Let $\Lambda = \{m\omega_1 + n\omega_2 : m,n \in \mathbb{Z}\}$ where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

Prop. $\mathbb{C}/\Lambda$ is homeomorphic to a torus $S^1 \times S^1$.

Proof. Let $L : \mathbb{R}^2 \to \mathbb{C}$ be defined by ~~$L(\xi)$~~

$L\begin{pmatrix} r \\ s \end{pmatrix} = r\omega_1 + s\omega_2$. L is $\mathbb{R}$ linear and invertible, since (ω_1, ω_2) is an $\mathbb{R}$ basis for $\mathbb{C}$.

$L(\mathbb{Z}^2) = \Lambda$ so L induces a homeomorphism from $\mathbb{R}^2/\mathbb{Z}^2$ to $\mathbb{C}/\Lambda$.

$$\mathbb{R}^2/\mathbb{Z}^2 \cong \frac{\mathbb{R}}{\mathbb{Z}} \times \frac{\mathbb{R}}{\mathbb{Z}}$$

We want to construct meromorphic functions f on $\mathbb{C}/\Lambda$. Note that if $f: \mathbb{C}/\Lambda \to \mathbb{C} \cup \infty$ is meromorphic then $\tilde{f} = f \circ \pi : \mathbb{C} \to \mathbb{C} \cup \infty$ is meromorphic and Λ invariant. Conversely if $\tilde{f}: \mathbb{C} \to \mathbb{C} \cup \infty$ is meromorphic and Λ invariant then there is a meromorphic function

$f: \mathbb{C}/\Lambda \to \mathbb{C} \cup \infty$ with $f(q) = \tilde{f}(p)$ when $\pi(p) = q$.

We call such a function $\tilde{f}$ doubly periodic.

Let us attempt to build a Λ-periodic meromorphic function on $\mathbb{C}$. We know that this function cannot have a single simple pole. Let us attempt to build a Λ-periodic fcn. with a pole of order $\ell \geq 2$ at the points of the lattice Λ.

Consider $\ell \geq 2$.

Write $\mathcal{E}_\ell(z) = \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^\ell}$. Formally $\mathcal{E}_\ell$ looks like a good candidate. We need to worry about convergence.

Does this expression converge?

Thm. (Weierstrass M-test) Let $W \subset \mathbb{C}$, let $f_n : W \to \mathbb{C}$ be a sequence of holomorphic functions on W. Suppose there is a sequence of positive real numbers M_n such that $|f_n(z)| \leq M_n$ and $\sum M_n < \infty$ then $\sum_n f_n(z)$ converges uniformly to a holomorphic function f and $f'(z) = \sum_n f_n'(z)$

Now return to

$$\mathcal{E}_\ell(z) \stackrel{?}{=} \sum_{\omega} (z-\omega)^{-\ell}.$$

This function will be meromorphic rather than holomorphic but we have a wa… to deal with t…

Let's fix a disk $|z| < R$ and establish convergence in **the disk** $D(0,R)$. Now f will have some poles in this disk.

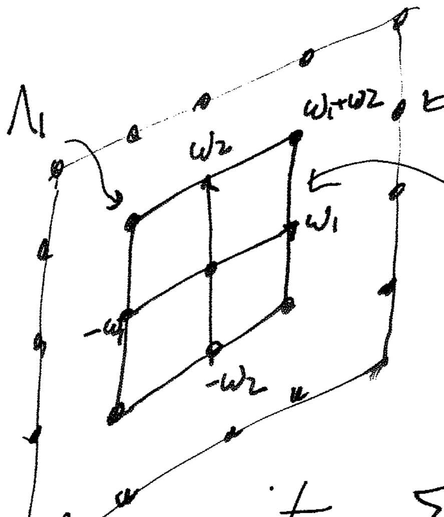
We deal with these separately. Let $\Lambda' = \{\omega \in \Lambda : |\omega| < 2R\}$. So

$$\sum_{\omega} (z-\omega)^{\ell} = \sum_{\omega \in \Lambda'} (z-\omega)^{\ell} + \sum_{\omega \in \Lambda - \Lambda'} (z-\omega)^{\ell}$$

The first sum is finite so convergence is no problem.

To make our estimates come out in a manageable form it is useful to choose Λ' to be larger. Let $P = \{r\omega_1 + s\omega_2 : |r| \le 1, |s| \le 1\}$.

Let $\Lambda_k = \{m\omega_1 + n\omega_2 : \max\{m, n\} = k\}$.



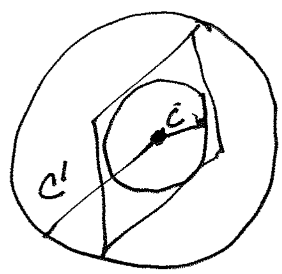
Note $\dfrac{\Lambda_k}{k} = \left\{\dfrac{m}{k}\omega_1 + \dfrac{n}{k}\omega_2 : \max\left\{\dfrac{m}{k}, \dfrac{n}{k}\right\} = 1\right\} \subset \partial P$.

We can write $\displaystyle\sum_{\omega \in \Lambda} (z-\omega)^{-\ell} = \sum_{k \ge 0} \sum_{\omega \in \Lambda_k} \frac{1}{(z-\omega)^\ell}$.

Let P = convex hull of $\Lambda_1 = \{r\omega_1 + s\omega_2 : \max\{r, s\} \le 1\}$.

Let c be the radius of the largest disk $D(0,c) \subset P$.

