Let R be the topological surface constructed from the regular octagon. R is a surface of genus 2.

~~Let v be~~

Let v be the point on R corresponding to the vertex. We described how to construct a translation atlas on $R - \{v\}$.

We would like to construct a conformal atlas on all of R.

Question: What does the ~~flat~~ translation structure look like in a nbd. of R? Metrically? In terms of horizontal trajectories?

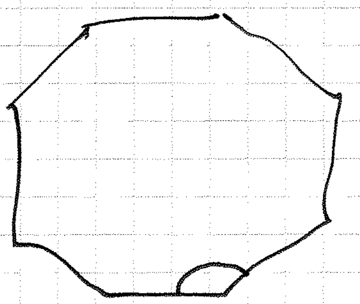
Let Δ be an ε disk around v. What does Δ look like in the octagon?

Let $F: \Delta \to D$ be the function obtained by ~~gluing them~~ translating these sectors so the vertices match up.

We can extend the atlas on R to a <u>conformal</u> atlas on $R \cup \{v\}$ which is <u>not</u> a translation atlas.

polar coordinates

Write U as $\{(r,\theta) : 0 \le r < 1/4,\ 0 \le \theta \le 6\pi\}$.

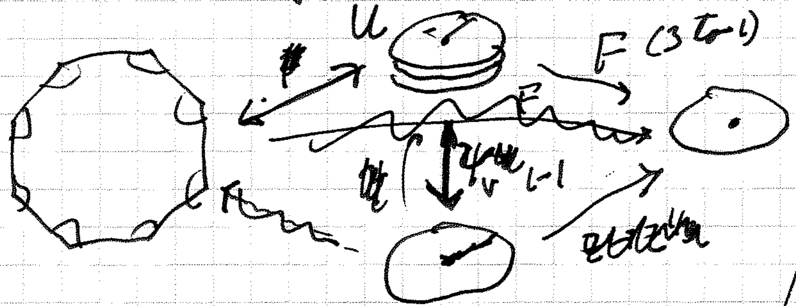


$F(r,\theta) = (r,\theta)$

Define a coordinate chart $\psi_v : U \to \Delta$ by
$\psi_v(r,\theta) = (r^{1/3}, \theta/3)$.

① takes the circle of length 6π (in U) to the circle of length 2π in Δ.

②



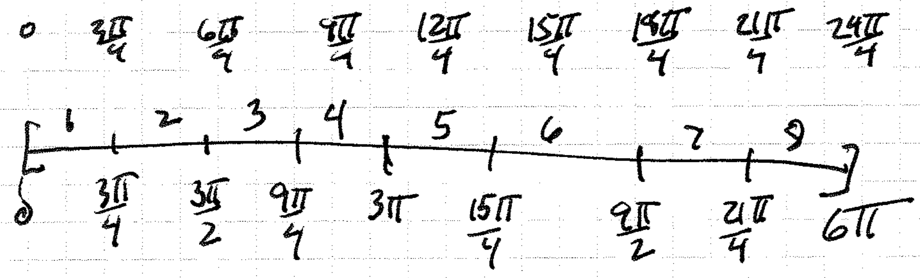
Trajectory structure:

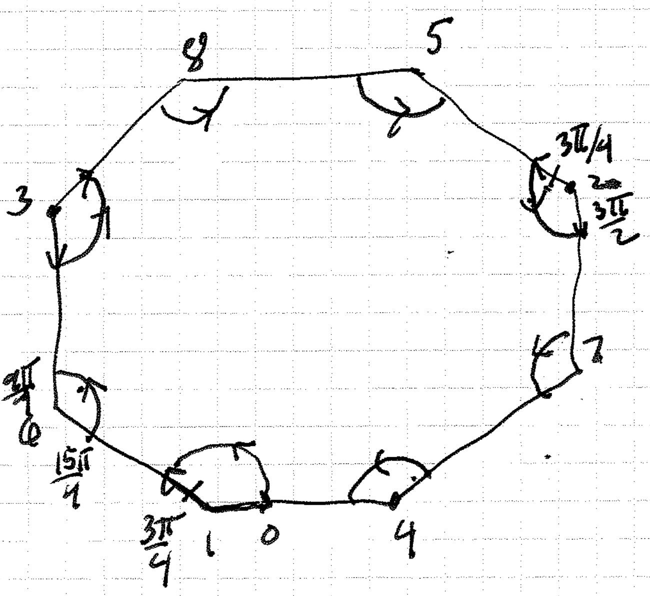
$\mathcal{A}' = \{\phi_j, \psi\}$ on $R \cup \{v\}$.

$F = \psi_v^3$. In terms of the local coordinate $w = \psi_v$, $z = \psi(w)$ $F = w^3$.

