(Uniformization thm.)

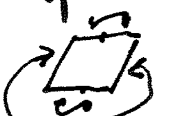
Thm.: A simply connected Riemann surface is conformally equivalent to

1) $\mathbb{CP}^1$

2) $\mathbb{C}$

3) the unit disk Δ.

Special case:



$\simeq \mathbb{CP}^1$ ← quotient compact & s.c.

~~Cubic curve $\simeq \mathbb{C}/\Lambda$.~~

Special case: A proper subset of $\mathbb{C}$ is conformally equivalent to Δ. (Riemann Mapping Thm.) (proper → simply connected)

Thm. requires a substantial investment in analysis on Riemann surfaces. See Beardon.

Cor. Let R be a Riemann surface then either ~~$R = \mathbb{CP}^1$ or~~ $R = X/\Gamma$ where X is $\mathbb{C}$ or Δ, $\mathbb{CP}^1$, and $\Gamma \subset Aut(\mathbb{C})$ or $Aut(\Delta)$ or $Aut(\mathbb{CP}^1)$ and Γ acts freely and properly discontinuously on X.

$$\begin{array}{c} \tilde{R} \\ \downarrow \pi \\ R \end{array}$$

Γ = deck trans. group.

Proof. Let $\tilde{R}$ be the universal covering space of R. $\tilde{R}$ inherits a Riemann surface structure from R so that $\pi : \tilde{R} \to R$ is holomorphic. The deck translation group preserves this Riemann surface structure. $\tilde{R}$ is simply connected so is 1), 2) or 3). Let

What are these automorphism groups?

① $\mathbb{CP}^1 \rightsquigarrow PSL(2,\mathbb{C})$.

Element α of $PSL(2,\mathbb{C})$ corresponds to a linear automorphism A of $\mathbb{C}^2$. A fixed point of α corresponds to an eigenvector of A. Every A has eigenvectors so every α has fixed points. Only the identity group acts freely. If $\tilde{R} = \mathbb{CP}^2$ then $\Gamma = \{1\}$, $R = \mathbb{CP}^2$.

② $\mathbb{C}$ $\qquad \mathrm{Aut}(\mathbb{C}) = \{z \mapsto az+b\}$.

~~αz~~ $z \mapsto az+b$ has a fixed point if $z = az+b$ has a solution $\quad (1-a)z = b \quad z = \frac{b}{1-a}$.
If $a \neq 1$ we have a fixed point.

So $\Gamma \subset \{z \mapsto z+b\}$. 3 possibilities $\Gamma = \{1\}$

$\Gamma = \{z \mapsto n\omega : \omega \neq 0\}$ $\qquad \Gamma = \{z \mapsto m\omega_1 + n\omega_2 \quad \frac{\omega_1}{\omega_2} \text{ not real}\}$

In particular $\pi_1(R)$ is abelian.
If R compact $\pi_1(R) \simeq \mathbb{Z}^2$.

R has a translation structure, a complete translation structure, R has a non-zero hol. 1-form.

let $G = \left\{ z \mapsto \frac{az + \bar{c}}{cz + \bar{a}} \right\}$. G is a group, iso. to $PSL(2,\mathbb{R})$.

3 $\tilde{R} = \Delta$.

~~In this case~~ Δ is conformally equivalent to the upper half-plane. Any conf. aut of $\mathbb{CP}^1$ which preserves ~~UHP~~ is a conf. aut of Δ. ~~$\mathrm{Aut}(\Delta) \supset PSL(2,\mathbb{R})$~~. $G \subset \mathrm{Aut}(\Delta)$

$PSL(2,\mathbb{R})$ acts transitively on UHP.

$PSL(2,\mathbb{R})$ preserves a metric on UHP given by

$$ds^2 = \frac{|dz|^2}{(\operatorname{Im} z)^2} = \frac{dx^2 + dy^2}{y^2}.$$

let $G = \left\{ z \mapsto \frac{az+\bar{c}}{cz+\bar{a}} \right\}$

~~Schwarz Lemma~~. On Δ $ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}$.

This is an example of a conformal metric.

On a Riemann surface ~~but~~ there is a notion of an angle between tangent vectors

hol. map with non-zero derivative preserves angles.

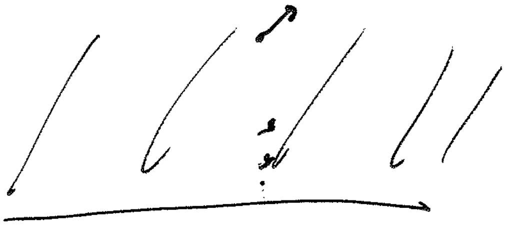
~~Charts and~~ ~~on~~ Change of coord. fns preserve angles.

but not a notion of length.

