Anosov

The next topic is a second example of a hyperbolic diffeomorphism of a surface exhibiting chaotic behavior. This example has a different flavor from the horseshoe. Suggesting that the notion of hyperbolicity has a range of applications.

Hyperbolic automorphisms of the torus.

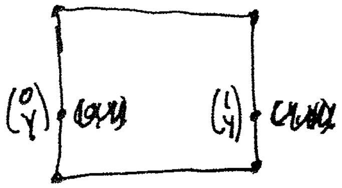
Let $\mathbb{T}^2$ be the ~~quotient~~ coset space $\mathbb{R}^2/\mathbb{Z}^2$.

An element of $\mathbb{T}^2$ is an equivalence class

$\{\binom{x+n}{y+m} : n, m \in \mathbb{Z}\}$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the projection. Write $\pi\binom{x}{y} = \binom{\bar{x}}{\bar{y}}$. $\pi\binom{x}{y} = \pi\binom{x'}{y'}$ iff $\binom{x}{y} - \binom{x'}{y'}$

Any such equivalence class has a representative in the square $[0,1] \times [0,1]$.

Certain points have more than one representative

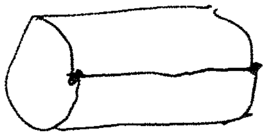


~~$(0,y) \sim (1,y)$~~ $\binom{0}{y} \sim \binom{1}{y}$

~~$(x,0) \sim (x,1)$~~ $\binom{x}{0} \sim \binom{x}{1}$

~~$(0,0) \sim (0,1) \sim (1,0) \sim (1,1)$~~

$\binom{0}{0} \sim \binom{0}{1} \sim \binom{1}{0} \sim \binom{1}{1}$.



Note that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, so $\pi\left(\binom{x}{y}\right) = \binom{x \bmod \mathbb{Z}}{y \bmod \mathbb{Z}}$

Prop. A 2×2 matrix A with integral entries induces a linear or affine map from $\mathbb{T}^2$ to $\mathbb{T}^2$.

Define $f_A\binom{\bar x}{\bar y}$ to be $\pi f_A\binom{x}{y}$ where $\pi\binom{x}{y} = \binom{\bar x}{\bar y}$. Want to check that the answer does not depend of our choice of $\binom{x}{y}$.

Say $\pi\left(\binom{x}{y}\right) = \pi\left(\binom{x'}{y'}\right)$ so $\binom{x}{y} - \binom{x'}{y'} \in \mathbb{Z}^2$.

Want to show that $A\binom{x}{y} - A\binom{x'}{y'} \in \mathbb{Z}^2$.

But $A\binom{x}{y} - A\binom{x'}{y'} = A\left(\binom{x}{y} - \binom{x'}{y'}\right) \in A(\mathbb{Z}^2) \subset \mathbb{Z}^2$ since A has integral entries.

If A has det ± 1 then $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ has integral entries so A^{-1} induces a well defined map on the torus. Now the map induced by A composed with the map induced by A^{-1} can be constructed by taking $A \circ A^{-1}\binom{\bar x}{\bar y} = \pi\left(A \circ A^{-1}\binom{x}{y}\right) = \pi\binom{x}{y} = \binom{\bar x}{\bar y}$ so it is the identity.

Def. We say that a matrix A is hyperbolic if it has no eigenvalue on the unit circle.

If $A : V \circlearrowleft$ is hyperbolic then there are invariant subspaces V^u and V^s of V so that $V^u \cap V^s = \{0\}$, also $V^u + V^s = V$ and

V^u is the generalized eigenspace associated with all eigenvalues outside the unit circle.

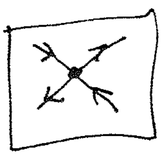
$$\lim_{n\to\infty} |A^n(v)| \to 0 \qquad \lim_{n\to-\infty} |A^n(v)| \to \infty \qquad \text{for } v \in V^s$$

$$\lim_{n\to\infty} |A^n(v)| \to \infty \qquad \lim_{n\to-\infty} |A^n(v)| \to 0 \qquad \text{for } v \in V^u.$$

In the particular case that A is 2×2 with $\det = \pm 1$ and A is hyperbolic" A has eigenvalues λ^u, λ^s with $|\lambda^u| > 1$, $|\lambda^s| < 1$. V^u is the eigenspace for λ^u, V^s is the eigenspace for V^s.

In general V^s and V^u are sums of generalized eigenspaces.

The image of V^u and V^s are the unstable and stable manifolds of 0.



$$\lim_{n\to\infty} |A^n(v)| = \lim_{n\to\infty} |\lambda^n \cdot v|$$

Def $W^s(p) = \{q : d(f^n(p), f^n(q)) \to 0 \text{ as } n \to \infty\}$

$W^u(p) = \{q : d(f^n(p), f^n(q)) \to 0 \text{ as } n \to -\infty\}$.

$W^s(0) = V^s + \mathbb{Z}^2$, $W^u(0) = V^u + \mathbb{Z}^2$.

Prop. $W^s(0)$ = eigenspace corresponding to λ^s

$W^u(0)$ = " λ^u.

Remark $W^s(0)$ and $W^u(0)$ are dense in the torus.

The head of arrow j corresponds to the location of S_j.

~~Suppose~~

Want to consider the ~~analogous~~ analogue of finite words, ~~or~~ which are finite paths.

~~We can write a path of length n as~~
Path of length 5:

$$R_1 \xrightarrow{4} R_2 \xrightarrow{5} R_2 \xrightarrow{2} R_1 \xrightarrow{3} R_1 \xrightarrow{1} R_1$$

This information ~~is recorded~~, can be recorded by listing just the arrows: 4 5 2 3 1.

~~It does not suffice to list just~~ directed edges

or the edges and vertices as above.

It does not suffice to list just the vertices.

~~If we We also want to record~~

We also want to record where we are "now" as ~~We can do this by underlyin~~ we did with the decimal point. We can do this by underlining a vertex or by putting a decimal point in the sequence of arrows

$$R_2 \xrightarrow{5} \underline{R_2} \xrightarrow{2} R_1 \quad \text{or} \quad 5.2$$

Image of $W^s(0)$ in the torus corresponds to all $\mathbb{Z}^2$ translates of the λ^+ eigenspace.

