Prop. Say R is a compact Riemann surface with a nonzero holomorphic 1-form θ. Then R is conformally equivalent to $\mathbb{C}/\Lambda$ for some lattice Λ.

~~Geometric translation~~

Geometric interpretation of 1-forms. They give metrics. If v is a tangent vector we define the length of v to be $|\theta(v)|$.

~~If θ is a holomorphic 1 form~~

In a local coordinate $\theta = f(z)dz$.

Identify v with a complex number

$$\theta(v) = f(z)\cdot dz(v) = f(z)\cdot v \quad \text{so} \quad |\theta(z)| = |f(z)|\cdot|v|.$$

$$ds^2 = |f(z)|^2\cdot(dx^2+dy^2).$$

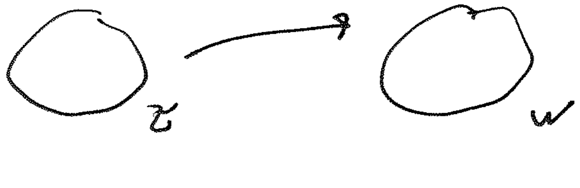
metric given by θ.

~~Conformal~~
~~Conformal~~
~~metric since~~
~~it is~~

Hol. 1-form is automatically closed.

What is the curvature of this metric?

It is flat: Choose $w = F(z)$ so that $F'(z) = f(z)$ then $F^*(dw) = \frac{dF}{dz}\cdot dz = f\cdot dz = \theta$.



~~thus F is~~ $\theta = F^*(dw)$.

~~since~~

since $DF(v)$

$\theta(v) = dw(F(v))$

$|\theta(v)| = |dw(F(v))|$

F is an isometry from the metric determined by θ to the metric determined by dw.

dw determines the standard Euclidean metric $ds^2 = dx^2 + dy^2$ ~~flt~~

$|dw(v)| = |v|$.

~~Returnia Retu~~

Returning to the proposition.

We have a global F not just a local F.

Let $\tilde{R}$ be the universal cover of R.

Pick $p_0 \in \tilde{R}$. Define $F(q) = \int_{p_0}^{q} \pi^*(\theta)$ $\quad F: \tilde{R} \to \mathbb{C}$

$\int_{\delta} = \int_{\delta_1}$ $\quad \pi^*(\theta)$ is the lift of θ to $\tilde{R}$ (local isometry).

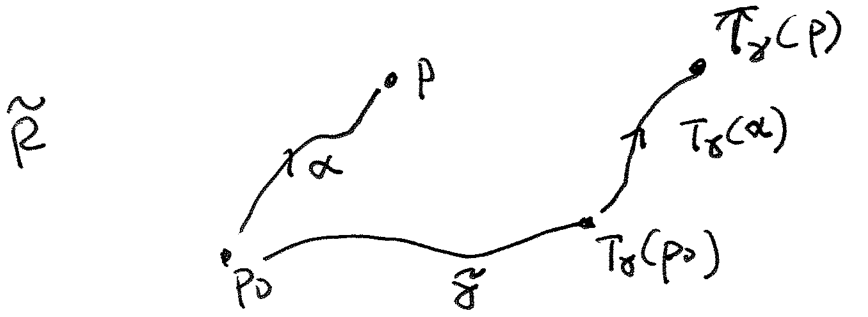
F makes sense since $\tilde{R}$ is simply connected and the integral does not depend on the homotopy class.

If we write $\theta = f(z)dz$ we see that $F'(z) = f(z)$

$\int_{p_0}^{q} f(z)dz = F(q) - F(p_0)$. This means that

$F^*(dw) = \theta$. In particular F is a local isometry.

Let T_γ be the deck translation corresponding to γ.

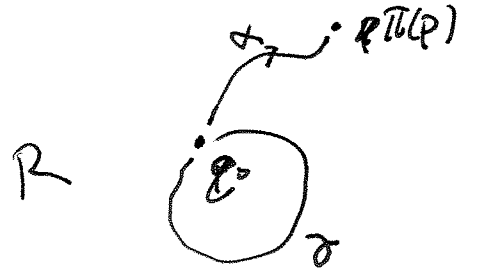


$$F(T_\gamma(p)) = \int_{T_\gamma(\alpha)\cdot\tilde{\gamma}} \theta$$

$$= \int_{T_\gamma(\alpha)} \theta + \int_{\tilde{\gamma}} \theta$$

$$= \int_{\alpha} \theta + \int_{\gamma} \theta$$

$$= F(p) + \int_\gamma \theta$$



$\gamma \in \pi_1(R, q_0)$

~~Let $T_\gamma(\alpha)$~~

Define $h : \pi_1(R, q_0) \to \mathbb{C}$ by $h(\gamma) = \int_\gamma \theta$.

Conclude: ~~Ekrnd~~ $F(T_\gamma(p)) = F(p) + h(\gamma)$.