Let c' be the radius of the smallest ball containing P.



$$\omega \in \Lambda_k \Rightarrow \frac{\omega}{k} \in \partial P \Rightarrow c \le \frac{|\omega|}{k} \le c'$$

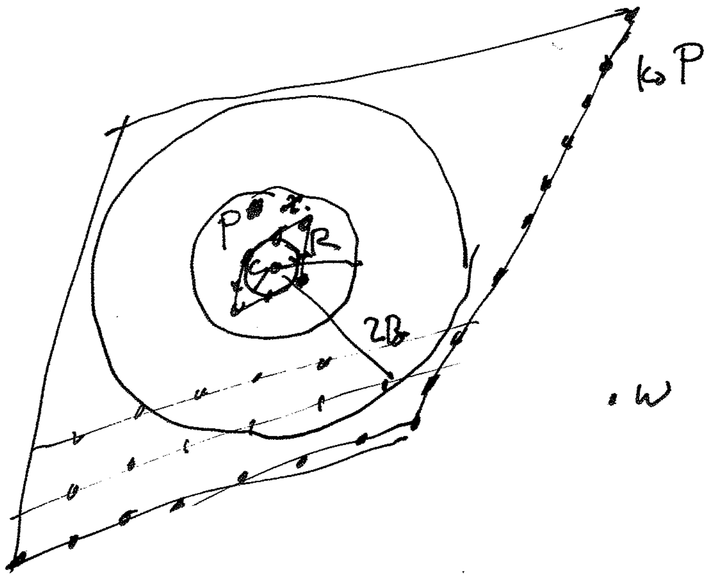
$$\Rightarrow c \cdot k \le |\omega| \le c' \cdot k.$$

Choose $k_0 \in \mathbb{N}$ so that $k_0 \cdot P \supset D(0, 2R)$.

It suffices that $k_0 \geq \frac{2R}{c_1}$

Let $\Lambda' = \bigcup_{k \leq k_0} \Lambda_k$.

Note that Λ' is the set of lattice points in $k_0 P$.



With these choices we have

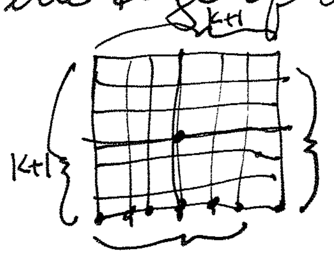
$|z| \leq R$
$|\omega| \geq 2R$

$\omega \in \Lambda - \Lambda'$, $\quad z \in D(0,R) \Rightarrow |z| \leq \frac{|\omega|}{2}$.

We want to apply the Weierstrass M-test to

$$\sum_{k \geq k_0} \sum_{\omega \in \Lambda_k} \frac{1}{(z-\omega)^\ell}.$$

Consider the size of the terms and the number of terms, ($\Lambda = \mathbb{Z}^2$ picture)



$\# \Lambda_k$

$k = 3 \quad \# \Lambda_3 = \cancel{8k}\ 4 \cdot 2k = 8k$

For fixed $z \notin \Lambda$ show $\mathcal{E}_\ell(z) = \mathcal{E}_\ell(z+\omega_0)$. ⑥

Given $\varepsilon > 0$ there is an L so that

$$\left| \mathcal{E}_\ell(z) - \sum_{\omega \in D(0,L) \cap \Lambda} (z-\omega)^{-\ell} \right| \le \varepsilon . \qquad \left| \mathcal{E}_\ell(z+\omega_0) - \underbrace{\sum_{\omega \in D(0,L) \cap \Lambda} (z-(\omega+\omega_0))^{-\ell}}_{\sum_{\omega \in (D(0,L)+\omega_0) \cap \Lambda} (z-\omega)^{-\ell}} \right| \le$$

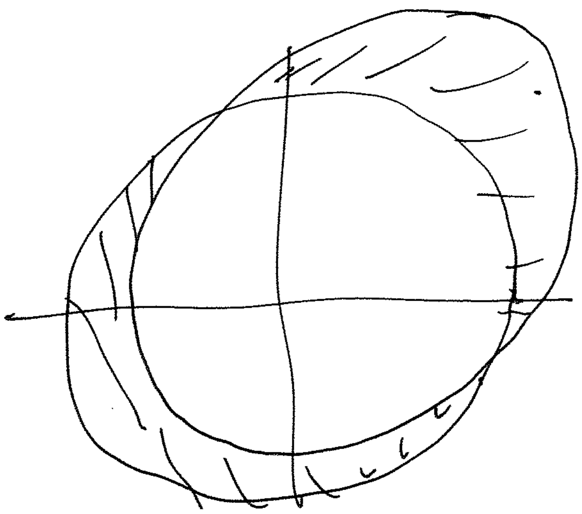
$$\left| \mathcal{E}_\ell(z) - \mathcal{E}_\ell(z+\omega_0) \right|$$

$$\le \underbrace{\sum_{\omega \in (D(0,L) - (D(0,L)+\omega_0)) \cup ((D(0,L)+\omega_0) - D(0,L))} (z-\omega)^{-\ell}}_{= D(0,L) \,\Delta\, (D(0,L)+\omega_0)} + 2\varepsilon$$

Let $L \to \infty$ $\qquad \to 0$

$$\left| \mathcal{E}_\ell(z) - \mathcal{E}_\ell(z+\omega_0) \right| \le 2\varepsilon .$$

Let $\varepsilon \to 0$.



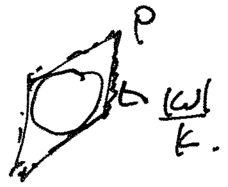
as $L \to \infty$ the complement of the symmetric difference contains larger and larger disks around the origin. Tails $\to 0$.

Size of terms:

If $|z|<R$, $|\omega|\ge 2R$ then $|z|\le \frac{|\omega|}{2}$. 8

$|\omega|\le|z-\omega|+|z|$ so $\underline{|z-\omega|}\ge|\omega|-|z|\ge|\omega|-\frac{|\omega|}{2}=\underline{\frac{|\omega|}{2}}$

If $|\omega|\in\Lambda_k$ then



$\underline{|\omega|\ge kC}$ so $\underline{|z-\omega|\ge\frac{kC}{2}}$

$$\frac{1}{|z-\omega|^{\ell}}\le\left(\frac{2}{C}\right)^{\ell}\cdot k^{-\ell}$$

$$\left|\sum_{\omega\in\Lambda_k}\frac{1}{(z-\omega)^{\ell}}\right|$$

$$\le\left(\frac{2}{C}\right)^{\ell}\frac{1}{k^{\ell}}\cdot 8k=\text{const.}\frac{1}{k^{\ell-1}}.$$

If $\ell-1\ge 2$ then $\frac{1}{k^{\ell-1}}$ is summable.

So $\mathcal{E}_\ell$ exists as long as $\ell\ge 3$.

"$\mathcal{E}_2$" is very problematic.