Exercise. $W^s(0)$ is a dense subset of the torus.

③

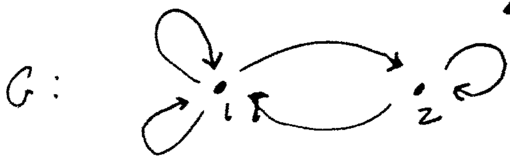
$$\text{Tr}=3 \quad \text{det}=1, \quad \lambda^2 - \text{Tr}\cdot\lambda + \det = 0$$

Example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. $\quad \lambda = \dfrac{3 \pm \sqrt{5}}{2}$.

2nd example of "hyperbolic behavior": if p, q suff close then $W^s_{loc}(p) \cap W^u_{loc}(q) = pt.$

$$\lambda^u = \frac{3+\sqrt{5}}{2} \qquad \lambda^s = \frac{3-\sqrt{5}}{2} \qquad |\lambda^u| > 1 > |\lambda^s|.$$

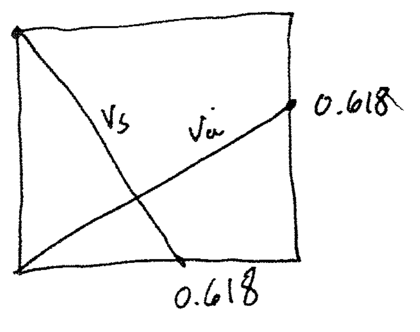
Thm. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ then $f_A : \mathbb{T}^2 \circlearrowleft$ is ~~Topologically equivalent~~ topologically semi-conjugate to ~~$\sigma: \Sigma_G$~~ the topological Markov chain $\sigma : \Sigma^c_G \circlearrowleft$ where

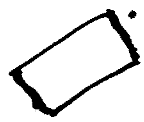


Proof. 0 is a fixed point for f_A.
We will use the local stable and unstable manifolds of 0 to divide $\mathbb{T}^2$ into two rectangles.

$$V_u = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}. \quad \text{Rescale to } \begin{pmatrix} 1 \\ 0.618.. \end{pmatrix}.$$

$$V_s = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}. \quad \text{Rescale to } \begin{pmatrix} -0.618.. \\ 1 \end{pmatrix}$$





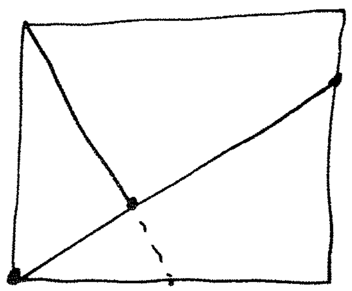
This is useful for insuring the Markov property in this setting.

Recipe for Markov partition.

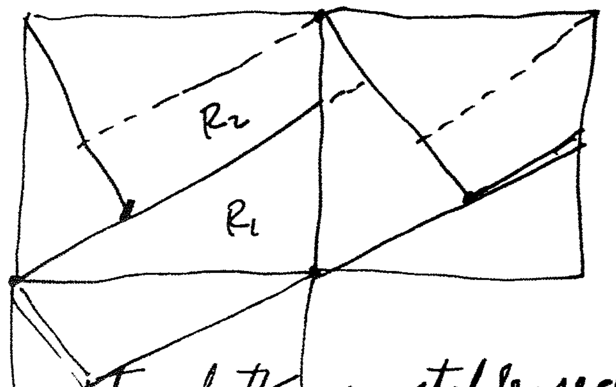
Expanding eigenvector: $\begin{pmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{pmatrix}$ Rescale $\begin{pmatrix}1\\ 0.618034\end{pmatrix}$

Contracting eigenvector: $\begin{pmatrix}\frac{1-\sqrt{5}}{2}\\ 1\end{pmatrix}$ $\begin{pmatrix}-0.6180\\ 1\end{pmatrix}$ $\begin{pmatrix}0.618\\ -1\end{pmatrix}$

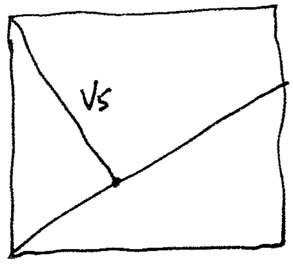
Slope of the expanding eigenvector is positive but less than 1. Take the branch in the first quadrant and extend it across the square.



The contracting eigenvalue has negative slope. Extend it until it hits the ~~ef~~ expanding segment

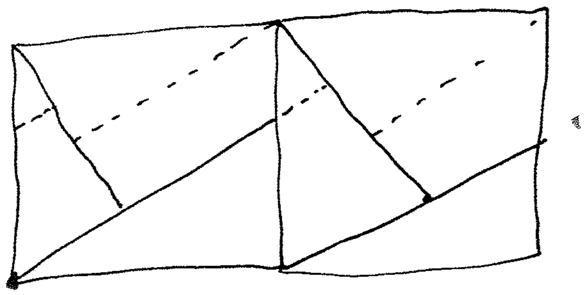


Now extend the unstable segment in both directions until it hits the stable segment. Call the resulting rectangles R_1 and R_2. The image of union of R_1 and R_2 is ~~th~~ all of $\mathbb{T}^2$. The images of the boundaries are not disjoint.

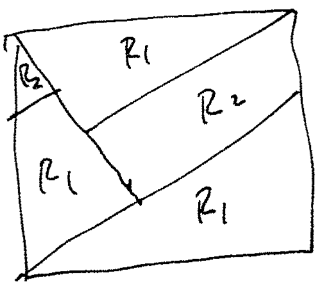


Cut off V_5 at the intersection point.

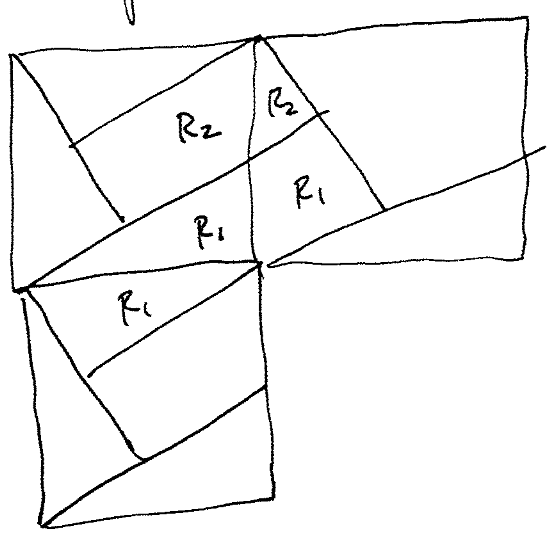
Now extend V_4 in both directions until it hits V_5 again.



