Let $P(z,w) = \sum_{m,n=0}^{k} a_{n,m} z^n w^m$ be a polynomial in two complex variables. [The degree of P is $\max\{n+m : a_{n,m} \neq 0\}$.]

P is an example of a holomorphic function from $\mathbb{C}^2$ to $\mathbb{C}$. In general a holomorphic function can be written as a convergent power series in two variables. [Defined in terms of power series expansion or C-R equations.]

~~If we write~~ We can think of P as a smooth function from $\mathbb{R}^4$ to $\mathbb{R}^2$ if we write $z = x+iy$ ~~and~~ $w = u+iv$ and $P = R+iQ$ then ~~is~~ the derivative of P is the 2×4 matrix:

$$DP = \begin{bmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial u} & \frac{\partial R}{\partial v} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial u} & \frac{\partial Q}{\partial v} \end{bmatrix}.$$

A holomorphic function from $\mathbb{C}^2 \to \mathbb{C}$ has the property that its derivative is complex linear. (C-R equations)
This means ~~it commutes~~ if you multiply by i (in $\underset{\mathbb{C}}{\mathbb{R}^2}$) then the result is multiplication by i in $\underset{\mathbb{C}^2}{\mathbb{R}^4}$. Explicitly this means that

$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $J_2 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & \\ -1 & 0 & & \\ \hline & 0 & 0 & 1 \end{array}\right)$ $J_1 D J_2 = D$,

$$DP = \begin{pmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial u} & \frac{\partial R}{\partial v} \\ -\frac{\partial R}{\partial y} & \frac{\partial R}{\partial x} & -\frac{\partial R}{\partial v} & \frac{\partial R}{\partial u} \end{pmatrix}$$

Cauchy-Riemann equations in 2 variables,

Equality of these expressions

If we write this as a complex matrix we get 1×2 $\begin{pmatrix} * & * \end{pmatrix}$ matrix
$\in \mathbb{C}$ $\in \mathbb{C}$

equal to $\begin{pmatrix} \frac{\partial R}{\partial x} + i \frac{\partial R}{\partial y} & \frac{\partial R}{\partial u} + i \frac{\partial R}{\partial v} \end{pmatrix}$

which we call $\begin{pmatrix} \overset{\in\mathbb{C}}{\frac{\partial P}{\partial z}} & \overset{\in\mathbb{C}}{\frac{\partial P}{\partial w}} \end{pmatrix}$

and we compute in the usual way.

Example: Say $P(z,w) = w^2 - z^3 + az^2 + bz + c$

then $\begin{pmatrix} \frac{\partial P}{\partial z} & \frac{\partial P}{\partial w} \end{pmatrix} = \begin{pmatrix} 2w & -3z^3 + 2az + b \end{pmatrix}$.

Claim: If

Algebraic curves.

Let $X = \{(z,w) \in \mathbb{C}^2 : P(z,w) = 0\}$.

X is called an affine variety (of dim) or affine "curve".

Claim. If at every point of X either $\frac{\partial P}{\partial z} \neq 0$ or $\frac{\partial P}{\partial w} \neq 0$ then we can construct a conformal atlas for X.

Before we prove this let me give an example.

~~Hagr~~ Say $X = \{w^2 = f(z)\}$ where f is a polynomial in z with no repeated roots then X ~~has~~ is a Riemann surface.

~~Check~~ $P(z,w) = w^2 - f(z)$.

$$\cancel{\frac{\partial P}{\partial z} \neq 0} \quad \frac{\partial P}{\partial z}(z,w) = f'(z)$$

$$\frac{\partial P}{\partial w}(z,w) = 2w.$$

Need to show that $w^2 - f(z)$, $f'(z)$ and $2w$ cannot vanish simultaneously.

Say $2w = 0$ then $w = 0$ and $f(z) = 0$.

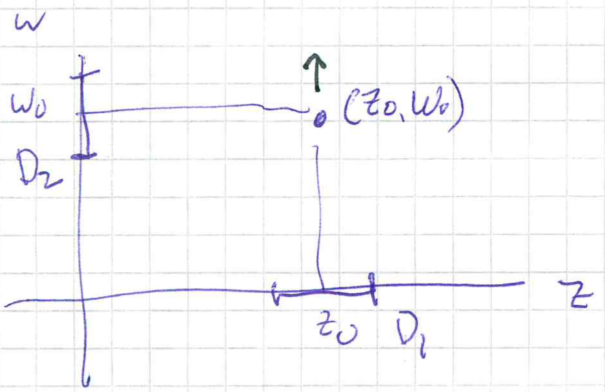
A z value for which $f(z)$ and $f'(z)$ vanish is a repeated root of f and we have assumed that these don't exist. QED.

