~~fix~~ Assume $C = \{y^2 = f(z)\}$ comes from $\mathbb{C}/\Lambda$. $u : \mathbb{C}/\Lambda \to C$. ①

$f(x)$ cubic.

"Prop."

$$\int_{x_0}^{x_1} \frac{dz}{\sqrt{f(z)}} = \pm\left(\wp^{-1}(x_1) - \wp^{-1}(x_0)\right)$$

Proof. Let γ be a path in $\mathbb{C} - \{a, b, c\}$ from x_0 to x_1. $\pi_x : C \subset \mathbb{C}^2 \to \mathbb{C}$ is a 2 sheeted covering map away from $\{\alpha, \beta, \gamma, \delta\}$. Choose ~~an~~ inverse image of $\pi_x^{-1}(x_0)$. i.e. choose a y so that $y^2 = f(x)$. There is a unique lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = (x_0, y_0, 1)$. Choose a $\tilde{\tilde{\gamma}} : [0,1] \to \mathbb{C}$ so that $u\pi\tilde{\tilde{\gamma}} = \tilde{\gamma}$.

$$\int_{\tilde{\gamma}} \frac{dx}{y} = \int_{\tilde{\tilde{\gamma}}} dz = \tilde{\tilde{\gamma}}(1) - \tilde{\tilde{\gamma}}(0)$$
$$= \wp^{-1}(x_1) - \wp^{-1}(x_0).$$

This integral is independent of the choice of $\tilde{\tilde{\gamma}}$ since any two lifts differ by the addition of $\omega \in \Lambda$.

In conclusion the answer depends only on γ and ~~the~~ $\tilde{\gamma}(0)$ i.e. the choice of a branch.

Using the maps u to identify $\mathbb{C}/\Lambda$ with C_Λ we can choose a further lift of $\tilde{\gamma}$ to $\tilde{\tilde{\gamma}} : [0,1] \to \mathbb{C}$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[rrr, bend left, dashed] & \mathbb{C}/\Lambda \arrow[r, "u"] & C_\Lambda \arrow[r, "\pi_x"] & \mathbb{C} \\
z \arrow[r, mapsto] & z+\Lambda \arrow[r, mapsto] & (\wp(z), \wp'(z), 1) \arrow[r, mapsto] & \wp(z+\Lambda)
\end{tikzcd}$$

$$\wp$$

$$\begin{tikzcd}
\mathbb{C} \arrow[rr] & & C_\Lambda \arrow[r] & \mathbb{C} \\
& & {[0,1]} \arrow[ull, "\tilde{\tilde{\gamma}}"] \arrow[u, "\tilde{\gamma}"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

$\gamma(0) = x_0$, $\gamma(1) = x_1$, $\wp(\tilde{\tilde{\gamma}}(0)) = \wp(\gamma(0)) = \wp(x_0)$

$\wp(\tilde{\tilde{\gamma}}(1)) = \wp(x_1)$

$$\begin{tikzcd}
\mathbb{C} \arrow[rr] & & C_\Lambda \arrow[r] & \mathbb{C}
\end{tikzcd}$$

pullbacks of 1-forms

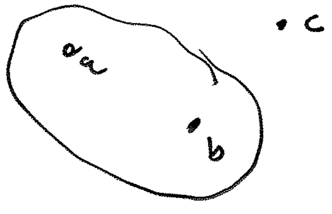
$$\int_{\tilde{\tilde{\gamma}}(0)}^{\tilde{\tilde{\gamma}}(1)} dz \qquad \int_{\tilde{\gamma}} \frac{dx}{y} \qquad \int_{\gamma} \frac{dx}{\sqrt{f(x)}}$$

$$\int_{\tilde{\tilde{\gamma}}} dz = \tilde{\tilde{\gamma}}(1) - \tilde{\tilde{\gamma}}(0) = \wp^{-1}(z_1) - \wp^{-1}(z_0).$$

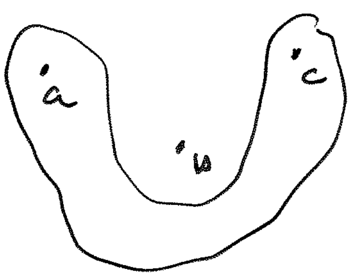
~~Let's be more~~

Roughly speaking the ambiguity in the integral corresponds to integrals over loops, γ in $\mathbb{C}-\{a,b,c\}$

Consider a loop which encircles an even number of pts. $\{a,b,c\}$ in the plane.