This divides $\mathbb{T}^2$ into two rectangles R_1 and R_2.



or



This picture shows that R_1 and R_2 fill $\mathbb{T}^2$.

This picture shows that R_1 and R_2 are rectangles.

Keep track of the image as follows:

Stable boundaries

Keep track of the image of R_1. The top boundary is part of the unstable manifold of $\begin{pmatrix}0\\0\end{pmatrix}$. This gets mapped to a longer part of the unstable manifold of $\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$. The r.h. stable boundary is part of the stable manifold of $\begin{pmatrix}1\\1\end{pmatrix}$. This gets mapped to part of the stable manifold of $\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$.

The bottom unstable boundary of R_1 goes through $\begin{pmatrix}1\\0\end{pmatrix}$. This gets mapped to an unstable segment through $\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$.

Left hand stable segment gets mapped into itself.

7

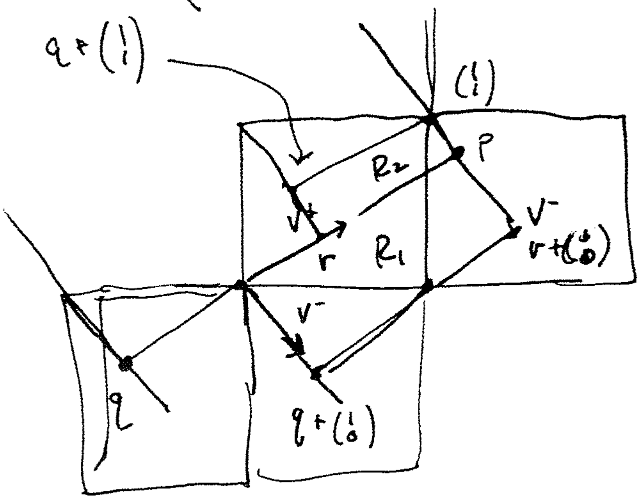
~~Let $v^+ = \begin{pmatrix} \\ \end{pmatrix}$~~

The eigenvectors of A are

$\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$

$$\frac{1+\sqrt{5}}{2} \cdot \frac{1-\sqrt{5}}{2} = \frac{1-5}{4} = -1$$

$\begin{pmatrix} \phi \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1.618.. \\ 1 \end{pmatrix}$ $\begin{pmatrix} -0.618.. \\ 1 \end{pmatrix}$



Eigenvectors give local stable and unstable manifolds of 0.

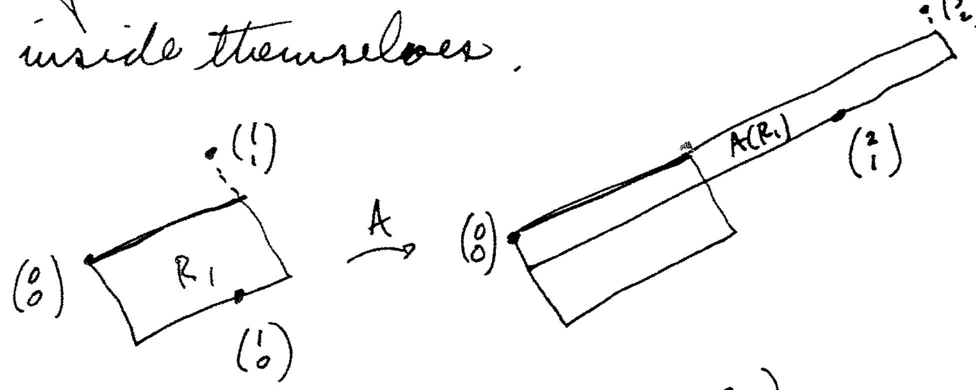
~~Note that the boundaries~~

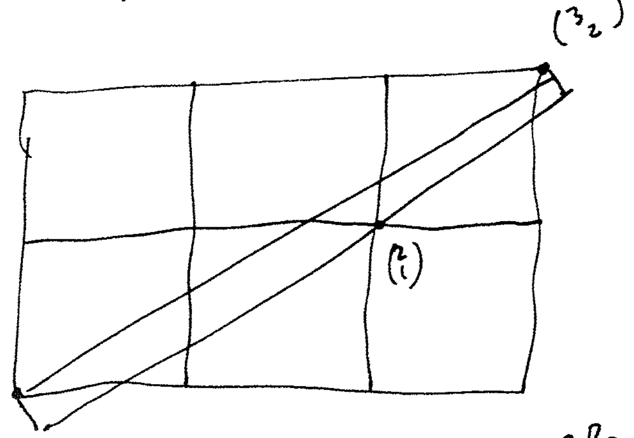
$R_1 \cup R_2$ is the whole torus.

