~~Definitions~~ Recall $\wp, \wp'$ satisfy $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. (*)

Recall $C_\Lambda = \{ Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3 \}$ $\quad g_2 = g_2(\Lambda)$, $g_3 = g_3(\Lambda)$.

Prop. Let $u : \mathbb{C}/\Lambda \to C_\Lambda$ be defined by

$$u(z+\Lambda) = \begin{cases} [\wp(z) : \wp'(z) : 1] & \text{if } x \notin \Lambda \\ [0 : 1 : 0] & x \in \Lambda . \end{cases}$$

Then u is a conformal equivalence,

Proof. $\operatorname{Im} u \subset C_\Lambda$ by (*). If $z \notin \Lambda$ then $\wp(z)$, $\wp'(z)$ are finite and $\cap \{[X:Y:1]\}$. ~~Let~~ $z \in t$

~~Sup~~ To show injectivity say $u(z) = u(z')$ but $z \neq z'$. So $\wp(z) = \wp(z')$ which implies that $z' = L(z)$. Since $L(z) \neq z$ $\quad z \notin \{\alpha, \beta, \gamma, \delta\}$ which implies $\wp'(z) \neq 0$. Now

$\wp'(z') = \wp'(L(z)) = -\wp'(z) \neq \wp'(z)$ so $u(z)$ and $u(z')$ are distinct.

If $u(z) \notin \{[X:Y:1]\}$ then $\wp(z)$ or $\wp'(z)$ are infinity. This occurs only on lattice pts.

Want to ~~show that~~ analyze u for z near a lattice pt. Consider $z \neq 0$ near 0.

P has a pole of order 2 at 0 so $P(z) = \frac{g(z)}{z^2}$

g hol, $g(0) \neq 0$. $P'(z)$ ~~sh~~ has a pole of order 3

so $P'(z) = \frac{h(z)}{z^3}$. h hol. $h(0) \neq 0$.

$$u(z) = [P(z) : P'(z) : 1] = \left[\frac{g(z)}{z^2} : \frac{h(z)}{z^3} : 1\right]$$

$$= [z \cdot g(z) : h(z) : z^3]$$

$$\lim_{z \to 0} u(z) = [0 : 1 : 0].$$

u is a ~~non~~. degree 1 map between compact Riemann surfaces so it is a conformal equivalence.

We started with a geometric picture of $p: \mathbb{C}/\Lambda \to \mathbb{CP}^1$. This is related to the fact that $\mathbb{C}/\Lambda$ has a translation structure, $\mathbb{C}/\Gamma$ has a half-translation structure.

Considered $\mathbb{P}^1$ related to trans. structure.

We can also relate these geometric structures to integration and holomorphic 1-forms.

Recall that hol. 1-forms $f(z)dz$ (say on $\mathbb{C}$) are related to integration $\int f(z)dz$.

They are also (unsurprisingly) related to differentiation. They give a coordinate free way of differentiating. Operator d from functions to 1-forms which is called the exterior derivative.

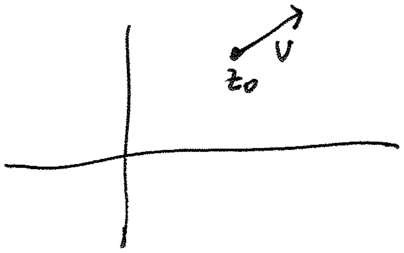
Unlike differentiation in a coordinate system this is coordinate independent.

$df(v) = D_v f$ ← directional derivative of f in the direction v.

↑ tangent vector at a pt. p

Recall the meaning of $f(z)\,dz$ in $\mathbb{C}$.

This is a function on ~~vectors. Say v is~~ tangent vectors in $\mathbb{C}$. Say v is a tangent vector based at z_0. View v as a complex number.



$$dz(v) = v$$

vector   number

$$f(z)\,dz(v) = \underbrace{f(z_0)\cdot \cancel{dz}\cdot v}_{\text{number}}.$$

Say f is a function in $\mathbb{C}$ then

$$df(v) = \text{directional derivative of } f \text{ in direction } v = f'(z_0)\cdot v$$

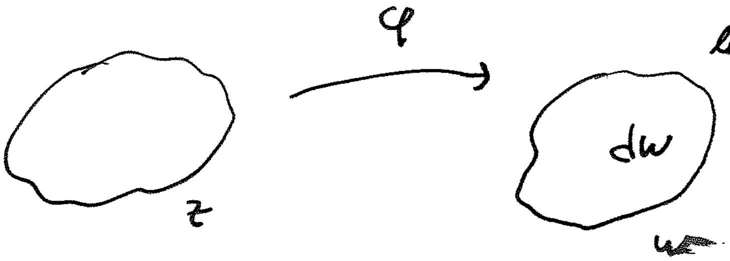
So $df(v) = f'(z_0)\cdot dz(v)$ and

$$\cancel{df = \tfrac{df}{dz}}\cdot \quad df = f'(z)\cdot dz \quad \text{or} \quad df = \frac{df}{dz}\cdot dz.$$