Theorem 1

Let $P(z,w)$ be a polynomial in 2 complex variables

Let $X = \{(z,w) \in \mathbb{C}^2 : P(z,w) = 0\}$.

Theorem. Suppose $(z_0, w_0) \in X$ and $\frac{\partial P}{\partial w}$ does not vanish at (z_0, w_0). Then there is a disk D_1 centred at z_0 in $\mathbb{C}$ and a D_2 centered at w_0 and a holomorphic map $\phi : D_1 \to D_2$ with $\phi(z_0) = w_0$ such that



$$X \cap (D_1 \times D_2) = \{(z, \phi(z)) : z \in$$

Proof requires 2 facts from 1 complex variable.

1. If f is a holomorphic function defined in an open set containing $\bar{D}$ then a disk D then

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(w)}{f(w)}\, dw$$

is the number of solutions to $f(w) = 0$ in D counted with multiplicity.

Recall the proof. ~~the~~ Integral of a meromorphic function only contribution comes from the ~~poles~~ residues at the poles. Poles only occur at solutions of $f(z)=0$. Fix one such solution, call it z_0.

$f(z)=(z-z_0)^m g(z)$ with $g(z_0)\neq 0$ and g hol.

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int \frac{m(z-z_0)^{m-1}g(z)+(z-z_0)^m g'(z)}{(z-z_0)^m g(z)}\,dz$$

If $f'(z_0)\neq 0$ then $m=1$.

$$= \frac{1}{2\pi i}\int_{\partial D}\frac{m}{(z-z_0)}+\frac{g'(z)}{g(z)}\,dz = m.$$

~~Ecm~~

m = multiplicity of z_0 as a soln. of $f(z)=0$. (inside D)

Counts the solution ~~z_0~~ z_0 with multiplicity.

② $$\int\frac{zf'(z)}{f(z)}\,dz = \sum_{f(z_0)=0} z_0 \quad \text{Counted with multy}$$

Proof. As before we consider the local contribution at z_0.

$z=(z-z_0)+z_0$

$$\frac{1}{2\pi i}\int_{\partial D}\frac{zf'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int\frac{mz}{(z-z_0)}+\frac{zg'(z)}{g(z)}\,dz$$

$$= \frac{1}{2\pi i}\int_{\partial D} m+\frac{mz_0}{(z-z_0)}+\frac{zg'(z)}{g(z)}\,dz$$

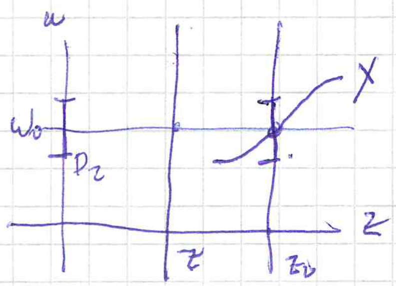
$$= m\cdot z_0$$

z_0 is a solution of $f(z)=0$ inside D.

~~Now consider~~

In particular if there is just one solution z_0 to $f(z)=0$ in D then $\frac{1}{2\pi i}\int_{\partial D} \frac{z f'(z)}{f(z)}\, dz = z_0$ is that solution.

Proof of thm. Fix z and consider



$f_z(w) = P(z,w)$ as a function of w. It is a polynomial in w so it is holomorphic.

~~The hypothesis that~~

First consider $z=z_0$. The hypothesis that $\frac{\partial P}{\partial w} \neq 0$ means that f'_{z_0} does not vanish at w_0. Thus we can find a small ~~disk~~ disk D_z centered at w_0 so that f_{z_0} has no other zero in D_z.

$$\phi(z) = \int_{\partial D_z} \frac{w}{P} \frac{\partial P}{\partial w}\, dw$$

integrand depends holomorphically on z $\Rightarrow$ value of the integral depends holomorphically on z.

$$\phi(z) = \int_0^1 \frac{w}{P} \cdot \frac{\partial P}{\partial w} \cdot \gamma'(t)\, dt$$

Differentiation under the integral sign implies $\phi(z)$ is satisfies C-R equations and is holomorphic.

The fact that $\frac{\partial P}{\partial w} \neq 0$ means that $f'_{z_0} \neq 0$ at w_0.