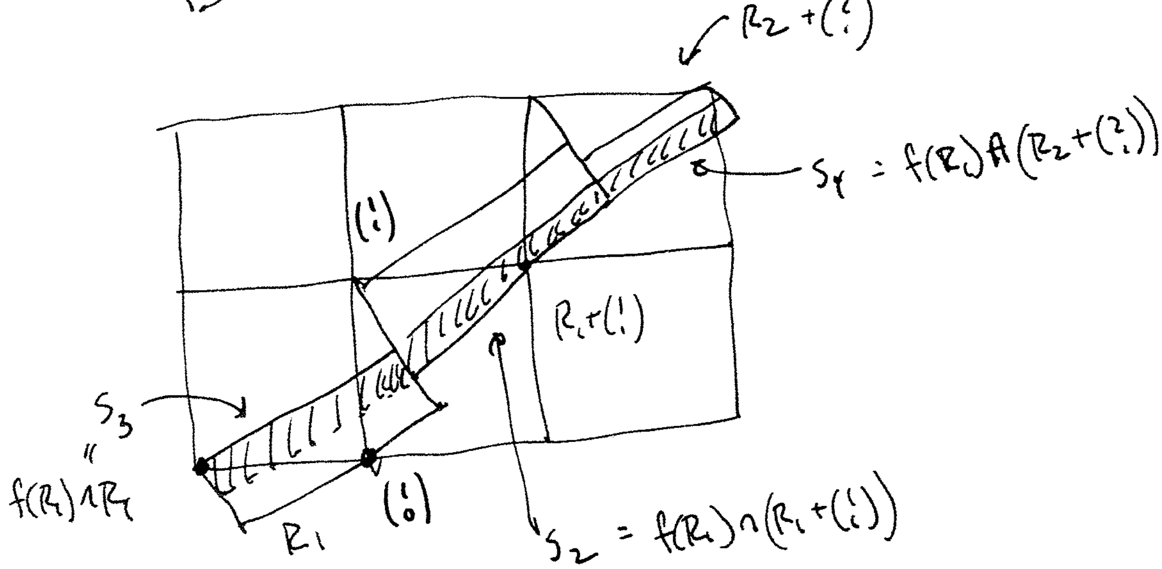
Now ~~Note that~~

Let $\partial^+ R_1$, $\partial^+ R_2$ be the pieces of the boundaries which intersect the unstable manifolds. Let $\partial^- R_1, \partial^- R_2$ be the intersections of boundaries with stable manifolds. Then $f(\partial^- R_1 \cup \partial^- R_2) \subset \partial^- R_1 \cup \partial^- R_2$ and $f^{-1}(\partial^+ R_1 \cup \partial^+ R_2) \subset \partial^+ R_1 \cup \partial^+ R_2$. This is the Markov condition in the invertible case.

We will now calculate the images of R_1 and R_2 in $\mathbb{R}^2$. We use 3 facts. A fixes $\binom{0}{0}$ and takes integral points to integral points. A takes unstable segments outside themselves and stable segments inside themselves.



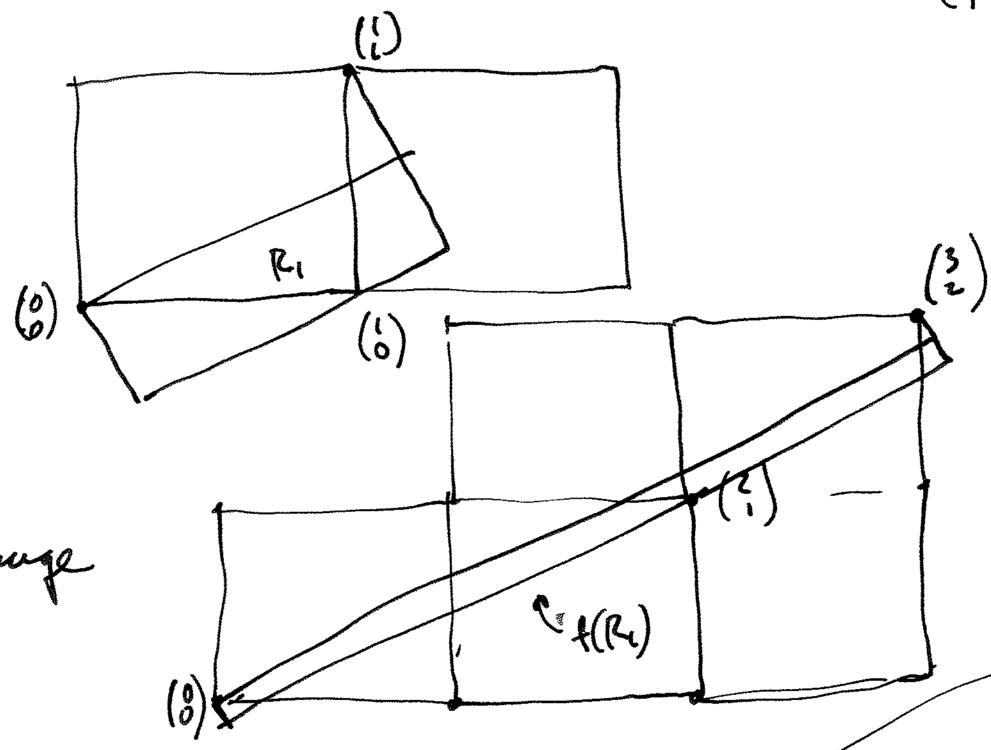




$$\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$\begin{pmatrix}1\\0\end{pmatrix} \mapsto ($

$\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$



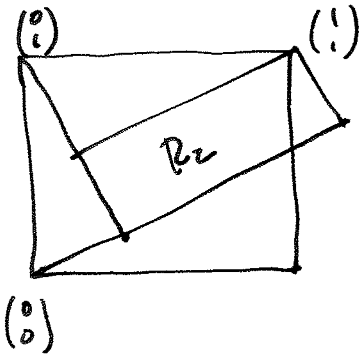
Image

top of R_1

Image of the unstable segment lies above $\binom{2}{1}$ and below $\binom{3}{2}$.

Image of the bottom segment goes through $\binom{2}{1}$ and ends on a stable segment through $\binom{3}{2}$.

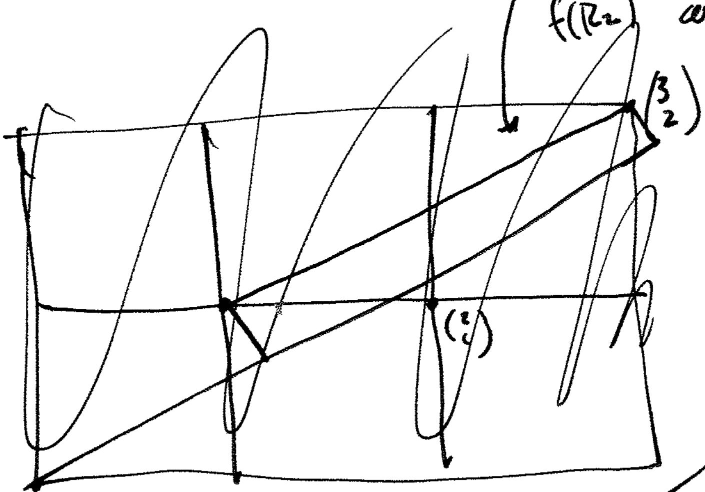
Top of $f(R_1)$ coincides and extends top of R_1 because this is an eigendirection with an expanding eigenvalue.