Or $\gamma \sim l_a^m \cdot l_b^n \cdot l_c^p$ with $m+n+p$ even.



$\gamma \sim l_a l_b l_c l_b^{-1} l_a l_b^{-1} \cdots$
with the sum of the exponents even

~~If~~ $\mathrm{ind}(\gamma,a)+\mathrm{ind}(\gamma,b)+\mathrm{ind}(\gamma,c)$

$$\mathrm{ind}(\gamma,p) = \frac{1}{2\pi i}\int \frac{dx}{(x-p)}$$

Such a loop lifts under π_x to a $\tilde{\gamma}$ loop in C.

$$\int_\gamma \frac{dx}{\sqrt{f(x)}} = \int_{\tilde{\gamma}} \mathbb{C}/\Lambda \, dz = \tilde{\gamma}(1)-\tilde{\gamma}(0) \in \Lambda.$$

Regular covering Λ is the deck group. (Universal covering.)

~~We recover Λ from the~~

~~We recover the lattice~~ Λ Don't need to worry about base points since Λ is abelian.

~~$\Lambda = \{\int_\gamma \omega\}$~~ $\Lambda = \left\{\int_\gamma \frac{dx}{y} : \gamma \text{ a loop in } C\right\}$.

Cusps.

Now we consider a general curve $C = \{Y^2 = f(x)\}$ f has degree 3. Claim that any such curve has a non-vanishing holomorphic 1-form given by ~~the~~ $\frac{dx}{Y}$.

Convenient method for calculating is to consider $P(x,Y) = Y^2 - f(x)$. $P \equiv 0$ on C. The 1-forms dx and dy on $\mathbb{C}^2$ both restrict to 1-forms on C and these 1-forms are related.

To see the relation calculate:

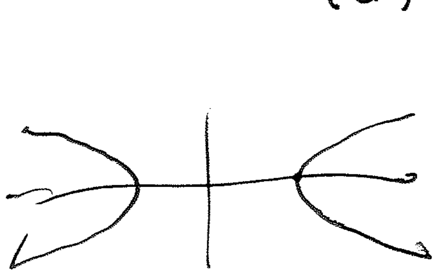
$$dP = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy$$

$$= -f'(x)\,dx + 2y\,dy$$

Since $P = 0$ on C, $-f'(x)dx + 2y\,dy = 0$ on C.

So $2y\,dy = f'(x)dx$. $\quad \frac{dy}{f'(x)} = \frac{dx}{2y}$

So $\frac{dx}{Y} = \frac{2dy}{f'(x)}$ is not zero at $x = 0$ since $f'(x)$ does not vanish at the when $f(x) = 0$.



What about the point at ∞?

~~Choose a coordinate~~ $x = \sim$

Monodromy around ∞ is non-trivial

π_x is locally a double cover.

$x = \frac{1}{\tau}$ gives a coord. at ∞.
$x = (\frac{1}{\tau})^2$ gives lifts to a coord. for C.

let $x = \tau^{-2}$ $\qquad y = \sqrt{f(x)} = \sqrt{4x^3 - g_1}$

near $\tau = 0$. $\qquad y = \sqrt{f(x)} = \sqrt{a_3 x^3 + a_2 x^2 + a_1 x + a_0}$

$$= \sqrt{a_3 \tau^{-6} + a_2 \tau^{-4} + a_1 \tau^{-2} + a_0}$$

$$= \tau^{-3} \sqrt{a_3 + a_2 \tau^2 + a_1^4 + a_0 \tau^6}$$

$$\frac{dx}{y} = \underset{\substack{\uparrow \\ dx}}{\left(-2 \frac{d\tau}{\tau^3}\right)} \underset{\substack{\uparrow \\ y}}{\left(\frac{\tau^3}{\sqrt{\cdots}}\right)} = \frac{-2\, d\tau}{\text{hol. fcn.}}$$

not zero.

