Corollary. A meromorphic function on a compact Riemann surface R has the same number of zeros as poles (counted with multiplicity.)

[applying deg formula to $F: R \to S^2$. This # is the degree of F. mult. of a zero p is $\nu_F(p)$, if $F(p)=0$. mult of a pole is $\nu_F(p)$, if $F(p)=\infty$.]

~~This number is just the degree of the map.~~ ~~Let's~~ work out an example.

Prop. If d is the number of poles ~~on a~~ ~~compact Riem~~ of a meromorphic function on a compact Riemann surface R then

① $d > 0$

② If $d = 1$ then R is the 2-sphere.

$F: R \to S^2$

① $d=0$ implies no poles so F is a hol. fcn. on a compact Riem. surface.

② $d=1$ implies F is a hol. bijection so F has a hol. inverse.

Let's consider an example.

Write $S^2 \subset \mathbb{R}^3$ with NP, SP for ~~the~~ north and south poles.
~~Exercise~~ in charts.
(Have also written ~~$\mathbb{CP}^1$~~ $\mathbb{P}^1$. Could write $\hat{\mathbb{C}}$, $\mathbb{C} \cup \{\infty\}$
$\mathbb{CP}^1$ ...

Consequence of the fact that Riemann surfaces intersects many fields.

15.6

Degree formula says that if f is hol. and non-const and proper

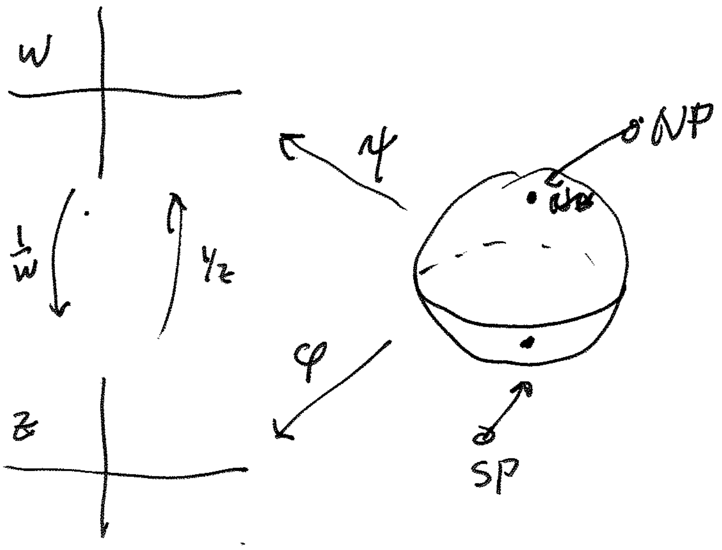
$$\sum_{f(p)=q} \nu_f(p) = d$$ is independent of q.

~~Take $q=0$~~

If we take $q=0$ we get the number of zeros counted with multiplicity.

If we take $q=\infty$ we get the # of poles counted with multiplicity

Recall that we have charts ψ, φ for S^2



Following typical practice we associate each of these charts with a "local variable"

In this language

$$\varphi \circ \psi^{-1}(w) = \frac{1}{z}$$

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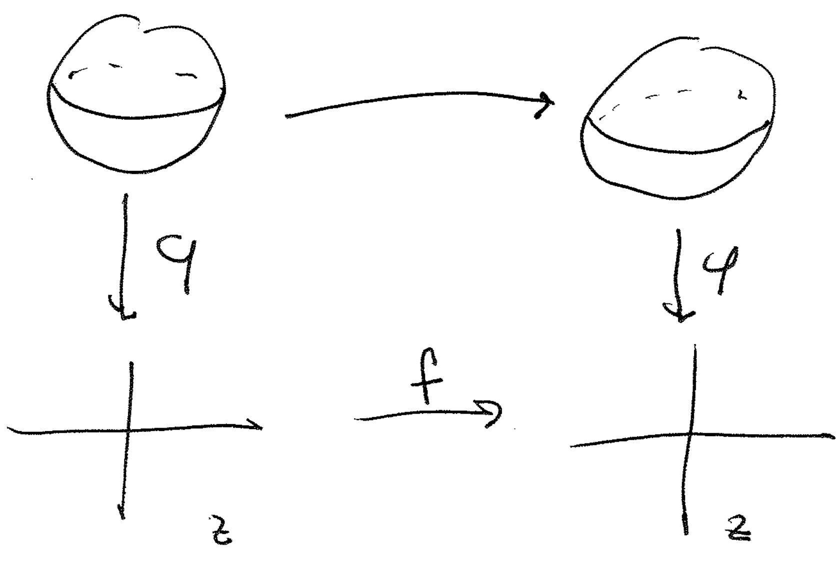
Let $f(z) = \dfrac{P(z)}{Q(z)}$ where P, Q are polynomials of degree m and n with no common factors.