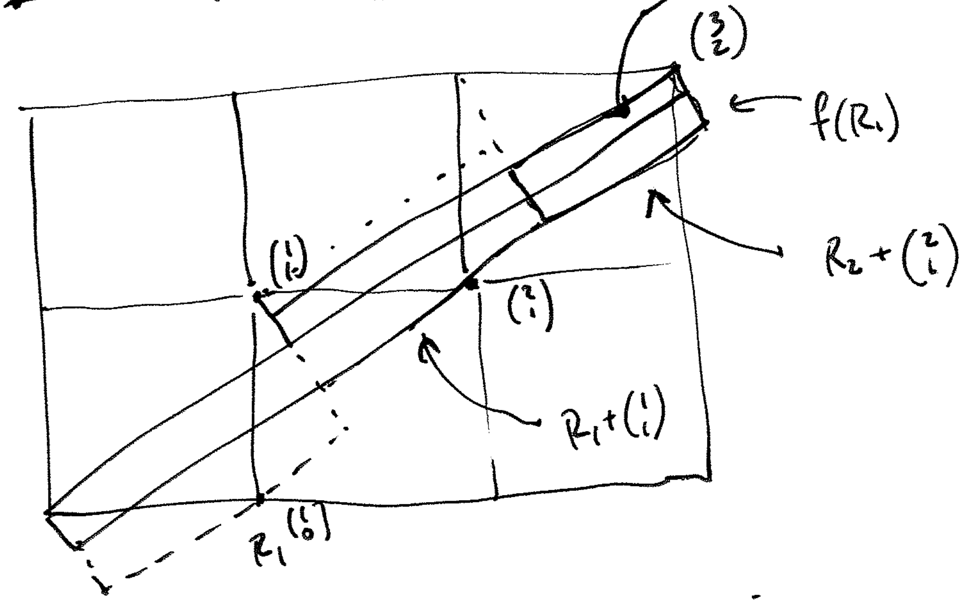


12:30

$\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

We can draw pictures in $\mathbb{R}^2$ as long as we remember ~~as~~ that we ~~should~~ are really thinking about all the $\mathbb{Z}^2$ translates as well.

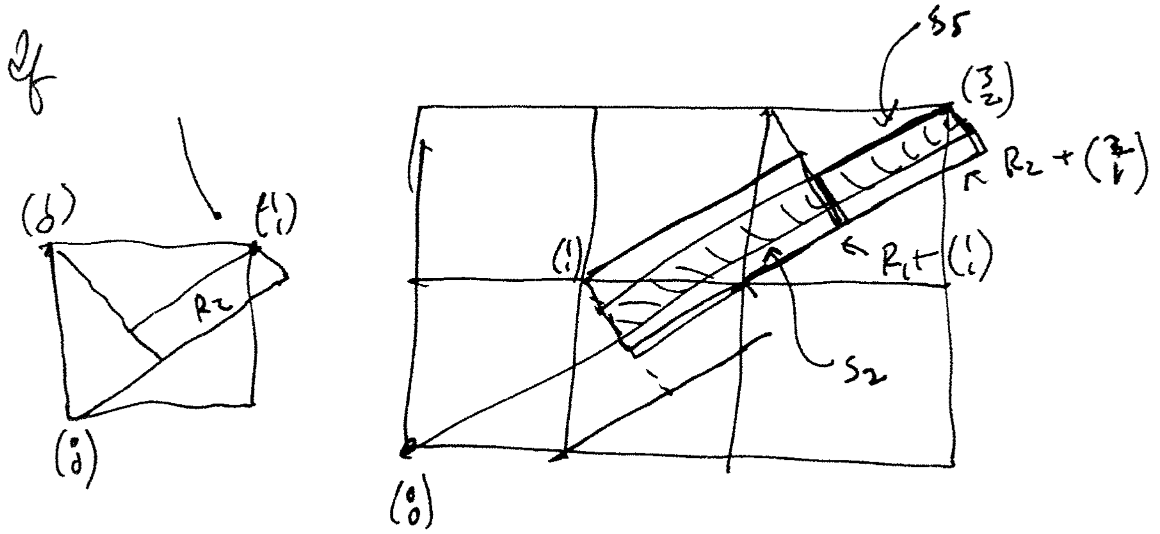


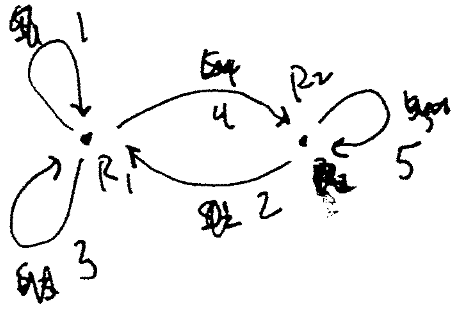


Geometrically we should think of these images as winding through the torus. We get such a picture if we look at all the $\mathbb{Z}^2$ translates.

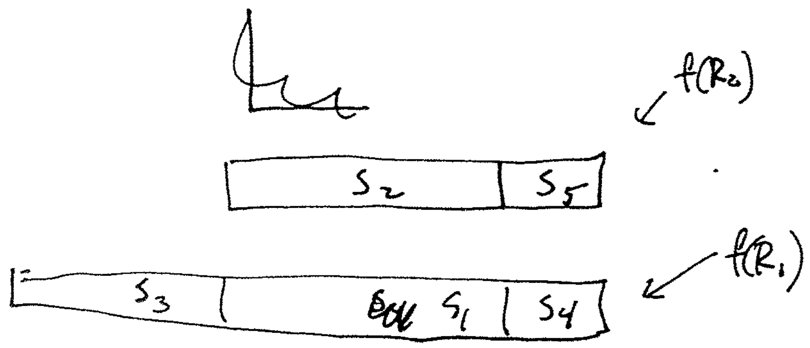
How do the images intersect R_1 and R_2?