Now consider $\tilde{R}$. Let $F(p) = \int_{p_0}^{p} \theta$.

The integral does not depend on the path since $\tilde{R}$ is simply connected.

$F: \tilde{R} \to \mathbb{C}$ and F is a local isometry.

since F is a local isomorphism...

Given any $p \in R$ we can find a r_p so that F takes a disk of radius r_p around p to a disk of radius r_p around $F(p)$.

(This is independent of the lift we choose.)

$$\begin{array}{ccc} \tilde{R} & \xrightarrow{F} & \mathbb{C} \\ \updownarrow & & \updownarrow \\ \pi_1(R) \text{ acts here} & & \pi_1(R) \text{ acts here via } h \end{array}$$

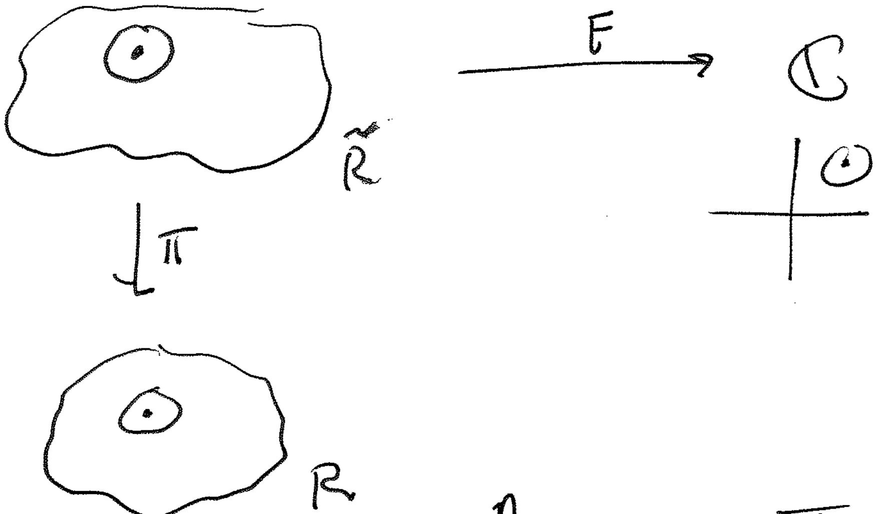
We have $h: \pi_1(R) \to \mathbb{C}$. Group homomorphism.

$$\gamma \quad \int_\gamma \theta$$

We know that $\pi_1(R)$ acts ~~freely, properly~~ discontinuous on $\tilde{R}$ so that the quotient (R) is a compact manifold. It follows, since F is equivariant that $h(\pi_1(R))$ acts on $\mathbb{C}$ so that the quotient is a compact manifold. $\Rightarrow h(\pi_1(R))$ is a lattice Λ.

Now in general a local isometry need not be a global isometry: think about the inclusion of ~~an open set~~ the open disk in $\mathbb{C}$.

What saves us here is the compactness of R. ~~For each $q \in R$ we~~ Since F is a local isometry upstairs for ~~ea~~ each $q \in R$ we can find an $r_q > 0$ so that the r_q disk around

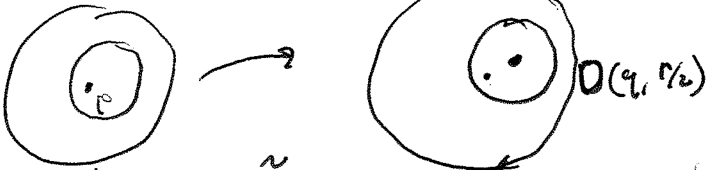


q lifts to an r_q disk in $\tilde{R}$. ~~w~~ (which maps to an r_q disk in $\mathbb{C}$).

By compactness there is some $r > 0$ which works for all points in R (hence ~~and~~ all points in $\tilde{R}$).

Claim that F is a covering map. In fact

Claim that a disk of radius $r/2$ in $\mathbb{C}$ is evenly covered.



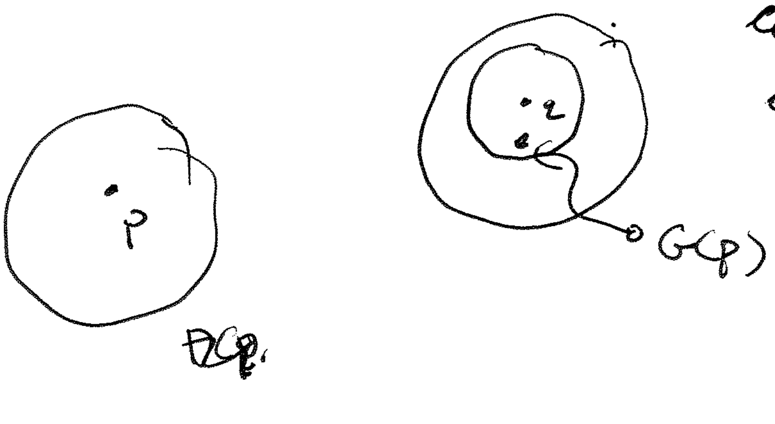
Any $p \in \tilde{R}$ maps to $D(q, r/2)$ then U_p maps to a ~~b~~ ~~be~~ disk of radius r around $F(p)$ which contains $D(q, r/2)$. It follows that F is a covering map and $\mathbb{C}$ is simply connected, F is 1-1.