Prop. For $\ell\ge 3$ $\mathcal{E}_\ell$ is meromorphic with poles of order ℓ precisely at lattice points. $\mathcal{E}_\ell$ is Λ-periodic.

Now we consider the case $l=2$.

$$\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^2}.$$

Is the situation as bad as it seems?
What do we get when $z=0$?

$$\sum_{\omega\in\Lambda-\{0\}}\frac{1}{\omega^2}.$$

This is already problematic.

Presumably if it does converge it is some form of conditional convergence since the sum of the absolute values of the terms diverge.

$$\sum_{\omega\in\Lambda_k}\left|\frac{1}{\omega^2}\right| \sim \sum_k \frac{\text{const.}}{k} \to \infty.$$

If we could make sense of convergence we would still ~~for~~ have trouble with ~~convergence~~ invariance.

We can try to deal with this by subtracting the offending terms. $P(z)=\sum_{\omega\in\Lambda}\frac{1}{(z-\omega^2)}-\sum_{\omega\in\Lambda'}\frac{1}{\omega^2}$

(Grouping of terms is critical).

Let $P(z)=\frac{1}{z^2}+\sum_{\omega\in\Lambda-\{0\}}\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}.$

No particular reason to expect invariance but we might get convergence.

We can proceed as before. Choose R and consider convergence on $D(0,R)$.

Write our function as a meromorphic part with finitely many terms and a holomorphic part with ∞ many terms. May assume $|z| \le R \le \frac{|\omega|}{2}$, as before.

Evaluate the size of a typical term:

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{\omega^2 - (z-\omega)^2}{(z-\omega)^2 \cdot \omega^2} \right| = \left| \frac{-z^2 + 2z\omega}{(z-\omega)^2 \omega^2} \right|$$

Using $|z| \le R$. Using $|z| \le \frac{|\omega|}{2}$

$$\le \frac{|z| \cdot |z+2\omega|}{|z-\omega|^2 \cdot \omega^2} \le \frac{R \cdot \frac{5}{2}|\omega|}{\left|\frac{3}{2}\omega\right|^2 \cdot |\omega|^2} \le \text{const} \cdot \frac{1}{|\omega|^3}.$$

Using $|z| \le \frac{|\omega|}{2}$

eg $|z+2\omega| \le |z| + 2|\omega| \le \frac{|\omega|}{2} + 2|\omega| = \frac{5}{2}|\omega|$.

Using our previous analysis the series converges and it has poles of order 2 exactly at the lattice points.

Invariance is far from clear. Not just a question of reordering terms.

Proposition

$\wp(z) = \wp(-z)$.

Proof. $\wp(z) = \sum_k \sum_{\omega \in \Lambda_k} (z-\omega)^{-2}$.

$$\wp(-z) = \sum_k \sum_{\omega \in \Lambda_k} (-z-\omega)^{-2} = \sum_k \sum_{-\omega \in \Lambda_k} (-z+\omega)^{-2}$$

$$= \sum_k \sum_{-\omega \in \Lambda_k} (z-\omega)^{-2}.$$

Proposition, $\wp'(z) = \wp'(z+\omega)$.

Proof. The Weierstrass M test lets us differentiate term by term. So

$$\wp'(z) = \sum_{\omega \in \Lambda_k} -2(z-\omega)^{-3}.$$ This is convergent by the M test. We can rearrange terms by absolute convergence.

Prop. $\wp(z) = \wp(z+\omega)$.

Proof. $\wp(z) - \wp(z+\omega)$ is constant. Set

$\wp(z+\omega) - \wp(z) = c(\omega)$. We use the fact

that $z \mapsto -z+\omega$ acts with fixed points

on $\mathbb{C}$. The fixed point

is the z such that $z = -z+\omega$ or $2z = \omega$ or $z = \frac{\omega}{2}$.

If $\omega \in \Lambda - 2\Lambda$ then $\frac{\omega}{2}$ is an nol in Λ so

$\wp(\frac{\omega}{2})$ is finite.

Now $\wp(\frac{\omega}{2}) = \wp(-\frac{\omega}{2}) + C(\omega)$ since $\frac{\omega}{2} - (-\frac{\omega}{2}) = \omega$.

But $\wp(\frac{\omega}{2}) = \wp(\frac{-\omega}{2})$.

So $C(\omega) = 0$.

For $z \notin \Lambda$

we also have: $\wp(z+2\omega) = \wp(z+\omega+\omega) = \wp(z+\omega) + C(\omega)$

$= \wp(z) + 2C(\omega)$. Any $\omega \in \Lambda$ can be written

as $2^n \cdot \lambda$ with $\lambda \in \Lambda - 2\Lambda$. So $\wp(z+\omega)$

$= \wp(z + 2^n \cdot \lambda) =$

$= \wp(z) + C(2^n \cdot \lambda)$

$= \wp(z) + 2^n \cdot C(\lambda)$

$= \wp(z)$. by previous argument.

Thus $\wp$ is invariant.

Def. $\wp$ is called the Weierstrass $\wp$. function.

We have constructed $\wp$ analytically using convergent series. Now we want to analyse this function geometrically. Recall the group $\Gamma = \{z \mapsto \pm z + w : w \in \Lambda\}$ acts on $\mathbb{C}$. Even though the action is not free the surface $\mathbb{C}/\Gamma$ still has a conformal atlas hence a Riemann surface structure.

Let $\pi : \mathbb{C}/\Lambda \to \mathbb{C}/\Gamma$ be defined by $z + \Lambda \mapsto \pm z + \Lambda$.

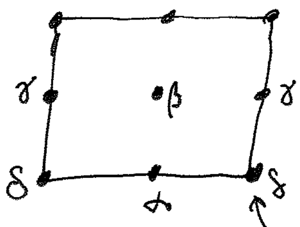
π takes Λ orbits to Γ orbits. π is holomorphic so according to our theorem on proper ~~mappin~~ holomorphic mappings π is a branched cover. In fact in an appropriate sense this branched covering map is normal.

Let $\iota : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ send $z + \Lambda$ to $-z - \Lambda = -z + \Lambda$.

Claim. ι permutes the inverse images of π.
Fixed points of ι correspond to points where π has mult. 2.