To parametrize the curve $y=f(x)$ we need to be able to construct a local square root at ∞. ~~Since~~

We want to be able to lift a local parametrization near the point at ∞ to a parametrization of the curve.

Since the monodromy in ~~[illegible]~~ in ~~$\mathbb{Z}_2$~~ Symm(2) is non-trivial and order 2 ~~we need in order~~

$y^2 = x^3$

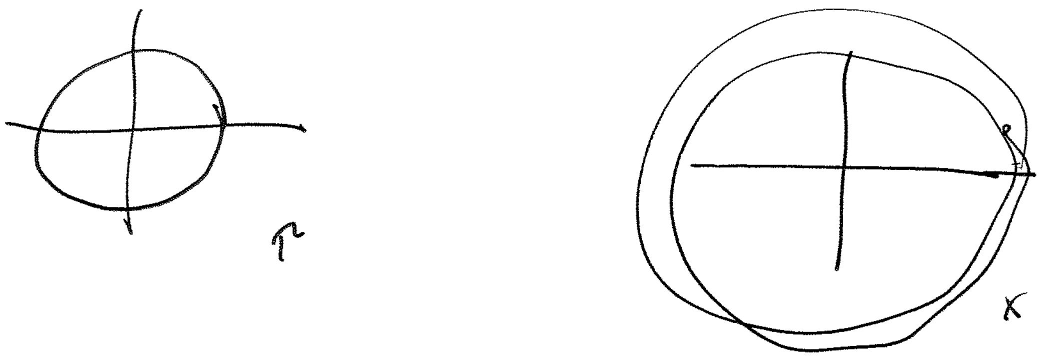
~~we need a~~ in order to lift a path it needs to go around the pt. at ∞ twice.

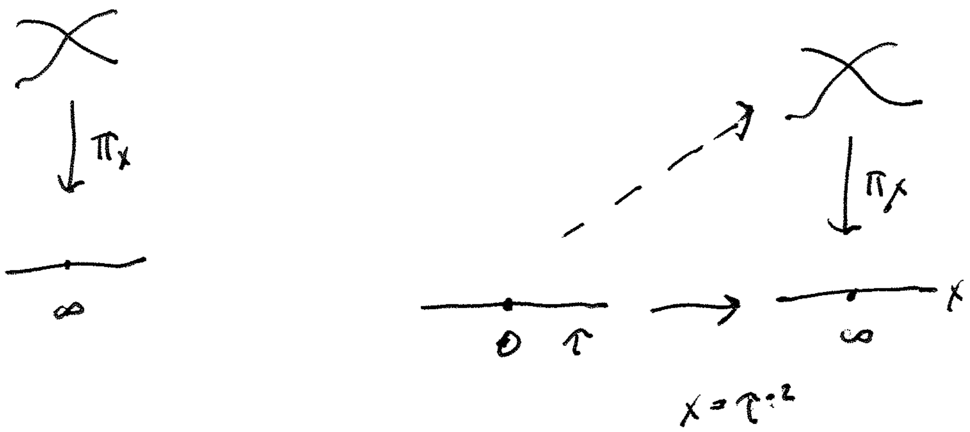
Cohn

A priori could try $x = \tau^{-1}$. This fails for topological reasons so we use $x = \tau^{-2}$.

$x = (\tau^{-1})^2 = \tau^{-2}$.

Locally the ~~map~~ projection map is a covering of degree 2. In order that our parametrization has a chance of lifting we use $x = \tau^{-2}$.





non-trivial monodromy picture

Parametrizing the curve is finding a squareroot for $f(x)$.

Prop. If a compact Riemann surface has a ~~hol~~ holomorphic 1-form with no zeros then ~~any two differ by mult. by a constant.~~ it is essentially unique: any 2 differ by a multiplicative constant. Proof. The quotient of the 2 1-forms in a chart has the form $\frac{f(z)dz}{g(z)dz}$; $\frac{f(z)}{g(z)}$ is a well defined function on R.

Prop. If a compact Riemann surface has a holomorphic 1-form with no zeros then it ~~has the fo~~ is conformally equivalent to $\mathbb{C}/\Lambda$ for some lattice Λ.

