Example Sheet 2 is out. MA 3L01 ①

$\mathbb{CP}^2$ is the set of complex 1-dim subspaces in $\mathbb{C}^3$.

It is a 2-dim complex manifold with charts given by

$$X, Y \mapsto (X, Y, 1)$$
$$X, Z \mapsto (X, 1, Z)$$
$$Y, Z \mapsto (1, Y, Z).$$

A projective curve in $\mathbb{CP}^2$ is the zero set of a homogeneous ~~pl~~ polynomial ~~in~~

$P(X,Y,Z)$. Homogeneity implies $P(\lambda X, \lambda Y, \lambda Z) = \lambda^d P(X,Y,Z)$
$\sum a_{m,n,v} X^m Y^n Z^v$ $m+n+v=d$.

A homogeneous polynomial P determines affine curves in each chart eg.

$\{(X,Y) : P(X,Y,1)=0\}$. $\quad \mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$

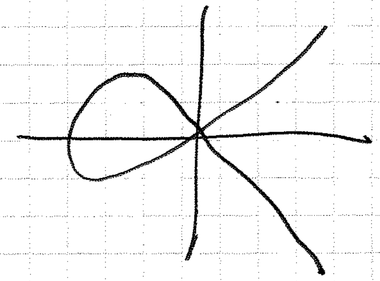
A projective curve can be thought of as an affine curve together with points at ∞ $\{(X,Y) : P(X,Y,0)=0\}$.
(Homogen. poly in 2 variables is a product of linear factors)
Points at ∞ correspond to asymptotes of the affine curve.

Prop. A projective curve is compact.

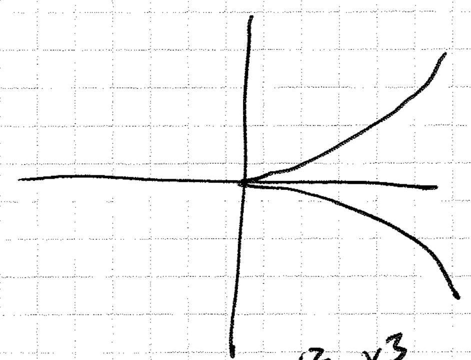
Proof. Closed subset of a compact space.

Remark. Rescaling can be done by zooming in as well as zooming out.

Examples



$Y^2 = X^3 + X^2$

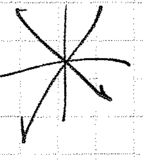


$Y^2 = X^3$

Only the lowest order terms contribute. ~~We get~~ At regular points we get ~~line~~ the tangent line. At singular points we get unions of lines (~~products~~ of corresponding to products of linear factors.

$Y^2 = X^2$

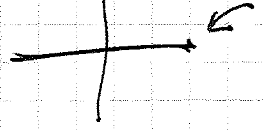
$(X-Y)(X+Y)=0$



Collection of tangent lines at a sing. pt,
~~Cross with~~
Order of a sing pt.

$Y^2 = 0$

mult. 2



(3)

Proposition. Suppose $P(z_0, z_1, z_2)$ is a homogeneous polynomial of degree $d \geq 1$ and the only solution of

$$\frac{\partial P}{\partial z_0} = \frac{\partial P}{\partial z_1} = \frac{\partial P}{\partial z_2} = 0$$

Want to know that any line intersects $P=0$.

is $z_0 = z_1 = z_2 = 0$. Then the ~~solutions~~ of projective ~~surface~~ ~~variety~~ curve determined by $P=0$ in $\mathbb{CP}^2$ is a compact Riemann surface.

Euler's identity
$$\sum_{j=0}^{2} z_j \frac{\partial P}{\partial z_j} = d\,P. \qquad (*)$$

Proof of $(*)$. $P(\lambda z_0, \lambda z_1, \lambda z_2) = \lambda^d P(z_0, z_1, z_2)$

$$\frac{d}{d\lambda} P(\lambda z_0, \lambda z_1, \lambda z_2) \Big|_{\text{at } \lambda=1} = z_0 \frac{\partial P}{\partial z_0}(z_0, z_1, z_2) + z_1 \frac{\partial P}{\partial z_1}(z_0, z_1, z_2) + z_2 \frac{\partial P}{\partial z_2}(z_0, z_1, z_2)$$

$$= z_0 \frac{\partial P}{\partial z_0} + z_1 \frac{\partial P}{\partial z_1} + z_2 \frac{\partial P}{\partial z_2} \quad \text{at } \lambda = 1.$$

$$\frac{d}{d\lambda} \lambda^d P = d\lambda^{d-1} P = dP \quad \text{at } \lambda = 1$$

Proof of Prop.

Say $P = z_0 Q$, where Q is non-zero on the line $z_0 = 0$.

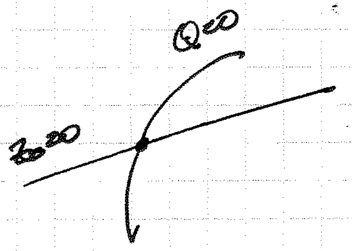
Restrict Q to the line $z_0 = 0$ $(\{(0, z_1, z_2)\})$ ← Homogeneous poly of 2 var.