If f = f(x, y) is a function in 2 variables

then $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$

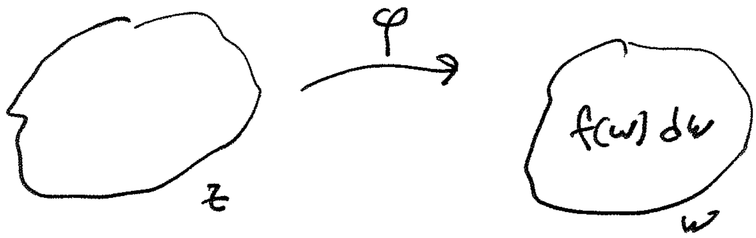
forms "pull back" just as functions do since tangent vectors map forward.



$w = \varphi(z)$

$\varphi^*(dw) = \varphi'(z)\,dz$

pullback.



$\varphi^*(f(w)dw) = f(\varphi(w))\cdot\varphi'(z)\,dz$

Expression for the change of variables formula in integration:

$$\int_\gamma \varphi^*(f(w)dw) = \int_{\varphi(\gamma)} f(w)\,dw.$$

$$\parallel$$

$$\int_\gamma f(\varphi(w))\cdot\varphi'(z)\,dz$$

We have seen that $\mathbb{C}/\Lambda$ gives rise to a conformally equivalent elliptic curve.

Is every elliptic curve obtained this way? ~~How do you interpret~~ Λ in terms

What is Λ in terms of this curve?

Say that $C = C_\Lambda$. How do you find Λ from C?

An elliptic curve is given by $Y^2 = f(X)$ where f is a polynomial of degree 3. We can also associate it with an elliptic integral $\int \frac{dX}{\sqrt{f(X)}}$.

These integrals arise in computing the arc length on an ellipse and this is the origin of the name elliptic curve.

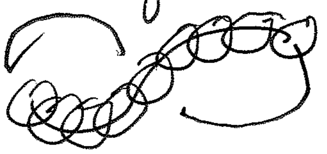
To what extent does the expression $\int \frac{dx}{\sqrt{f(x)}}$ give a well defined integral?

If we choose a branch $h(x)$ for $\sqrt{f(x)}$ $(h^2(x) = f(x))$ over an open set U then the integral $\int_\gamma \frac{dx}{h(x)}$ makes sense over U. If we choose

$\gamma \subset U$

the other branch $-h(x)$ then we change the value of the integral (multiplying by -1).

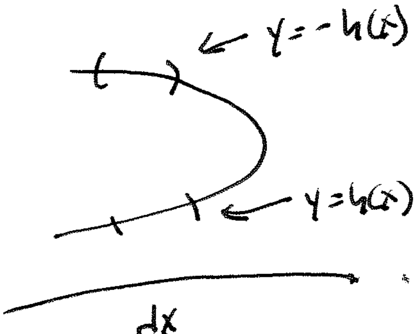
We can also think about choosing a branch of $\sqrt{f(x)}$ <u>along</u> γ if γ does not go through the roots of f.



A branch $h(x)$ of $\sqrt{f(x)}$ gives a parametrization of the curve $y^2=f(x)$ by $x \mapsto (x, h(x))$.

$C = \{ P(x,y) = y^2 - f(x) \}$.

So on the curve $C \subset \mathbb{C}^2$ we can think of the ~~integral as being well defined~~ function $\sqrt{f(x)}$ as being well defined and given by the y coordinate.



On C we have a well defined holomorphic 1-form which ~~corresponds to~~ resolves the ambiguity of the expression $\int \frac{dx}{\sqrt{f(x)}}$.

It is given by $\int \frac{dx}{y}$.

(note $\frac{dx}{y}$ makes sense on $\mathbb{C}^2$ but only on the curve C is it giving us the integral we want.)

We have an alternative parametrization of C_Λ

$x \qquad y \qquad\qquad C_\Lambda : \{y^2 = f(x)\}$

$$w \overset{\text{"u"}}{\longmapsto} (\wp(w), \wp'(w))$$

Say we pull back the 1-form on C_Λ to $\mathbb{C}$ via this parametrization.

Pull back $\frac{dx}{y}$. dx pulls back to $\frac{d\wp}{dw}\,dw$

y pulls back to the function $\wp'(w)$.

$\frac{dx}{y}$ pulls back to $\frac{\wp'(w)\,dw}{\wp'(w)} = dw$.

This is the 1-form ~~that~~ gives $\mathbb{C}/\Lambda$ its canonical translation structure.

"u" takes the lattice to $[0:1:0]$, pt. at ∞ on the curve?

Certain ~~integrals which look like the ones~~ improper integrals are in fact finite, since they can be transformed into integrals on the torus.