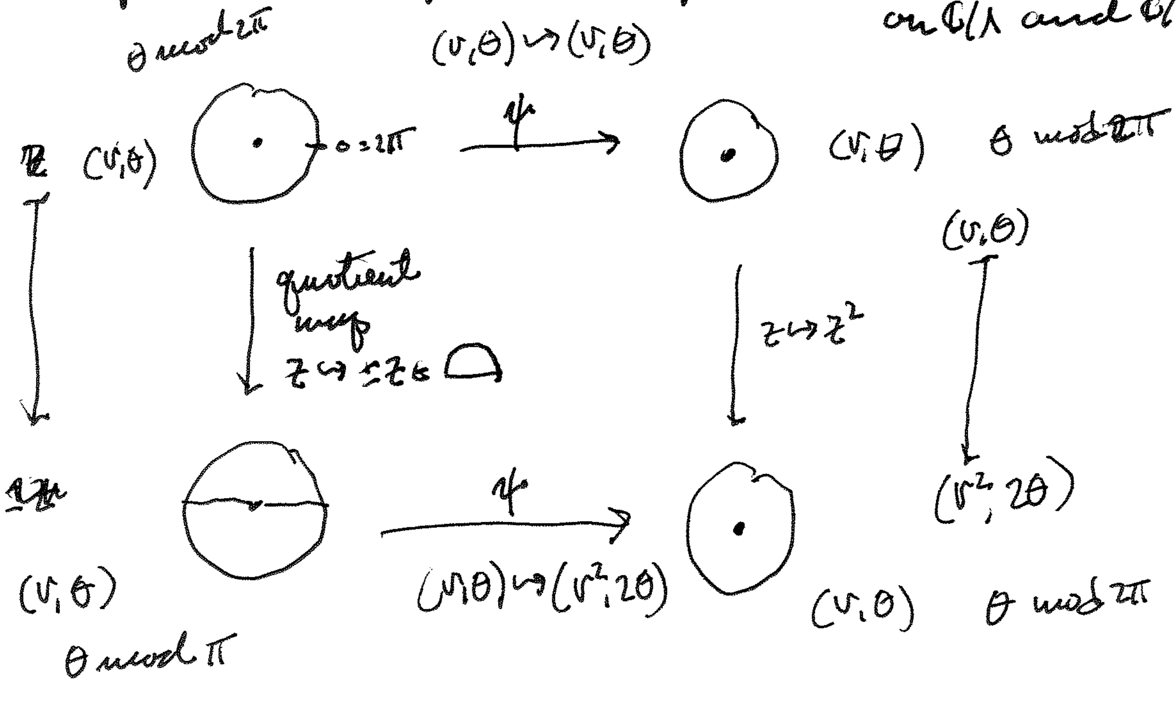
$\mathbb{C}/\Gamma$ has a non singular half-translation structure ~~on~~ away from the half-integer points.

At $\alpha, \beta, \gamma, \delta$ $\mathbb{C}/\Gamma$ has a Riemann surface structure.

$\theta \in [0,\pi]$ $\quad$ $(r,\theta) \mapsto (r^2, 2\theta)$ $\quad$ $\theta \in [0, 2\pi]$. $\quad$ Using polar coordinates



Locally the map from $\mathbb{C}/\Lambda \to \mathbb{C}/\Gamma$ is given as $z \mapsto z^2$ in this coordinate. In particular the quotient map is holomorphic w respect to the Riem. surface structures on $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Gamma$.



Proposition. The quotient map $\mathbb{C}/\Lambda \to \mathbb{C}/\Gamma$ is holomorphic of degree 2. $\wp : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ is holomorphic of degree 2. The induced map $\mathbb{C}/\Gamma \xrightarrow{\wp} \mathbb{CP}^1$ is holomorphic of degree 1.

Proof.

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[rr, "\wp"] \arrow[d, "\pi"] & & \mathbb{CP}^1 \\ \mathbb{C}/\Gamma \arrow[urr, "\wp_0"'] & & \end{tikzcd}$$

$\pi(z) = \pi(z')$ iff $z' = \iota(z)$. π is generically 2-1.

$\wp$ is well defined by previous result.

$\wp$ is degree 2 since $\wp$ has a unique pole of order 2.

Degrees multiply under composition.

$$\begin{tikzcd} \cdot \arrow[r, "2"] \arrow[d, "2"'] & \cdot \\ \cdot \arrow[ur, "?"'] & \end{tikzcd}$$

so degree of $\wp_0$ is 1.

Cor. $\wp_0$ is an analytic isomorphism from the half-torus surface $\mathbb{C}/\Gamma$ to the <u>standard</u> conformal 2 sphere $\mathbb{CP}^1$.