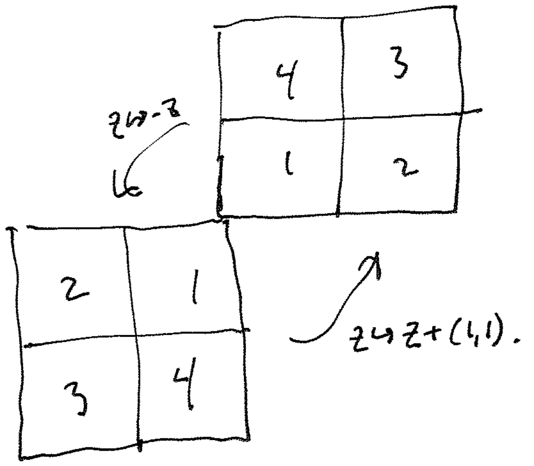
Example: $\Lambda = \{m + ni\}$.

Points in $\Lambda/2$ are labelled $\alpha, \beta, \gamma, \delta$.



pts. in $\Lambda/2$ are marked.

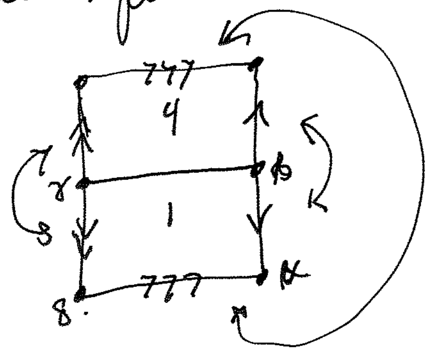
fund domain for Λ.

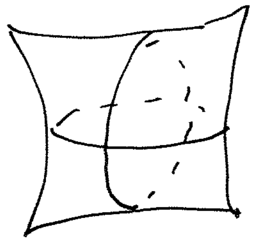


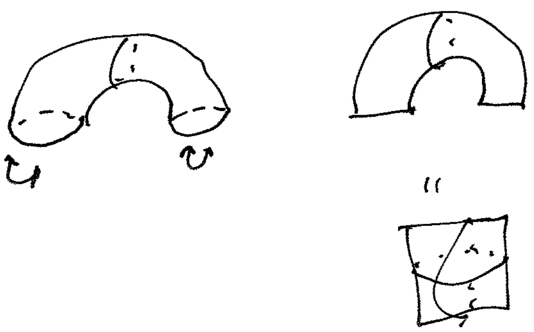
$L(z) = -z + (1,1)$

Note that L interchanges squares 1 and 3 and squares 2 and 4,

Get a fundamental domain for Γ by choosing small squares 1 and 4.

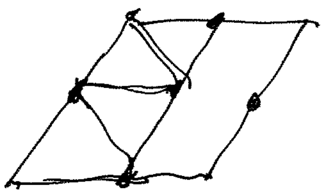


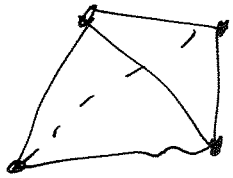




$\sqrt[3]{-1}$

Example 2 $\quad \omega_1 = 1 \quad \omega_2 = \dfrac{1+\sqrt{3}i}{2}$





Lemma. $\wp'(L(z)) = -\wp'(z)$. and $\wp'$ has a pole of order 3 at pts. of Λ and no other poles.

Derivative is not defined abstractly in Riemann surface. Depends on a chart. Use the chart coming from $\mathbb{C}/\Lambda$ structure. ①

Remark. We are thinking about $\wp$ as a function on $\mathbb{C}$ (not $\mathbb{C}/\Lambda$). We are using this chart and this z variable coming from $\mathbb{C}$ to differentiate.

Better to write $\frac{d\wp}{dz}(L(z)) = -\frac{d\wp}{dz}(z)$ to make this dependence explicit.

Recall $\wp(z) = \wp(-z)$. This gives

Proof. $\wp$ is an even function. $\wp = \sum a_{2n} z^{2n}$.

Can differentiate term by term

so $\wp' = \sum 2n \cdot a_{2n} z^{2n-1}$ is odd and $\wp'(-z) = \wp'(z)$

Corollary. $\wp'$ vanishes only at points of Σ. (in $\mathbb{C}/\Lambda$) $\Sigma = \{\alpha, \beta, \gamma\}$ S is a pole.

$\wp'$ has what degree? 3. $\wp'$ has 3 zeros at α, β, γ.

Proof. Σ is the set of z with points with $L(z) = z$. If $\wp'(z) = \wp'(L(z))$ and $z = L(z)$ then $\wp'(z) = -\wp'(L(z))$ so $2\wp'(z) = 0$. Conversely if $\wp'(z) = 0$ then $\wp(z)$ has local mult. 2 so $\{z, L(z)\}$ has 1 element.

(Proof not complete? What happens at ∞? Chart at ∞?)

Prop. $(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$
for constants g_2 and g_3.

Proof. $\wp(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$

vanishes at 0 (since we have subtracted off the pole) and is an even function (since $\wp$ is even and $\frac{1}{z^2}$ is even).

So $\wp(z) = z^{-2} + \qquad \lambda z^2 + \mu z^4 + z^6 \cdot h_0(z)$

missing term

$$\wp'(z) = -2z^{-3} + 2\lambda z + 4\mu z^3 + \underbrace{6z^5 h_0(z) + z^6 h_0'(z)}_{z^5 \cdot h_1(z)}$$

$2\cdot(-2)$

$$(\wp'(z))^2 = 4z^{-6} - 8\lambda z^{-2} - 16\mu + \ldots$$

$2\cdot(-2)\cdot 4$

(double the cross terms)

$$(\wp(z))^3 = z^{-6} + 3\lambda z^{-2} + 3\mu$$

$$(\wp'(z))^2 - 4(\wp(z))^3 = -20\lambda z^{-2} - 28\mu$$

$(a+b+c+\ldots)^2$
$= a^2+b^2+\ldots$
$+2ab+2ac+\ldots$

$(a+b+c+\ldots)^3$
$= a^3+b^3+\ldots$
$3ab^2+\ldots$
$6abc+\ldots$

We cancel the z^{-2} term by adding $20\lambda\wp(z)$

We adjust the constant by adding 28μ.