Thus we can find a small disk D_2 ~~around~~ centered at w_0 so that f_{z_0} has no other zero in D_2. Consider

$$(*) \quad \frac{1}{2\pi i}\int_{\partial D_2} \frac{f'_{z_0}(w)}{f_{z_0}(w)}\,dw .$$

Only contribution occurs at $w = w_0$ and that contribution is 1.

~~Now we can find~~

Since $f_{z_0}(w) \neq 0$ for $w \in \partial D_2$ we can find a small disk ~~around~~ D_1 centered at z_0 for which $f_z(w) \neq 0$ for $z \in D_1$ and $w \in \partial D_2$.

The integral $*$ varies continuously for $z \in D_1$ and takes on integral values so it is constant and equal to 1. Thus for ~~la~~ each $z \in D_1$ the equation $f_z(w) = 0$ has 1 solution in D_2.

To find this solution we use the integral

$$(**) \quad \phi(z) = \frac{1}{2\pi i}\int_{\partial D_2} \frac{w f'_z(w)}{f_z(w)}\,dw .$$

This integral varies uniformly continuously since the denominator remains non-zero. ~~Integrand~~

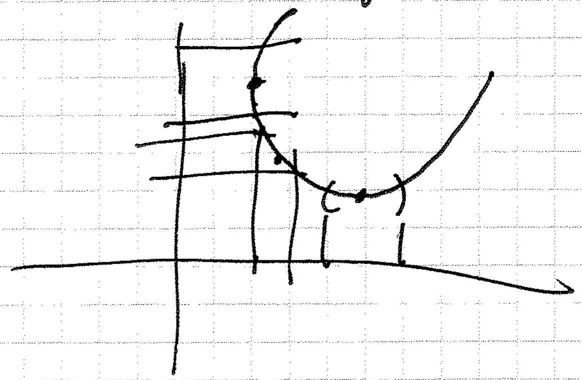
~~The integr~~

$$(**)\quad \phi(z) = \frac{1}{2\pi i}\int_0^1 \frac{\gamma(t)\, f_z'(\gamma(t))}{f_z(\gamma(t))}\cdot \gamma'(t)\,dt$$

Integrand is ~~not~~ differentiable in z. Then jdeberry on differentiating under the integral sign says that the integral is differentiable in z and in fact holomorphic.

Completes the proof.

Construction of the atlas.



at each $(z_0, w_0) \in R$ either $\frac{\partial P}{\partial z} \neq 0$ or $\frac{\partial P}{\partial w} \neq 0$ so

we can construct a local chart taking values in the z-axis or the w-axis using ~~projection onto the coordinate~~ coordinate projection π_z or π_w. These are homeomorphisms

Overlap functions have the form

$$z \mapsto (z, \phi(z)) \xrightarrow{\pi_w} \phi(z)$$

Definition. Affine variety
algebraic geometry viewpoint.

[Stress the connection between varieties with the same equation defined over different fields.
eg dimension should not change.]

An affine variety makes sense over a field. We will consider ~~the~~ fields $\mathbb{C}$ and $\mathbb{R}$.
(surface versus curve. ‒)

[Say "curve" for $\mathbb{C}$ fields]

Different flavor to studying equations over $\mathbb{C}$ and $\mathbb{R}$ ~~related to the fact that $\mathbb{C}$ is algebraically closed.~~

$$C = \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\}.$$

See this most directly with polynomial equations.

~~In $\mathbb{R}$~~ Over $\mathbb{C}$ a polynomial ~~of~~ degree d has d roots, if we count them with appropriate multiplicities

$P(x) = \prod (x - z_i)^{m_i}$ where $\sum m_i = d$.

Over $\mathbb{R}$ we only get an upper bound for the number of roots.

Analogous result about polynomials.

Say P has repeated factors if ~~$P(z,w)$~~

$P(x,y) = Q^2(x,y)\, R(x,y)$.

Thm. (Hilbert Nullstellensatz) If two polynomials P and Q determined the same variety and neither have repeated factors then P and Q differ by a constant.

Can there be compact components over $\mathbb{C}$?
(Over $\mathbb{R}$ rescaling takes compact components to 0.)