Let's assume ~~Q P, Q the~~ Q is not the zero polynomial and P, Q are not both constant.



Now $\varphi^{-1}\circ f\circ\varphi$ is a partially defined map from S^2 to S^2. We define $F: S^2\to S^2$ to be the continuous extension of $\varphi^{-1}\circ f\circ\varphi$.

In particular $F(\varphi^{-1}(z)) = \varphi^{-1}(f(z))$ if $\varphi^{-1}(z)\neq NP$ (∞).

~~$F(NP) = F(\infty) = \infty$ if Q~~

$$F(\varphi^{-1}(z)) = \infty \text{ if } Q(z)=0 \quad (\text{Note in this case } P(z)\neq 0.)$$

$$F(\infty) = \lim_{z\to\infty}\varphi^{-1}(f(z)) = \infty \text{ if } \overset{m}{\deg P} > \overset{n}{\deg Q}$$

$$= 0 \text{ if } \deg P < \deg Q$$

$$= \frac{a_d}{b_d} \text{ if } P(z) = a_d z^d + a_{d-1}z^{d-1}+\ldots$$

$$Q(z) = b_d z^d + \ldots$$

~~Now~~ Want to determine the degree of F.

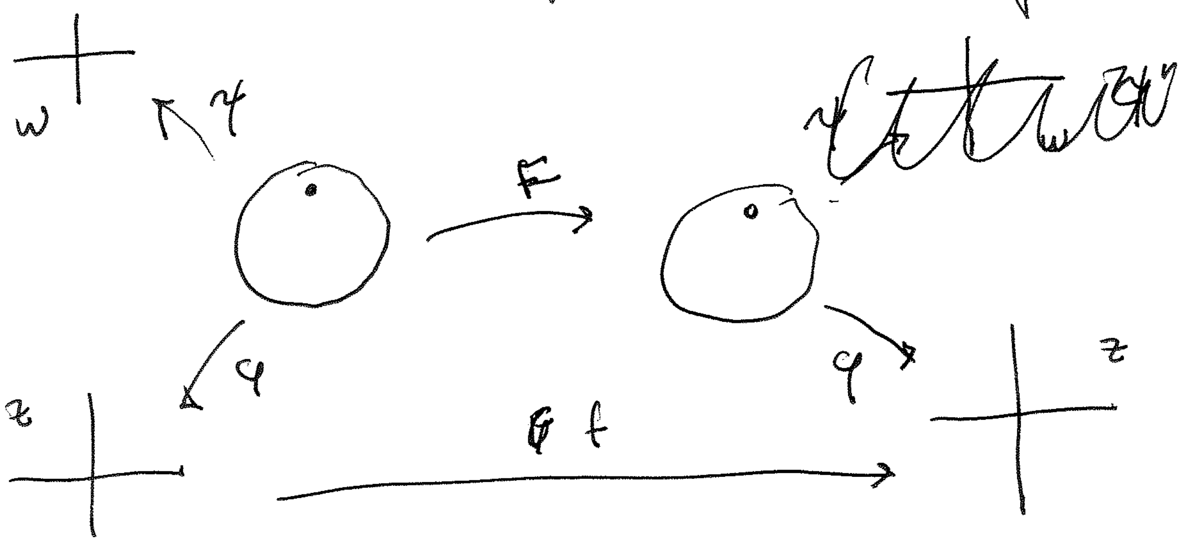
$\#$ of inverse images of 0 counted with multiplicity is the number of zeros of P which is m.

The number of inverse images of ∞ counted with multiplicity is the number of ~~inverse~~ images of zeros of Q counted with multiplicity $= n$.

Why is it $n=m$?

The missing multiplicity should come from ∞ ~~(P and Q are~~ (We are only counting multiplicities of finite points.)

Let's check this. If there are more poles than zeros say, then $n > m$. In this case ∞ maps to 0 so ∞ is a zero. Let's check that ∞ has the right multiplicity $n-m$



Want to look at $\varphi \circ F \circ \psi^{-1}(w)$. ~~Say~~

~~$\psi \circ F \circ \psi^{-1} \circ \psi \circ \varphi^{-1}\, w = \varphi^{-1}$~~ If $w \neq 0$ then

$w = \psi \circ \varphi^{-1}(z)$ for some z

so $\varphi \circ F \circ \psi^{-1}(w) = \varphi \circ F \circ \psi^{-1} \circ \psi \circ \varphi^{-1}(z)$

$= \varphi \circ F \circ \varphi^{-1}(z)$

$= f(z)$

$= \dfrac{P(z)}{Q(z)}$

$$\varphi \circ F \circ \psi^{-1}(w)$$

$$= \underbrace{\varphi \circ F \circ \varphi^{-1}}_{f} \circ \underbrace{\varphi \circ \psi^{-1}(w)}_{1/z}$$

$$= f(1/z)$$

$$= \frac{P(1/z)}{Q(1/z)} = \frac{a_m(1/z)^m + \dots a_0}{b_n(1/z)^n + \dots b_0}$$

$$= \frac{z^{-m}(a_m + a_{m-1}z + \dots a_0 z^m)}{z^{-n}(b_n + b_{n-1}z + \dots b_0 z^n)}$$

$$= z^{n-m} \cdot \underbrace{\text{hol non-zero}}_{h(z)} \quad \text{h hol non-zero.}$$

(F is hol. at ∞)

So $\nu_F(\infty) = n-m$ and ∞ is a zero

with ramification index $n-m$.