If

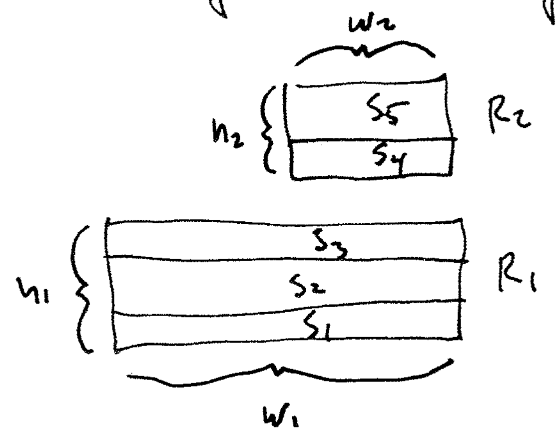




Write the images of R_1 and R_2 horizontally:



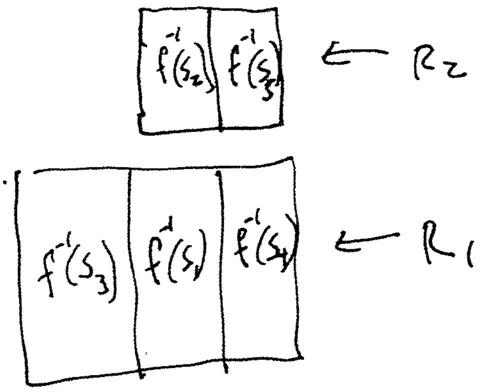
Arrange these rectangles inside R_1 and R_2



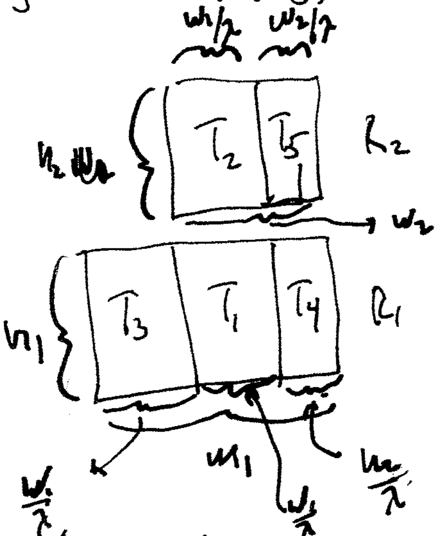
The S_j give a partition of R_i's based on the location of $f^{-1}(p)$.

Corresponds to arrows landing at R_1, R_2.

Apply f^{-1} to previous picture



Let T_j denote $f^{-1}(S_j)$ so $S_j = f(T_j)$.



The T's give a partition of the R's based on the location of $f(p)$.

Partition of R_j corresponds to arrows leaving R_j.

Note that the S's have full width in the R's and the T's have full height.

Coding of orbits:

We can define an itinerary for a point $p \in \mathbb{T}^2$ corresponding to a path in $\mathcal{A}$.

Say $p \in R_2$. p is in some set T_j, say T_2. Hence $f(p) \in R_1$. $f(p)$ is in some set T_j, say T_3.

$$I(p): \quad \underline{R_2} \xrightarrow{2} R_1 \xrightarrow{3} R_1$$

p ~~is~~ in some set S_j say S_5.

$$R_2 \xrightarrow{5} \underline{R_2} \xrightarrow{2} R_1 \xrightarrow{3} R_1 .$$

Now since ~~we s~~ a point may lie in more than one set T_j or S_j it may have more than one itinerary. If $\omega = \left(\underset{2}{\cdot} \xrightarrow{5} \underset{\underline{2}}{\cdot} \xrightarrow{2} \underset{1}{\cdot} \xrightarrow{3} \underset{1}{\cdot} \right)$ is a finite word we can define $A(\omega)$ to be the set of <u>all</u> points for which these compositions make sense. This is the closure of the set for which the itinerary is well-defined.

13.1

Coding of orbits by paths in G:

Any $p \in \mathbb{T}^2$. We set

$$\cdots\ \omega_{-3} \xrightarrow{\alpha_{-2}} \omega_{-2} \xrightarrow{\alpha_{-1}} \omega_{-1} \xrightarrow{\alpha_0} \underline{\omega_0} \xrightarrow{\alpha_1} \omega_1 \xrightarrow{\alpha_2} \omega_2 \cdots$$

Recall our conventions. A path in G is given by as above where ~~ω_j~~ ω's correspond to vertices and α's correspond to edges.

Let $p \in \mathbb{T}^2$. ~~Set ω_j~~ Define $\omega_j = \omega_j(p)$ and $\alpha_j = \alpha_j(p)$ as follows.

$$\omega_j(p) = \begin{cases} 1 \\ 2 \end{cases} \text{if } f^j(p) \in \begin{cases} R_1 \\ R_2 . \end{cases}$$

$$\alpha_j(p) = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{cases} \text{if } f^j(p) \in \begin{cases} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{cases} \text{ or equiv. } f^{j-1}(p) \in \begin{cases} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 . \end{cases}$$

$\alpha_j(p)$ determines $\omega_{j-1}(p), \omega_j(p)$

Applying f to p shifts the sequence to the left.

The sets $A(w)$ correspond to itinerary codings in a way but note that they ~~are~~ ~~are~~ are not always disjoint. This corresponds to points on boundaries which might have multiple codings.

Points in the interior of $A(w)$ have well defined codings (corresponding to the length of w).

Can mark the "zero" position by underlining a vertex or putting a dot in the sequence of arrows.

alt. $\omega = \alpha_{-j}\,\alpha_{-j+1} \ldots$

A(ω) is the set of points for which

$$\omega = \omega_{-j} \xrightarrow{\alpha_{-j}} \omega_{-j+1} \xrightarrow{\alpha_{-j+1}} \cdots \longrightarrow \omega_{-1} \longrightarrow \underline{\omega_0} \xrightarrow{\alpha_0} \omega_1 \cdots \xrightarrow{\alpha_{k-1}} \omega_k$$

makes sense.

A(ω) is the set of points in R_{ω_0} for which these compositions make sense.