$X^2+Y^2=-1$, $X^4+Y^4=-1$

Hilbert Nullstellensatz

Repeated factors $\quad P(X,Y) = Q^2(X,Y)\,R(X,Y)$

Degree of a curve

~~Pdt~~ Can say polynomials are equivalent if they are scalar mults of each other

Def. Degree of a curve C defined by $P(X,Y)$ is the degree of the polynomial P.
(since the poly is unique up to rescaling)

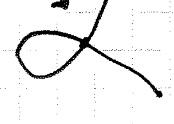
A point $(a,b)\in C$ is called a singular point of C if $\frac{\partial P}{\partial x}(a,b) = 0 = \frac{\partial P}{\partial y}(a,b)$.

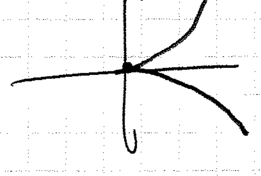
If there are no singular pts in C then C is non-singular

$X^2+Y^2=1$ is non-singular if $2X$, $2Y$ both vanish then $(a,b)=(0,0)$. not on the curve

$Y^2=X^3$ is singular

$Y^2=X^3+X$

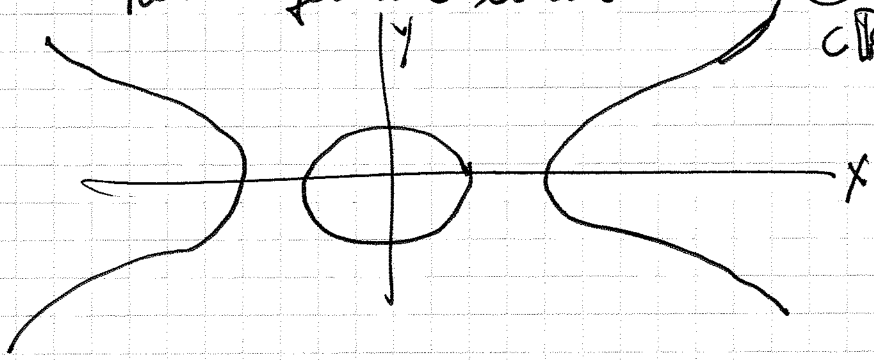


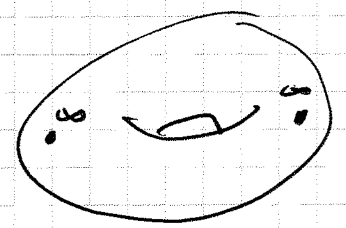


$Y^2 = (X^2-1)(X^2-k^2)$

Real algebraic curve

$\subset \mathbb{R}^2$





$\subset \mathbb{C}^2$

Complex algebraic curve.

$(X,Y)\in\mathbb{C}^2$: $\quad X, Y$ complex variables.

Let $C = \{P(X,Y)=0\}$

(or C)

Definition. A point $(a,b)\in R$ is called a singular point if $\frac{\partial P}{\partial X}(a,b)=0=\frac{\partial P}{\partial Y}(a,b)$.

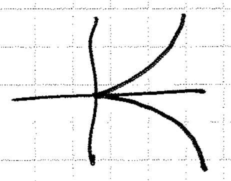
~~R is non-singular~~

R is non-singular if it contains no singular points.

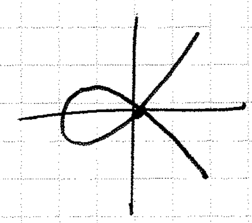
Example: $X^2+Y^2=1$ is non-singular:

$P(X,Y)=X^2+Y^2-1 \qquad \frac{\partial P}{\partial X}=2X,\ \frac{\partial P}{\partial Y}=2Y$ if both vanish then $(X,Y)=(0,0)$ but $(0,0)$ is not on R.

$Y^2=X^3$ is singular at $(0,0)$.



$Y^2=X^3+X$ singular at $(0,0)$.



Remark. We will focus on non-singular curves. Note that a ~~non-singular cu~~ polynomial defining a non-singular curve has no repeated factors.

$$\frac{\partial}{\partial x} P = \frac{\partial}{\partial x} Q^2 R = 2Q \cdot \frac{\partial Q}{\partial x} R + Q^2 \frac{\partial R}{\partial x}$$

$$\frac{\partial}{\partial y} Q^2 R = 2Q \frac{\partial Q}{\partial y} R + Q^2 \frac{\partial R}{\partial y}$$

~~If~~ Any pt. in $Q \equiv 0$ is singular.

~~For non-singular "curves" over $\mathbb{C}$ we expect a close connection~~

If P_x and P' both define the same non-singular curve then $P' = \lambda P$ for $\lambda \neq 0$.

In this case we expect a close connection between properties of the variety and properties of the polynomial that defines it.