Sum of ramification indices is total # of zeros ← mult of 0 at ∞

is ~~m = n-m~~ $\underset{\uparrow \text{finite zeros}}{m} + n - m = n$.

If $F(\infty) = \infty$ we consider

If $F(\infty)=\infty$ and $m>n$ then we consider F with respect to the charts ψ in the domain and ψ in the range

$$\psi\circ F\circ \psi^{-1}(w)$$

$$= \underbrace{\psi\circ\varphi^{-1}}_{1/w}\circ f\circ \underbrace{\varphi\circ\psi^{-1}}_{1/z}(w)$$

z

$$= \psi\circ\varphi^{-1}\, f(1/z)$$

$$= \frac{1}{f(1/z)} = \frac{Q(1/z)}{P(1/z)} = z^{m-n}\cdot h(z) \qquad h \text{ hol.}\ \ h(0)\neq 0.$$

So $\nu_F(\infty)=m-n$ and $d = \underset{\uparrow\ \#\text{ of finite poles}}{n} + \underset{\nwarrow\ \text{mult. of pole at }\infty}{\underbrace{m-n}} = m.$

Conclusion. Degree F = $\max\{\deg P, \deg Q\}$.

Consequence: We can construct holomorphic functions from $S^2 \to S^2$ with any ~~finite~~ collection of finite 0's and poles.

If the number of zeros is not equal to the number of poles (with multiplicity) then there is a zero or pole at ∞ with the appropriate multiplicity.

Proposition. Any holomorphic map F from $S^2 \to S^2$ is given by a rational function f.

Proof. Let F be a holomorphic map.
Let G be a rational function with the same (finite) zeros and poles as F counted with multiplicity. It follows from our previous analysis that F and G have the same value at ∞ (0, finite, ∞) and $\nu_F(\infty) = \nu_G(\infty)$.

Choose a chart in which both functions map 0 to 0.

Now the quotient $\frac{F}{G}$ is finite valued at each point since at each z has the form

$$\frac{a_j z^j + a_{j+1} z^{j+1} + \cdots}{b_j z^j + b_{j+1} z^{j+1} + \cdots} = \frac{a_j}{b_j} \cdot \frac{(1 + \cdots)}{(1 + \cdots)} \quad \text{where } j = \nu_F(p) = \nu_G(p).$$

Thus F/G is a hol. fcn. on a compact Riemann surface so constant.

Remark: Compactness + analyticity $\Rightarrow$ algebraic.
This is a very special case of a very general principal.

Prop. A holomorphic automorphism of ~~$S^2(\mathbb{C}P^1)$~~ S^2 is given by a ~~linear~~ Möbius transformation; ($z \mapsto \frac{az+b}{cz+d}$ with $ad-bc \neq 0$).

Proof. A holomorphic map is given by $z \mapsto \frac{P(z)}{Q(z)}$ with no common factors.

If it is an automorphism then $d=1$ so $\deg P(z) = \deg Q(z) \leq 1$. $z \mapsto \frac{az+b}{cz+d}$ where ~~$ad-bc\neq 0$~~.

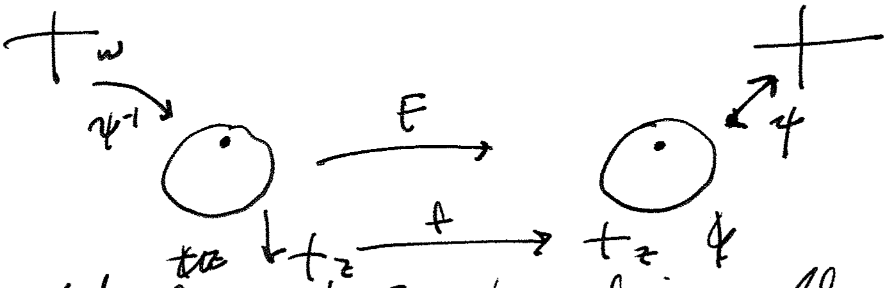
$az+b$ is not a ~~real~~ constant times $cz+d$.

See 21.5.

While we are discussing automorphisms let's consider $\mathbb{C}$.