Q is a non-zero polynomial so there is a $(0, z_1, z_2)$ for which Q vanishes. This is a sing. pt.

$\frac{\partial}{\partial z_0} z_0 Q = Q + z_0 \frac{\partial Q}{\partial z_0} = 0$ since $Q=0$

$\frac{\partial}{\partial z_1} z_0 Q = z_0 \frac{\partial Q}{\partial z_1}$ since $z_0=0$

$\frac{\partial}{\partial z_3} z_0 Q = z_0 \frac{\partial Q}{\partial z_2}$ since $z_0=0$.



$(0,0,0)$

Consider a point (z_0, z_1, z_2) where P vanishes.

Say $z_0=1$.

By assumption one of the partial derivatives of P does not vanish.

Claim one of $\frac{\partial P}{\partial z_2}$ or $\frac{\partial P}{\partial z_3}$ does not vanish.

If they both vanished then we get $z_0 \frac{\partial P}{\partial z_0} = 0$ so all 3 vanish.

Say $\frac{\partial P}{\partial z_2}$ ~~vanishes~~ does not vanish.

Now consider the affine polynomial

$P(z, w, 1)$. We have $\frac{\partial P}{\partial w} \neq 0$ so

our point is regular by the affine curve argument.

Remark: Compact, oriented surface is determined by its genus. Relation with degree?

Thm. Let f be a holomorphic function on an open nbd U of 0 in $\mathbb{C}$ with $f(0)=0$. Suppose $f'(0)\neq 0$ then there is a nbd $U' \subset U$ of 0 such that f is a homeomorphism onto its image $f(U') \subset \mathbb{C}$ and the inverse to $f|U'$ is holomorphic.

Proof. (This is the inverse fcn. theorem. We approach it in the same spirit as the implicit fcn. thm.)

Since 0's of non-constant functions are isolated there is a disk Δ with $0 \in \Delta \subset U$ where $f(z) \neq 0$ for $z \in \Delta - \{0\}$.

Now $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$ counts the # of solutions of $f(z)=0$ in Δ with mult.

Since $f'(0)\neq 0$ there is 1 soln of mult. 1

so $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = 1$.

$\mu(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)-w}\, dz$ counts the # of sol'ns of

(Replace $f(z)$ by $g(z) = f(z) - w$.
(Note that the $g'(z) = f'(z)$)
$g(z) = 0 \Leftrightarrow f(z) = w$

$f(z) = w$ in D.

By compactness $|f(z)| \geq \varepsilon$ on ∂W so for $|w| < \varepsilon$ $\mu(w)$ is continuous and

$\mu(w) = \mu(0) = 1$.

Let $W' = f^{-1}(\{|z| < \varepsilon\})$.

Δ_ε

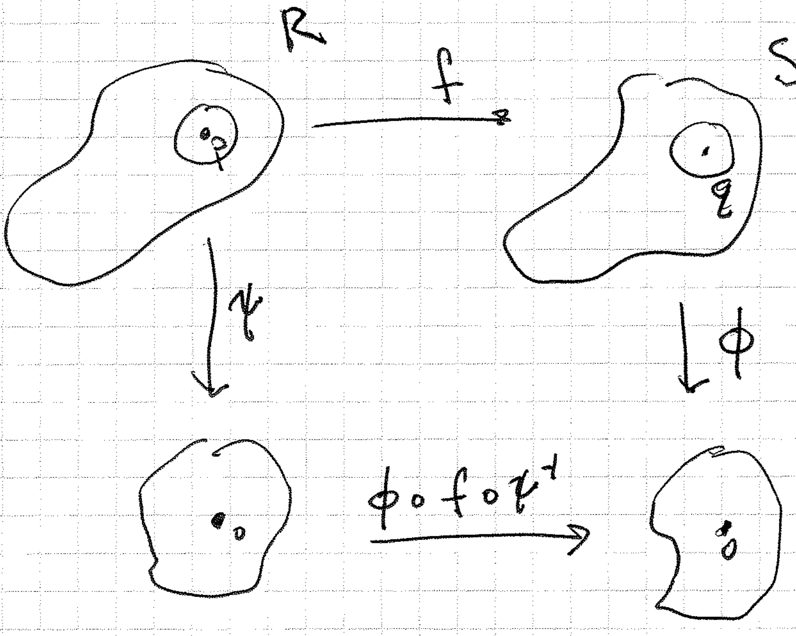
Let $\phi(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{z f'(z)}{f(z)-w}\, dz$. When there is

a unique sol'n to $f(z) = w$ it is given by

$\phi(w)$. (Replace $f(z)$ by $g(z) = f(z) - w$

$\phi(w) : \Delta_\varepsilon \to W'$. As before ϕ is holomorphic.

Remark. This result can be restated for Riemann surfaces.



Can say $f'(p) \neq 0$ even if we can't identify $f'(p)$.

~~Order of f at p~~

~~Any ψ φ f is regular at~~

Order of f at p is the order of the zero of $\varphi \circ f \circ \psi^{-1}$ at 0.

If f has order 1 then f is a local homeomorphism with hol. inverse.

Order 1 means $f'(p) \neq 0$ in any coord charts.