Thus $(\wp'(z))^2 - 4(\wp(z))^3 + 20\lambda\wp(z) + 28\mu$

has no pole at 0 and has the value 0 at 0. Since the only poles of this function occur at lattice points this meromorphic function has no poles. We can view it as a holomorphic function from

$$\mathbb{C}/\Lambda \to \mathbb{C} \to \mathbb{C}\cup\infty.$$

Since $\mathbb{C}/\Lambda$ is compact this function is constant. Since its value at 0 is 0 the function is 0 everywhere. So:

$$(\wp')^2 = 4\wp^3 - 20\lambda\wp + 28\mu.$$

Lemma. $g_2 = 60 \sum_{\omega \in \Lambda - \{0\}} \omega^{-4}$, $g_3 = 140 \sum_{\omega \in \Lambda - \{0\}} \omega^{-6}$.

Proof. See 2017 Notes 17 page 5 for this calculation

This page appears here:

Lemma. $g_2 = 60 \sum_{\omega \in \Lambda} \omega^{-4}$, $g_3 = 140 \sum_{\omega \in \Lambda} \omega^{-6}$ Eisenstein series.

$$\wp(z) - z^{2} = \lambda z^2 + \mu z^4$$

where $g_2 = 20\lambda$, $g_3 = 28\mu$.

$$= \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

Evaluated at 0.

$$(\wp(z) - z^{2})' = -2\sum_{\omega \in \Lambda - \{0\}} (z-\omega)^{-3} \qquad = 2\lambda z + 4\mu z^3$$

$$(\wp(z) - z^{2})'' = 6 \sum_{\Lambda - \{0\}} (z-\omega)^{-4} \qquad = 2\lambda + 12\mu z^2$$

$$(\wp(z) - z^{2})''' = -24 \sum (z-\omega)^{-5} \qquad = 24\mu z$$

$$(\wp(z) - z^{2})'''' = 120 \sum (z-\omega)^{-6} \qquad = 24\mu$$

$$120 \sum_{\omega \neq 0} \omega^{-6} = 24\mu$$

$$6 \sum_{\omega \neq 0} \omega^{-4} = 2\lambda$$

Set $g_2 = 20\lambda$

$g_3 = 28\mu$.

$g_2 = 20\lambda = 60 \sum \omega^{-4}$

$g_3 = 28\mu =$

$28 \cdot \frac{120}{24} \sum \omega^{-6}$

$= 28 \cdot 5 \cdot \sum \omega^{-6}$

Let C_Λ be the projective curve defined by in $\mathbb{CP}^2$
the polynomial

$$Q_\Lambda = Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3.$$

with $g_2 = g_2(\Lambda)$, $g_3 = g_3(\Lambda)$ as before.

Note that this is the homogenization of the curve as defined before.

Prop. C_Λ is a non-singular curve.

Proof. Let $a, b, c = \wp(\alpha), \wp(\beta), \wp(\gamma)$.
Recall $\wp(0) = \infty$ and a, b, c, ∞ are distinct
As we just saw $\wp'$ vanishes at α, β, γ.
As we have seen

$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ so
a, b, c are roots of $4X^3 - g_2X - g_3$ and they are distinct so

$$4X^3 - g_2X - g_3 = 4(X-a)(X-b)(X-c)$$

or so $Y^2 = 4(X-a)(X-b)(X-c)$.
Homogenize: $Y^2Z = 4(X-aZ)(X-bZ)(X-cZ)$.

Distinct roots $\Rightarrow$ smooth by the Gauss criterion.

Prop. Let $u: \mathbb{C}/\Lambda \to C_\Lambda$ be defined by

$$\cancel{u(z)} \quad u(z+\Lambda) = \begin{cases} [P(z) : P'(z) : 1] & \text{if } z \notin \Lambda \\ [0:1:0] & \text{if } z \in \Lambda. \end{cases}$$

Then u is a conformal isomorphism.

Proof. If $z \notin \Lambda$ then $P(z)$, $P'(z)$ are finite and Image $(u) \subset C_\Lambda$.

Show injectivity:

Say $u(z) = u(z')$ but $z \neq z'$. So $P(z) = P(z')$.

This implies that $z' = L(z)$.

Since $z \neq z' = L(z)$, $z \neq L(z)$ so $z \notin \{\alpha, \beta, \gamma, \delta\}$

In particular $P'(z) \neq 0$.

Now $P'(z') = P'(L(z)) = -P'(z) \neq P'(z)$ so $u(z) \neq u(z')$ since there second coordinates are distinct.

Our next objective is to make a connection between these two constructions and we will do this by analysing meromorphic functions on $\mathbb{C}/\Lambda$ and relations between them.

Note that a meromorphic function f on $\mathbb{C}/\Lambda$ gives a meromorphic function $f\circ\pi$ on $\mathbb{C}$ and that $f\circ\pi$ is invariant under translations by elements of Λ. Such a function is called doubly periodic.

$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi = \tilde{f}"] & \\ \mathbb{C}/\Lambda \arrow[r, "f"] & \mathbb{C}\cup\infty \end{tikzcd}$$

Let us attempt to build a doubly periodic meromorphic function on $\mathbb{C}$. We know that it has at least two poles. Let's try to construct a meromorphic function with a pole of order l at the origin for $l \geq 2$.

Write $\tilde{f}(z) = z^{-l}$. To make this invariant we can sum over $\omega \in \Lambda$

$$\tilde{f}(z) = \sum_{\omega\in\Lambda} (z-\omega)^{-l}.$$

2017 notes. 9
#17
(Stark)

Define $\iota : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ as follows

If $q = z + \Lambda$ is an element of $\mathbb{C}/\Lambda$ then $\iota(q) = -z + \Lambda = \pi'(-z)$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "z \mapsto -z"] \arrow[d, "\pi'"] & \mathbb{C} \arrow[d, "\pi'"] \\
\mathbb{C}/\Lambda \arrow[r, "\iota"] & \mathbb{C}/\Lambda
\end{tikzcd}$$

If $q \in \mathbb{C}/\Lambda$ then

Define $L: \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ by $\quad L(z+\Lambda)$

$= -z-\Lambda$

$= -z+\Lambda$

$z+\Lambda$

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \downarrow \pi' & & \downarrow \pi' \\ \mathbb{C}/\Lambda & \longrightarrow & \mathbb{C}/\Lambda \end{array}$$

L is well defined.