Prop. A holomorphic automorphism of $\mathbb{C}$ has the form $z \mapsto az+b$. (In fact a non-constant holomorphic injection has this form.)

Proof. Any $f: \mathbb{C} \to \mathbb{C}$ is injective. View F as a hol map from $S^2 - \infty \to S^2 - \infty$



Let ~~$f(w) = \psi \circ F \circ \psi^{-1}$~~ ~~f is well defined~~

$$g(w) = \frac{1}{f(1/w)}.$$ g is well defined away

from $f^{-1}(0) \in \mathbb{C}$. So g is well defined in a punctured disk. $\{|w| < \varepsilon\}$.

~~If~~ g is either meromorphic at 0 or it has an essential singularity at 0.

$\left(g(z) = \sum_{n=-\infty}^{\infty} a_n z^n \right)$ If g is meromorphic then

f has a meromorphic extension to $\overset{F}{S^2} \to S^2$

so $f(z) = \frac{az+b}{cz+d}$. Since $F(\infty) = \infty$, $c = 0$.

Since f is not constant $d \neq 0$.

Say that g has an essential singularity at 0.

(C3)

Proof. Say $f : \mathbb{C} \to \mathbb{C}$ is injective.

We can think of f as a (partially defined) map on ~~the~~ S^2 and look at it with respect to the coordinate around the northpole. ~~We~~ We get f.

$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ ~~or~~ or $\sum_{n=m_0}^{\infty} a_n z^n$

$f(w) = f(1/z)$ defined in a ~~nbd. of~~ deleted nbd of w. If $f(w)$ has a zero or pole at $w=0$ then f has a ~~continuous extension~~ to holomorphic extension to S^2 so f is given by a Moebius transformation taking ∞ to ∞ $z \mapsto \frac{az+b}{cz+d}$ and $c=0$. Say that 0 is an essential singularity then by the Casorati-Weierstrass theorem for any $w \in \mathbb{C}$ there is a sequence $z_j \to 0$ with $f(z_j) \to w$.

On the other hand since f is not constant there is some open set U in its image. Take well and we violate the injectivity of f.

Definition. We say that a Riemann surface R has finite type if it is conformally equivalent to a $M - \{p_1 \ldots p_k\}$ where M is a compact Riemann surface.

Example: Affine curve, regular at ∞. ~~Ethemae~~ M = proj. curve $\{p_1 \ldots p_k\}$ = pts. at ∞.

Example: $\mathbb{C}$

Non-example: Δ = unit disk.

If $\Delta \cong M - \Sigma$ then $M = S^2$ so Δ^* is conformally equivalent to S^2-pt $\cong \mathbb{C}$.

Conformal structure of $M - \{p_1 \ldots p_k\}$ captures M

Prop: If $M - \{p_1 \ldots p_k\}$ is conformally equivalent to $N - \{q_1 \ldots q_j\}$ then $M \cong N$ by means of an equivalence $F: M \to N$ that takes $F\{p_1 \ldots p_k\}$ ~~to~~ = $\{q_1 \ldots q_j\}$ bijectively. (Of course the converse is also true.) so $k = j$.

Proof. Let F: ~~$N-\{q_1\ldots\}$~~ $M - \{p_1 \ldots p_k\} \to N - \{q_1 \ldots q_j\}$ be a conformal equivalence. Choose ~~charts~~ ~~arou~~ open sets $V_1 \ldots V_j$ equiv. to disks around $q_1 \ldots q_j$. Choose ~~charts~~ nbhds $U_1 \ldots U_k$ around $p_1 \ldots p_k$ ~~equiv~~ equiv. to disks. C = Complement of $V_1 \ldots V_j$ in N is compact so $F^{-1}(C)$ is compact. There is an

ε so that an ε nbd around ~~q~~ pt each of $p_1 \cdots p_k$ is disjoint from $F^{-1}(C)$.

Thus $\Delta_k^\varepsilon - \{0\}$ maps into $\bigcup V_j$. So $\Delta_k^\varepsilon - \{0\}$ maps into some specific V_{j_k}. Arguing as before $F|\Delta_k^\varepsilon - \{0\}$ ~~is holomorphic~~ has a holomorphic extension to Δ_k^ε. So F has a holomorphic extension to $\bar{F}: M \to N$. ~~Since~~ Since F is 1-1, $\bar{F}$ is 1-1 ~~on~~ from $M - \{p_j\} \to N - \{q_k\}$ so $\bar{F}$ has degree 1. In particular $\bar{F}$ is a bijection with a holomorphic inverse. Conclude $\bar{F}$ takes $\{p_1 \cdots p_j\}$ to $\{q_1 \cdots q_k\}$ bijectively.

Define local coordinates z_j at p_j where $p_j = \{z_j = 0\}$.