That is

$$\omega = \omega_0 \xrightarrow{\alpha_0} \omega_1 \cdots \underline{\omega_k} \cdots \xrightarrow{\alpha_{j-1}} \omega_j$$

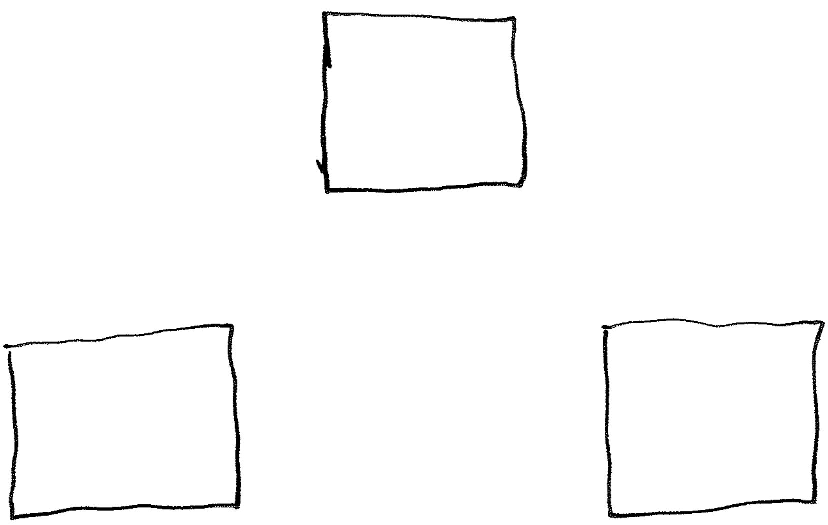
String

Consists of $j+1$ vertices and j edges. Call it a word of length j.

~~Examples~~

A(ω) consists of $p \in R_{\omega_k}$ so that

$$\omega_{k-1} \xrightarrow{\alpha_{k-1}} \omega_k \xrightarrow{\alpha_k} \omega_{k+1}$$



f expands ~~stable~~ unstable manifolds by λ and contracts unstable manifolds by λ.

Now say that we have a finite word

$$w = \overset{\alpha_{j+1}}{\omega_j} \cdots \omega_{-1} . \omega_0 \cdots \overset{\alpha_k}{\omega_k} \qquad j \le 0 \quad k \ge 0,$$

w determines a cylinder set $C(w)$ in G_∞ of infinite words which agree with w where w is defined.

w determines a set $A(w)$ in $\mathbb{T}^2$ of points whose codings agree with the entries of w where they are defined.

Example:

$w = .1 \qquad A(w) = R_1$

$w = .2 \qquad A(w) = R_2$

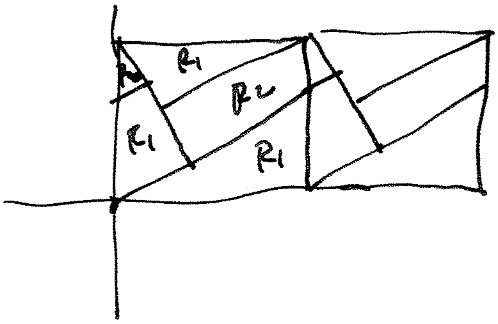
$w = 1 \overset{1}{.} 1 \qquad A(w) = S_1$

$w = 2 \overset{2}{.} 1 \qquad A(w) = S_2$

$w = .1 \overset{1}{\,} 1 \qquad A(w) = T_1$

$w = .2 \overset{2}{\,} 1 \qquad A(w) = T_2.$

Recall that last time we constructed rectangles R_1 and R_2 inside $\mathbb{R}^2$. Map to regions in $\mathbb{T}^2$.



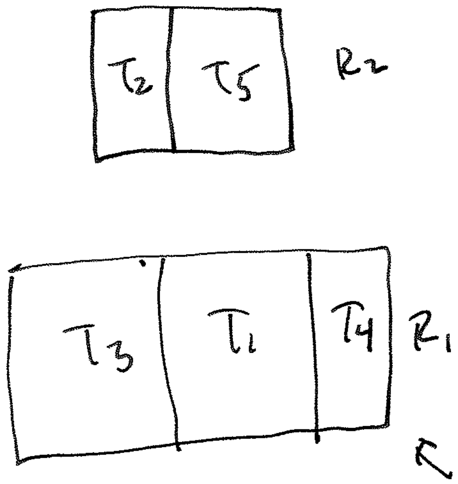
~~The map is an~~ The image rectangles touch along their boundaries but the interiors are disjoint.

Rotate coordinates so that x-axis corresponds to the λ^u eigenspace and the y-axis corresponds to the λ^s eigenspace. (In general these need not be perpendicular. Related to symmetry of matrix.)

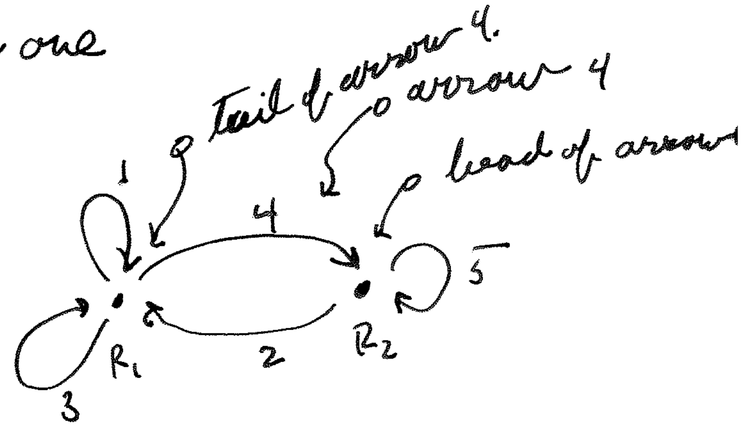
Consider sets $f(R_i) \cap R_j$ in $\mathbb{R}^2$. $f(R_i) \cap (R_j + \binom{m}{n})$.

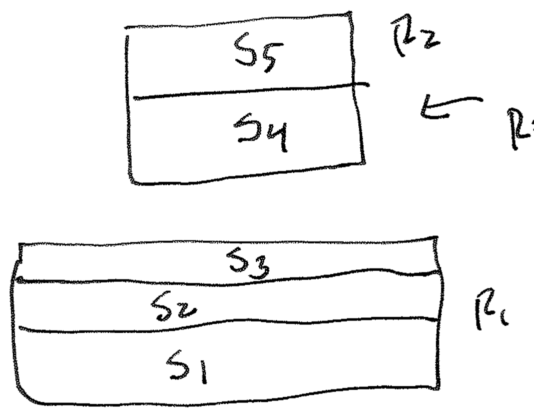
These are rectangles.