$G(w)dw = \frac{G\,dw}{dz}\cdot dz$

non-zero hol. fcn is constant

Proof. Let R be the surface and θ the 1-form. ~~Let $\check{R}$ be the universal cover~~

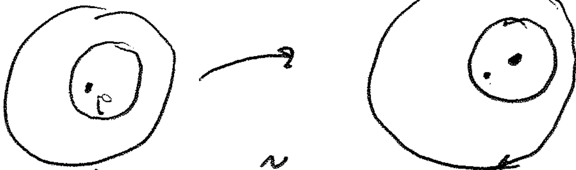
Given $p \in R$, U a s.c. nbd of p, $q \in U$ | $F(q) = \int_p^q \theta$ defines a map ~~from $\check{R} \to \mathbb{C}$~~ to $\mathbb{C}$, $F: U \to \mathbb{C}$.

Since ~~F~~ $F(q) = \int_p^q f(z)dz$

$F' = f \neq 0$ this map is locally invertible. We can find an open set U_p mapping to a disk of radius r_p. ~~Fo~~ By compactness there is a lower bound for $r_p \geq r > 0$.

Let $\tilde{R}$ be the universal cover of R. Fix $p_0 \in \tilde{R}$. We have a map ~~$\int_{p}$~~ $G(q) = \int_{p_0}^q \theta$ from $\tilde{R} \to \mathbb{C}$.

Claim that a disk of radius $r/2$ is evenly covered.



$D(q, r/2)$

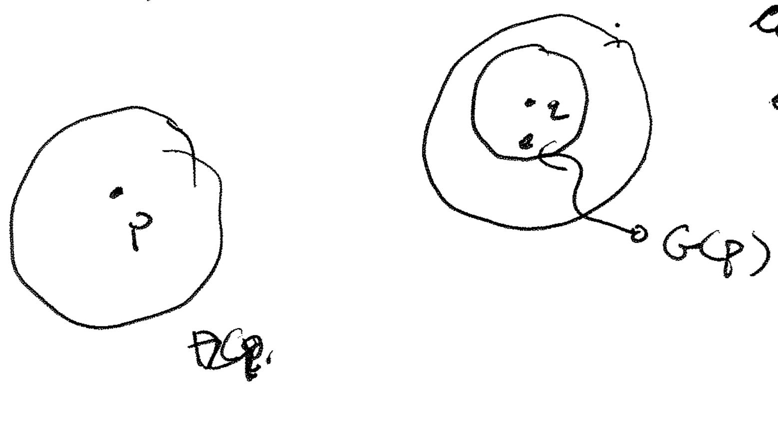
Any $p \in \tilde{R}$ maps to $D(q, r/2)$ then U_p maps to a ~~f la~~ disk of radius r around $F(p)$ which contains $D(q, r/2)$. It follows that F is a covering map and $\mathbb{C}$ is simply connected, F is 1-1.

Need to show that $G^{-1}(D(q, r/2))$ is a disjoint union of disks ~~mappings~~ so that G is a bijection on each disk.

Any ~~$q \in \tilde{R}$ maps to~~ $p \in \tilde{R}$, $G(p) \in D(q, r/2)$.

There is a set U_p ($p \in U_p$) which maps to the disk of radius r in $\mathbb{C}$. By the triangle inequality

$G(U_p) = D(G(p), r)$ contains $D(q, r/2)$. So G^{-1}



~~Prop. If a compact Riemann surface~~

~~Prop.~~

~~Note that~~

$\tilde{R} \longrightarrow \mathbb{C}$

~~Deck group of~~ $\pi_1(R)$ acts freely and properly discontinuously on $\tilde{R}$. $\gamma \in \pi_1(R)$ acts by translation by $\int_\gamma \theta$. ~~The group of~~

Let Λ be the group of $\int_\gamma \theta$, $\gamma \in \pi_1(R)$.

We know that Λ acts freely and prop. disc. on $\tilde{R}$ but G is equivariant so Λ acts the same way on $\mathbb{C}$.

with compact quotient

Λ is a lattice.

~~Furthermore~~

G induces an isomorphism