If p, p' map to q then

$p' = p+\omega \quad \omega \in \Lambda$

$-p' = -p-\omega \qquad -p = -p$ differ by an element of Λ so they map to the same point of $\mathbb{C}/\Lambda$.

Now let $\pi: \mathbb{C}/\Lambda \to \mathbb{C}/\Gamma$ take $\underbrace{z+\Lambda}_{\text{a } \Lambda \text{ orbit}}$ to $\underbrace{\pm z+\Lambda}_{\text{a } \Gamma \text{ orbit}}$

$\pi(p) = \pi(p')$ iff $\quad p = p'$ or $p = L(p')$.

π is well defined?

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto z} & \mathbb{C} \\ \downarrow \pi' & & \downarrow \pi'' \\ \mathbb{C}/\Lambda & \overset{\pi}{\dashrightarrow} & \mathbb{C}/\Gamma \end{array}$$

$q = \pi'(p)$

$q = \pi(p')$ $\qquad p' = p+\omega$

$\pi''(p) = \pi''(p')$.

If π were an actual covering space then.

Observe that Λ is a normal subgroup of Γ,

$\gamma^{-1}(z) = -z+\omega$

If $\underset{z \mapsto -z+\omega}{\gamma \in \Gamma}$ ~~then~~ and $\underset{z \mapsto z+\lambda}{\lambda \in \Lambda}$ then

$\gamma^{-1}(-z+\omega) = z-\omega+\omega = z$

$$\gamma \lambda \gamma^{-1}$$

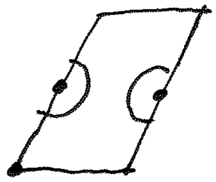
$$\Gamma / \Lambda \simeq \mathbb{Z}/2\mathbb{Z}.$$

L corresponds to a generator of the deck group.

This would be literally true if Γ were acting freely. Language of "orbifolds" was constructed to extend the Galois correspondence to non-free actions.

It is literally true if we remove all "branching points".

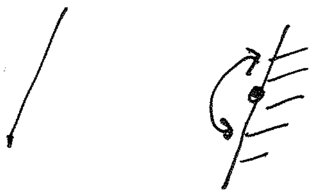
$\mathbb{C}/\Gamma$ has a Riemann surface atlas



If $p \in \mathbb{C}/\Gamma$ is not a cone point then

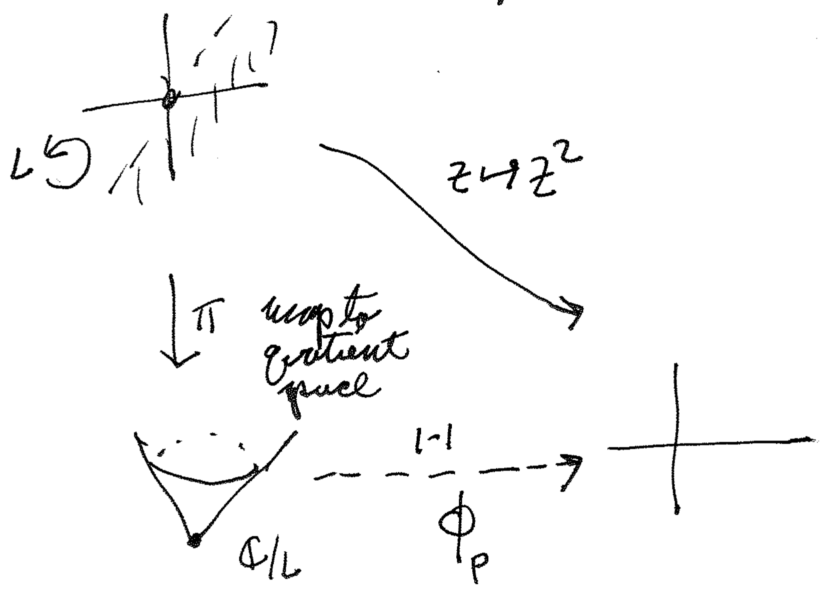
$\phi(z) = z$ is a chart around p.

If p is a cone point then $\phi(z) = z^2$ induces a chart on the quotient.



$(r,\theta) \mapsto (r^2, 2\theta)$

local picture.



Choose charts taking point p to 0.

Same picture holds at any fixed point of L.

ϕ is well defined and a homeomorphism.

We can check that the overlap functions are continuous.

ϕ_p is 1-1. If $\phi(q) = \phi(q')$ then $q = \pi(p)$ $q' = \pi(p')$

$\phi(\pi(p)) = \phi(\pi(p'))$
$p^2 = (p')^2$ so $p = \pm p'$

Lemma. Let $Q \xrightarrow{f} R \xrightarrow{g} S$ be hol. maps between compact Riemann surfaces then

$\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

Proof. Exercise.

Since $\wp : \mathbb{C}/\Lambda \longrightarrow \mathbb{CP}^1$ is actually constant on Γ orbits ($\wp(z) = \wp(-z)$) $\wp$ induces a map

$$\begin{tikzcd} T^2 = \mathbb{C}/\Lambda \arrow[rr, "\wp"] \arrow[dr, "\pi"'] & & \mathbb{C} \cup \infty \\ & \mathbb{C}/\Gamma \arrow[ur, "\wp_0"'] & \end{tikzcd}$$

where

$\wp_0(\pi(q)) = \wp(q)$.

Proposition. $\wp_0$ is a conformal isomorphism.

Proof. $\wp_0$ is holomorphic. This is clear away from the fixed points of ι.

In a nbd. of the ~~fixed~~ points we have charts so that π and $\wp$ both have the form $z \mapsto z^2$.

$$\begin{tikzcd} \odot \arrow[rr, "z \mapsto z^2"] \arrow[dr, "z \mapsto z^2"'] & & \odot \\ & \odot \arrow[ur, "z \mapsto z"'] & \end{tikzcd}$$

Local picture.

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[rr, "P"] \arrow[dr, "\pi"'] & & \mathbb{C}\cup\infty \\ & \mathbb{C}/\Gamma \arrow[ur, dashed, "P_0"'] & \end{tikzcd}$$

Since $P(L(z)) = P(z)$ we can define P_0 by

$P_0(\pi(z)) = P(z)$.