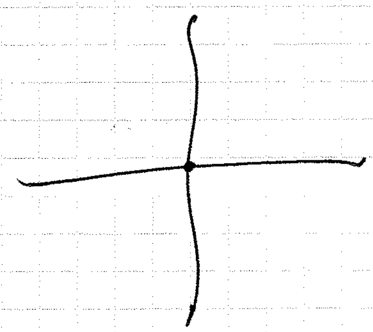
1-form is $3w^2\,dw$.

Change of coordinate functions have the form $z \mapsto z^3$ or $z \mapsto z^{1/3}$ locally.





lifting of the angle function in polar coordinates.



3-fold branched cover of a disk is still a disk and it has a natural conformal structure even at the origin.

Define $F : \Delta \to D$ by $F(r,\theta) = (r, \theta \bmod 2\pi)$.

$\Delta = \{(r,\theta) : 0 \le \theta \le 6\pi\}$, $D = \{(r,\theta) : 0 \le \theta \le 2\pi\}$

F is 3 to 1.

Define $\psi : \Delta \to \mathbb{C}$ by $\psi(r,\theta) = (r^{1/3}, \theta/3)$.

ψ is 1-1.

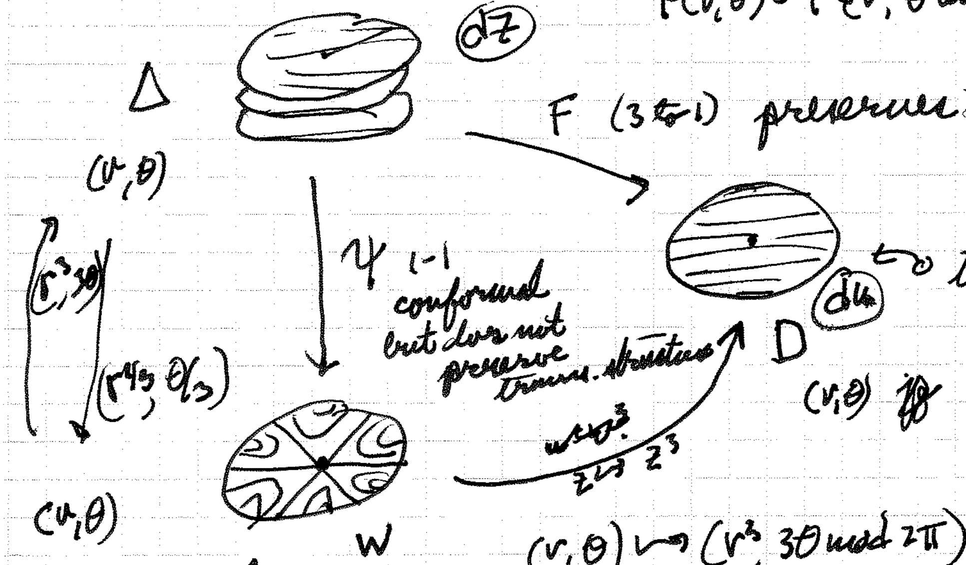
Let $\mathcal{A}'$ be the atlas for R where we add the chart psi ψ to the charts ϕ_j.

$\mathcal{A}' = \{\phi_j, \psi\}$.

geometrically correct picture of u (dz)

$F(r,\theta) = F(r, \theta \bmod 2\pi)$

③



to trajectory structure

$(r,\theta) \mapsto (r^3, 3\theta \bmod 2\pi)$

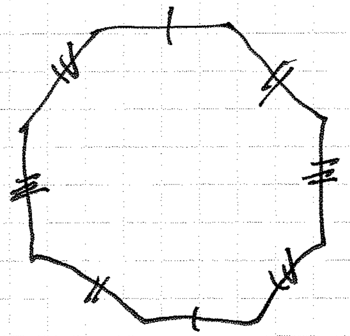
$z \mapsto z^3$.

correct horizontal trajectory structure in terms of $w = \Psi(z)$

(not metrically correct)

Change of coordinate functions in $\mathbb{C}'$ have the form $z \mapsto z^{1/3}$, $z \mapsto z^3$ where they make sense.

Recall



We constructed a translation structure on ~~R which~~ a Riemann surface R which is a surface of genus 2 with a puncture.

We would like to show that ~~we~~ R is included in a Riemann surface ~~R*~~ $R \cup \{V\}$ which is a closed surface, ie genus 2 with no puncture.

Translation atlas $\mathcal{U} = \{\varphi_j\}$ on R.

~~We can't extend~~

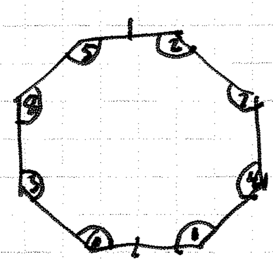
~~The~~ First observation is that we cannot extend the ~~flat~~ translation structure any further.