Riemannian metric has notions of angle and length. A conformal metric on a Riemann surface is a metric with the same notion of angle.

~~Sch~~ Schwartz Lemma. If $f : \Delta \to \Delta$ is a holomorphic map taking 0 to 0 then $|f'(0)| \leq 1$. If $|f'(0)| = 1$ then f is a ~~rotation of the disk~~, $f(z) = e^{i\theta}(z)$ for some real θ.

$PSL(2,\mathbb{R})$ acts transitively on UHP.



$z \mapsto z + r$

$z \mapsto \lambda z.$

Prop. $\mathrm{Aut}(\Delta) = \{$Mobius trans taking Δ to $\Delta\} = G$

Proof. ~~So~~ Say $f : \Delta \to \Delta$ is hol. and invertible. By composing with an element g of G we may assume that $gf(0) = 0$ and $gf : \Delta \to \Delta$. ~~By Schwarz lemma and $f'(0) = 1$. By~~ By Schwartz lemma $|gf'(0)| \le 1$. Applying SL to f^{-1} gives $|gf'(0)| = 1$.

~~gf' $|(gf)'(0)| = 1$ $|g'| = 1 \Rightarrow |f'(0)| = 1$.~~

So $gf(z) = e^{i\theta}z \Rightarrow gf \in G \Rightarrow f \in G$.

~~Cor~~ Cor. ~~Every~~ ~~Remarks~~ Cor. Schwartz-Pick thm. ~~Any~~ So a holomorphic map from Δ to Δ does not increase distance with respect to the hyperbolic metric.

Remark. ~~Every Riemann surface has a constant~~ ~~Δ has~~ The metric we have described on Δ has constant curvature.

Prop. Every Riemann surface has a metric of constant curvature.

measured w.r.t metric

Proof. $|f'|$. By pre and post composing with isometries may as

$gfh(0) = 0$
$|gfh'(0)| = 1$
$|g'| = 1$ $|h'| = 1$.

Theorem. Two Riemann surfaces $R_1 = \tilde{R}_1/\Gamma_1$ and $R_2 = \tilde{R}_2/\Gamma_2$ are ~~conjugate~~ conformally equivalent iff $\tilde{R}_1 \sim \tilde{R}_2$ and Γ_1 is conjugate to Γ_2 inside $\mathrm{Aut}(\tilde{R}_1)$.

Proof. A conformal automorphism $f: R_1 \to R_2$ lifts to a conformal automorphism $\tilde{f}: \tilde{R}_1 \to \tilde{R}_2$. ~~Since~~ ~~the~~ ~~Wenger~~
If $\gamma \in$ deck group of R_1 then $\tilde{f} \circ \gamma \circ \tilde{f}^{-1}$ is in the deck group of R_2.

$$\begin{array}{ccc} \tilde{R}_1 & \xrightarrow{\tilde{f}} & \tilde{R}_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ R_1 & \xrightarrow{f} & R_2 \end{array}$$

Remark: Since $\mathbb{CP}^1$, $\mathbb{C}$, Δ are not conformally equivalent the "type" of $\tilde{R}$ is an invariant of R. We say that R is hyperbolic iff $\tilde{R} \simeq \Delta$.

Cor. A holomorphic map ~~from oth~~ between hyperbolic surfaces does not increase distances with respect to the hyperbolic metric.

Cor. ~~Hol. map of a torus or or sphere may increase distances.~~

Cor. A holomorphic automorphism of a hyperbolic surface is an isometry.

spheres and tori pr

Thm. If R is a closed surface then $R \simeq \mathbb{C}P^1$ iff $\chi(R)>0$, $R \simeq \mathbb{C}/\Lambda$ iff $\chi(R)=0$. $R \simeq \Delta/\Gamma$, $\Gamma \subset PSL(2,\mathbb{R})$ iff $\chi(R)<0$.

Proof. By Gauss-Bonnet & $2\pi\chi(R) = \int_R \text{curv.}$

~~so the si~~ $\chi(R)$ determines the sign of the curvature.

Cor. The curvature of the metric $\frac{|dz|}{1-|z|^2}$ on Δ is negative.

Proof. There exist ~~close~~ compact Riem. surfaces of neg. Euler characteristic.

Def. Let us say that R is hyperbolic if $\tilde{R} \simeq \Delta$.

A hyperbolic surface has a complete metric of curvature -1.

Remark. It is also interesting to identify the "type" of a surface which is not closed. ~~W~~ ~~If $\pi_1(R)$~~