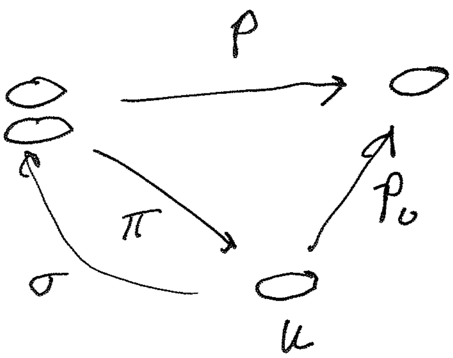
$\pi(z) = \pi(z')$ then we get $P_0(\pi(z')) = P(z')$.

But $z' = z$ or $z' = L(z)$ and $P(L(z)) = P(z)$

so this recipe gives a well defined function.

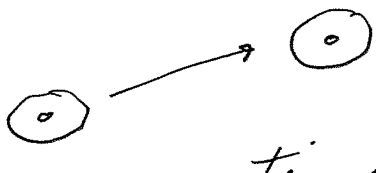
We further claim that P_0 is holomorphic.

At a non-cone point of $\mathbb{C}/\Gamma$ the map π is a covering so we can define a holomorphic section σ from u to $\pi^{-1}(u)$.



Then $P_0|u = P \circ \sigma$ is holomorp

At a branch point this construct



gives a continuous map

holomorphic on the punctured disk.

Since degrees multiply $\deg(\wp) = \deg(\pi) \cdot \deg(\wp_0)$.

$\deg(\wp) = 2$ since $\wp$ has a unique order 2 pole.

$\deg(\pi) = 2$ since π is a 2-1 covering away from $\frac{1}{2}$.
$\{\alpha, \beta, \gamma, \delta\}$. Conclude that $\deg(\wp_0) = 1$ so

$\wp_0$ is a conformal isomorphism.

Cor. $\mathbb{C}/\Gamma$ is conformally $\mathbb{CP}^1$.
This was <u>not</u> clear before.

Cor. $\wp$ has multiplicity 2 exactly at
since these are the fixed points of ι.
$\alpha, \beta, \gamma, \delta$.
Sum of multiplicities is 2.
$\wp(z) = \wp(w) \Rightarrow z = \iota w$ or $z = w$.

$\wp(\alpha), \wp(\beta), \wp(\gamma), \wp(\delta)$ are distinct points in $\mathbb{CP}^1$.
$(\wp(\delta) = \infty)$.

$\pi^{-1}(q) = \{p, \sigma(p)\}$
If $\pi^{-1}(q)$ has a single point then it has mult. 2. If there are two points then both have mult. 1.

Lemma. $\wp'(-z) = -\wp'(z)$.

Proof. $\wp(z)$ is an even function of z since $\wp(z) = \wp(-z)$ so $\wp'$ is odd.

Given a lattice Λ we have defined a Riemann surface $\mathbb{C}/\Lambda$ and a projective curve C_Λ given by $Q_\Lambda(X,Y,Z) = Y^2Z - 4X^3 + g_2(\Lambda)XZ^2 + g_3(\Lambda)Z^3$.

This curve has an "affine part" where $Z=1$ and points on the line at ∞ where $Z=0$. To calculate the points at ∞ we set $Z=0$ in Q and get $Q_\Lambda(X,Y,0) = -4X^3$.
So the intersection of C_Λ with the line at ∞ is the intersection $X=0$ with $Z=0$ which is $[0:1:0]$.

We are showing that $u(z) = \begin{cases} [\wp(z) : \wp'(z) : 1] & z \notin \Lambda \\ [0:1:0] & z \in \Lambda \end{cases}$
gives a parametrisation of C_Λ, ie a conformal isomorphism.

This is analogous to the parametrisation $V = \{X^2+Y^2=1\}$ given by $z \mapsto (\cos z, \sin z)$. $\mathbb{C}/2\pi i\mathbb{Z} \to V$.

A difference is that in our case there are additional parameters involved.

Last time we almost showed that u maps into $\mathbb{C}-\Lambda$ into

$C_\Lambda - \{[0:1:0]\}$.

Needed the fact that $\wp'(z)=0 \Rightarrow z \in \{a, b, c\}$. (Showed $\Leftarrow$.)

Showed that $\wp'$ has a unique pole of order 3.

But the sum of the orders of the poles is the same as the sum of the orders of the zeros. The order of each zero is at least 1 so

$\wp'$ has exactly 3 zeros and no more.

End of proof:

What happens when z is near a lattice point? ③

Say z is near 0.

Now P has a pole of order 2 at 0 so $P(z) = \frac{g(z)}{z^2}$

with g hol, $g(0) \neq 0$. $P'(z)$ has a pole of order 3 so $P'(z) = \frac{h(z)}{z^3}$.

$$u(z) = [P(z) : P'(z) : 1] = \left[\frac{g(z)}{z^2} : \frac{h(z)}{z^3} : 1\right]$$

$$= [z \cdot g(z) : h(z) : z^3]$$

$$\lim_{z \to 0} = [0 : 1 : 0].$$

So u is holomorphic and injective.

By the degree formula $\deg u = 1$ (choose a point in the image of u, calculate the sum of local degrees). A degree 1 map between compact Riemann surfaces is a conformal equivalence.

Construct a space of lattices and a space of cubic curves. Show that they are equivalent.

This is called the "modular curve".

Backwards map is given by integration.

Sin and cos are given by trig integrals. $\int \frac{dz}{\sqrt{z^2-1}}$

$\wp, \wp'$ are given by "elliptic" integrals $\int \frac{dz}{\sqrt{(z-a)(z-b)(z-c)}}$.

Inverses of trig. functions evaluate integrals of the form

$\int \frac{dz}{\sqrt{z^2-1}} \ldots$

Can lift this integral to C_Λ. Equivalent (under u) to dz on $\mathbb{C}/\Lambda$.