What does the translation structure look like "near the "vertex"?

Glue 8 vertex nbds. together to construct a function $F : \mathcal{U} \subset R \to \mathbb{C}$.

Cone type singularity with cone angle 6π.

$1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 6 \sim 7 \sim 8 \sim 1$



If we move the images of all these sectors to a single point we construct a map

$F : \mathcal{U} \longrightarrow \Delta^*$ = punctured disk

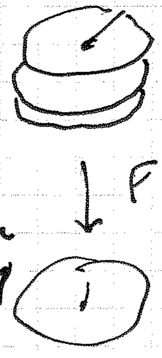
punctured

angle of $\frac{3\pi}{4}$ at every vertex

total angle: 6π

Define cone angle.

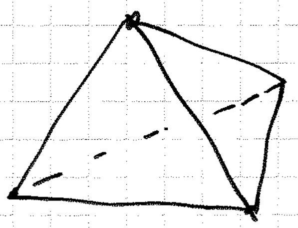
Cone angle $\neq 2\pi$ is an ~~obs~~ obstruction to extending the transl. structure



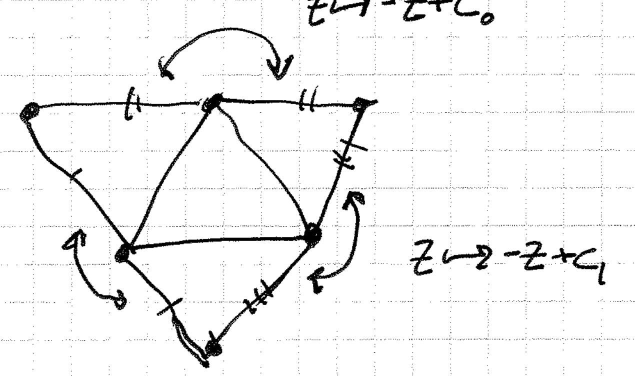
This map "gives the translation structure" ~~where~~ away from the vertex but does not extend to a ~~ma~~ map which is [illegible] at the vertex.

Two more quick examples.

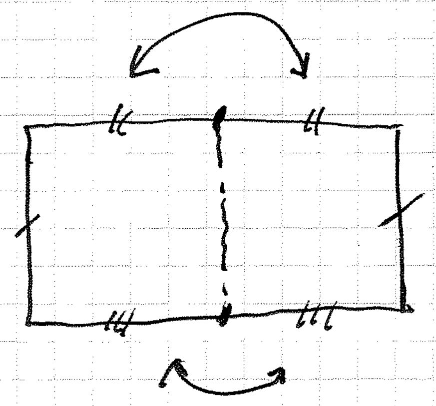
Surfaces with half-translation structures.



metrically correct pictures.

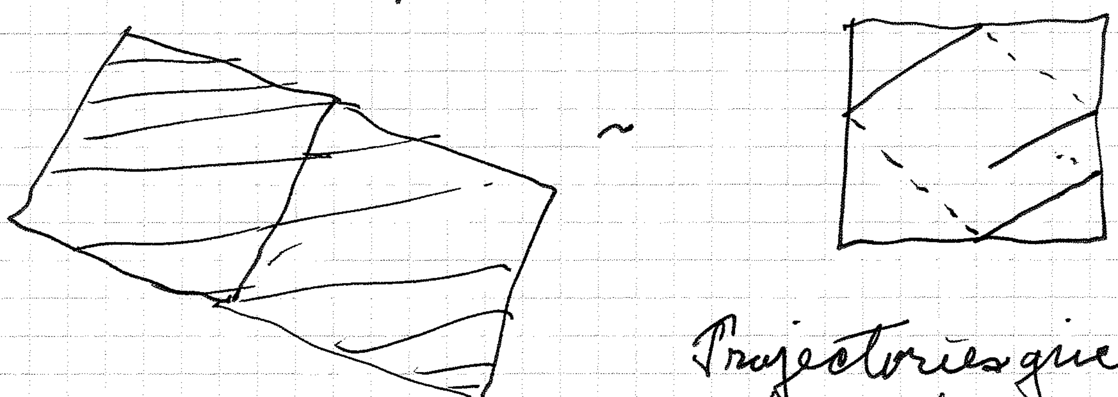


Cone angles at vertices are equal to π. Use $(r, \theta) \mapsto (r^{1/2}, \theta/2)$ coordinates at vertices.



Cone angle of π at each of 4 vertices.

What does the trajectory structure look like if we tip over example?



Trajectories give billiard paths on the square.