Need to show that $G^{-1}(D(q, r/2))$ is a disjoint union of disks ~~mapping~~ so that G is a bijection on each disk.

Any ~~$p \in \tilde{R}$ maps to~~ $p \in \tilde{R}$, $G(p) \in D(q, r/2)$.

There is a set U_p ($p \in U_p$) which maps to the disk of radius ~~2~~ r in $\mathbb{C}$. By the triangle inequality

$G(U_p) = D(G(p), r)$ contains $D(q, r/2)$, so G^{-1}



~~Prop. If a compact Riemann surface~~

~~Prop.~~

~~Note that~~

$\tilde{R} \longrightarrow \mathbb{C}$

~~Deck group of~~ $\pi_1(R)$ acts freely and properly discontinuously on $\tilde{R}$. by deck translations. $\gamma \in \pi_1(R)$ acts by translation by $\int_\gamma \theta$. ~~The group of~~

Let Λ be the group of $\int_\gamma \theta$, $\gamma \in \pi_1(R)$.

We know that Λ acts freely and prop. disc. on $\tilde{R}$ with compact quotient but G is equivariant so Λ acts the same way on $\mathbb{C}$.

Λ is a lattice.

~~Fixed now~~

G induces an isomorphism

Def. Complete translation structure.

Complete (G,X) structure

$\tilde{R} \xrightarrow{\sim} X$

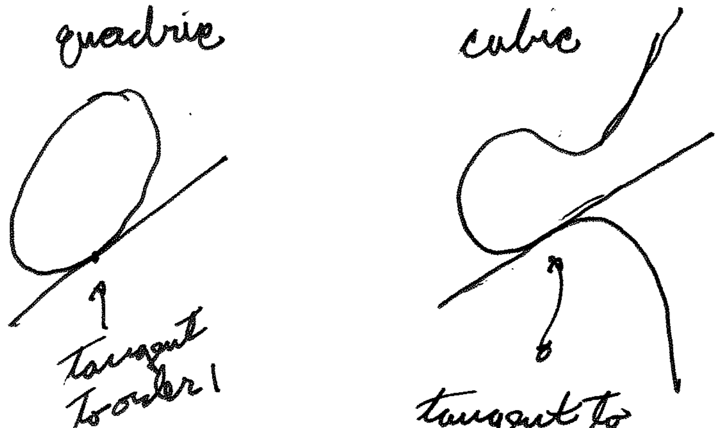
F induces an isomorphism from R to $\mathbb{C}/\Lambda$.

Cor. Every cubic curve ~~has the form~~ of the form $\mathbb{C}/\Lambda$.
(Weierstrass form of $y^2 = f(x)$).

Cor. Every cubic curve has the structure of an Abelian group.

What does this group structure mean geometrically?

One difference between ~~quartics~~ quadrics and cubics is that cubics can have ~~lines~~ points of inflection.



tangent to order 2

Interesting because new phenomenon. (Newton ... quadrics)

In Weierstrass form ∞ is a point of inflection

Fact. If we choose the origin of the group structure to be a point of inflection then

$\Rightarrow$ iff $z_0+z_1+z_2 \in \Lambda$ iff $u(z_0), u(z_1), u(z_2)$ lie on the same line.

$2z_0+z_1 \in \Lambda \Rightarrow$ tangent line through

$u(z_0)$ & intersects the curve at $u(z_1)$

$3z_0 \in \Lambda \Rightarrow u(z_0)$ is a point of inflection.

Cor. 9 points of inflection on a cubic.
(complex)

Uniformization theorem: Every simply connected Riemann surface is conformally equivalent to $\mathbb{CP}^1$, $\mathbb{C}$ or ~~U~~ the upper half-plane.

Cor. If R is compact and $\chi(R)=0$ then $\tilde{R}$ is $\mathbb{C}$ ~~in particular~~ and R is $\mathbb{C}/\Lambda$. ~~If $\chi(R)<0$ then~~ ~~then~~ In particular R has a ~~non zero 1-fo~~ non-vanishing hol 1-form.

~~Moduli spaces.~~

Cor. If R is a Riemann surface then R has a conformal metric of constant curvature.

$\mathbb{CP}^1$ has a metric of pos. curvature

$\mathbb{C}$ has a metric of 0 curvature invariant under fixed point free conformal automorphisms.

$\mathbb{H}$ has a metric of negative curve invariant under all ~~autos~~ conformal automorphisms.