Inverses of $\wp$ functions evaluate integrals of the form

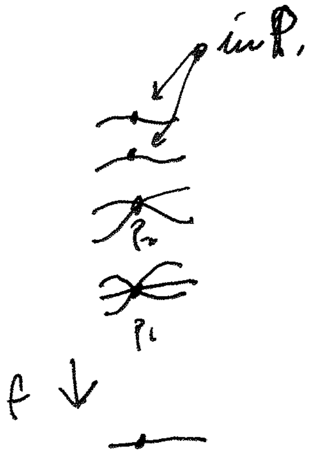
$\int \frac{dz}{\sqrt{(z-a)(z-b)(z-c)}}$, Another global corollary of our local form theorem.

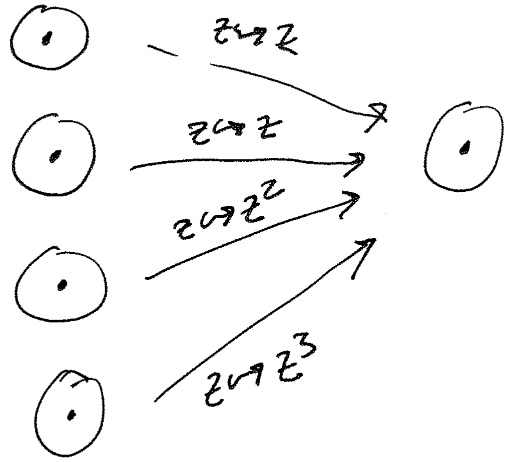
Thm. (Riemann-Hurwitz). Let $f: R \to S$ be a non-constant holomorphic map between compact Riemann surfaces of degree d. Then

$$\chi(R) = d \cdot \chi(S) - \sum_{\substack{p \in R \\ \nu(f,p) > 1}} \nu(f,p) - 1$$

Proof. We can calculate Euler characteristics in terms of triangulations. Let $p_1 \dots p_m \in R$ be points with $\nu(f,p) > 1$. Let $q_1 \dots q_m \in S$ be their images. Choose a triangulation of S whose vertices include the points $q_1 \dots q_k$.
~~At~~ Away from $p_1 \dots p_j$ the map f is a covering so the inverse image of a cell is ~~a collection~~ d-copies of that cell. Inverse image of a point q_i is a collection of points p_k.

Let $v(S), e(S), f(S)$ be the number of vertices edges and faces of points in the





Inverse image of a vertex is a point.

Inverse image of an interval or a triangle in S consists of d intervals or triangles in R.

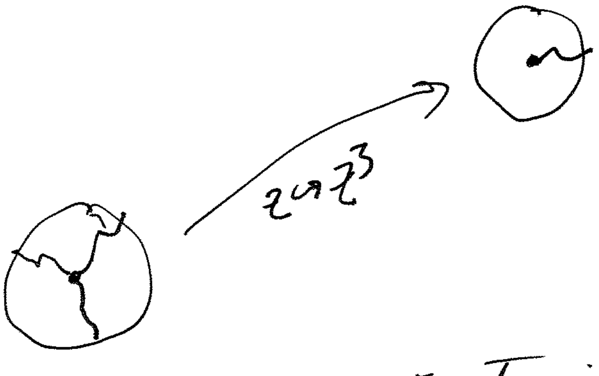
Let $p_1 \dots p_n \in R$ be the points with $\nu(f, p_j) > 1$. (critical points.)

Let $Q = f(\{p_1 \dots p_n\})$. Let $P = f^{-1}(Q)$.

$f: R - P \to S - Q$ is a proper holomorphic map with no critical points so ~~the~~ it is a covering of degree d.

Given an edge e_1 downstairs with endpoints v_1, v_2 we can lift $e_1 - \{v_1, v_2\}$ to d open intervals upstairs.

In a nbd. of v_1 or v_2 we can show that the lift extends to the endpoint. In local coordinates this is just extending a branch of the n-th root function.



Same analysis applies to triangles

$\chi(R) = v(R) - e(R) + f(R) \qquad \chi(S) = v(S) - e(S) + f(S)$

$$\chi(R) - d\chi(S) = v(R) - d\,v(S) + e(R) - d\cdot e(S) + f(R) - d\cdot f(S)$$

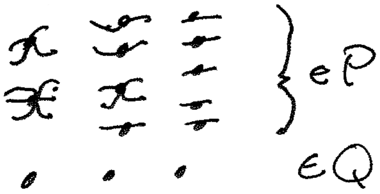
$$= v(R) - d\cdot v(S)$$

$$= \sum_{p\in P} 1 - \sum_{q\in Q} d$$

$$= \sum_{q\in Q}\left(\left(\sum_{f(p)=q} 1\right) - d\right)$$

$\#P - d\cdot\#Q$

Partition points in P depending on their image under f.



$$= \sum_{q\in Q}\ \sum_{p:\, f(p)=q} 1 \; - \; \sum_{q\in Q} d.$$

Recall that $\displaystyle\sum_{p:\, f(p)=q} \nu(f,p) = d$ so write

$$= \sum_{q\in Q}\ \sum_{p:\, f(p)=q} 1 \; - \; \sum_{q\in Q}\ \sum_{p:\, f(p)=q} \nu(f,p)$$

$$= \sum_{q\in Q}\ \sum_{p:\, f(p)=q} 1-\nu(f,p)$$

$$= \sum_{p\in P} 1-\nu(f,p)$$

$$= \sum_{\substack{p\in P\\ \nu(f,p)>1}} 1-\nu(f,p).$$

$$= \sum_{q_j \in Q} \sum_{p_k : f(p_k) = q_j} (1 - \nu(f, p_k))$$

$$= \sum_{p_k \in P} q \cdot 1 - \nu(f, p_k).$$

Can use this to calculate the genera of algebraic curves explicitly.

Prop. If M is a connected topological surface and M is connected then $g(M) = \frac{(2-\chi)}{2}$

$\chi = 2 - 2g$ Check. $g=0$ $M = S^2$ $\chi = 2$,
$g=1$ $M = T^2$ $\chi = 0$

In general a surface of genus g is a connected sum of g tori.
Taking a connected sum with a torus reduces χ by 2.

Theorem. A non-singular algebraic plane curve is connected.

"Proof". If not it is represented by at least 2 disjoint surfaces with degrees m, n. Using coh. cup product or Bezout's then this is not possible.