Each one



$f^{-1}(R_i) \cap R_i$





$R_i \cap f(R_j)$

$f(T_j) = S_j$

Location of T_j and S_j with respect to R_1 and R_2 determines the transition graph

Tail of arrow j corresponds to the location of T_j.

Also

For completeness we can define a path of length 0 to be a vertex.

Now let $\omega = \omega_{-j} \ldots \omega_{-1} . \omega_0 \ldots \omega_{k-1}$ be a finite word with $j \geq 0$ and $k \geq 0$. $\omega_i \in \{1 \ldots 5\}$

~~We ... defn~~

$$R_{-j} \xrightarrow{\omega_{-j}} \quad \xrightarrow{\omega_{-2}} R_{\alpha_{-1}} \xrightarrow{\omega_{-1}} R_{\alpha_0} \xrightarrow{\omega_0} R_{\alpha_1} \xrightarrow{\omega_1} \quad \xrightarrow{\omega_{k-1}} R_{\alpha_k}$$

We define $A(\omega)$ to be the set of points $p \in R_{\alpha_0}$ so that $f^i(p) \in T_{\omega_i}$ for $i = 0 \ldots k$ and

$f^i(p) \in S_{\omega_i}$ for $i = -j \ldots -1$.

~~Obvious facts about~~

If we were to define itineraries they would be well defined on the interior of $A(\omega)$ but possibly ambiguous on the boundary of $A(\omega)$. Recall that for expanding maps ~~in on~~ of circles we ~~cons~~ defined itineraries and took a closure.

Elementary facts about the sets $A(\omega)$.

If ω is a ~~subword of ω' then~~ subpath of ω' then $A(\omega) \supset A(\omega')$.

Example $\omega' = 2\,1\,1.4\,5\,2\,4$

$\omega = 1\,1.4$

If two words overlap

$\omega = 11.4$

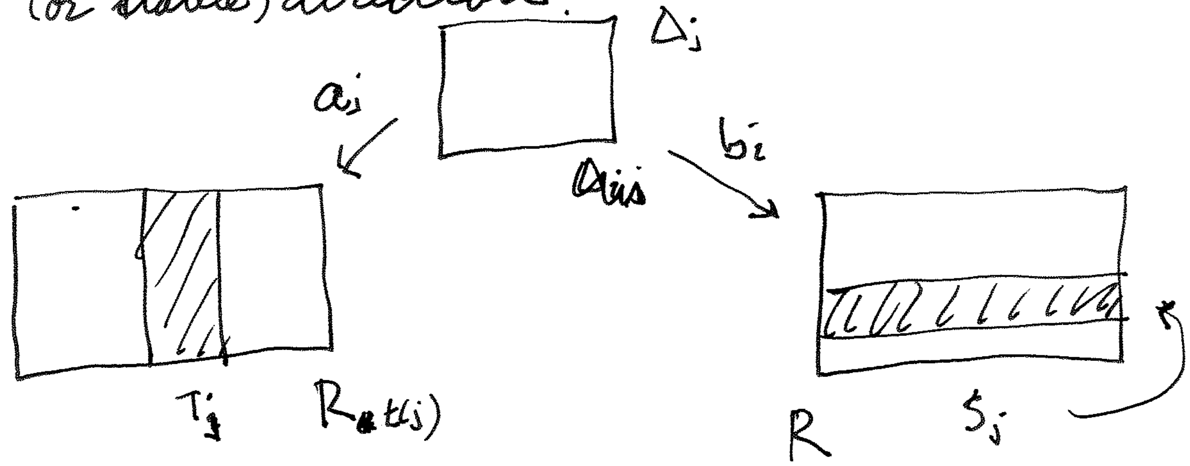
$\omega' = .452$ or

$\omega'' = 11.452$

~~then $A(\omega'')$ is exact~~ corresponds to $A(\omega) \cap A(\omega')$.

f acts like the left shift on words. $f(A(\omega)) = A(\omega')$ where $\omega'_i = \omega_{i+1}$.

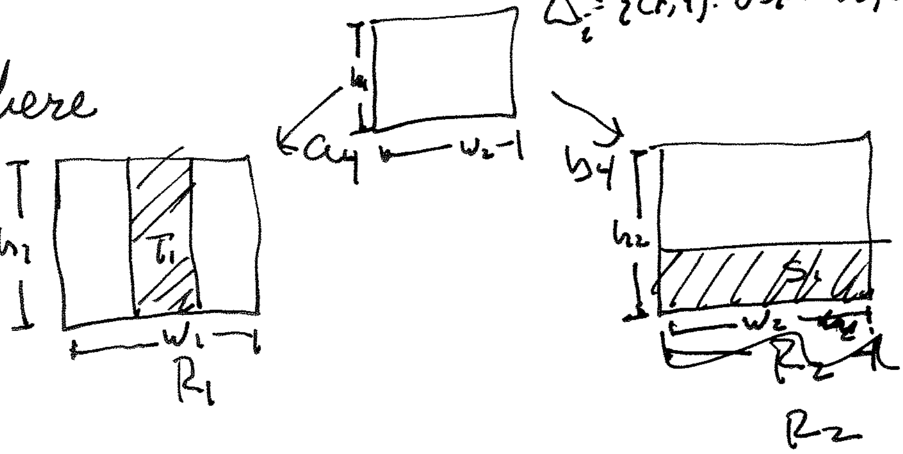
When restricted to a rectangle T_j the map f_A expands the unstable direction and contracts the unstable direction. As before we factor we factor this map $f|T_j$ as $b_j a_j^{-1}$ where a_j contracts the (unstable) x-direction and b_j contracts the y (or stable) direction.



$$a_j\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\lambda^s\cdot x + c_j\\ y\end{pmatrix} \qquad b_j\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\ \lambda^s\cdot y + d_j\end{pmatrix}$$

$\Delta_i = \{(x,y) : 0 \le x \le w_2,\ 0 \le y \le h_1\}$

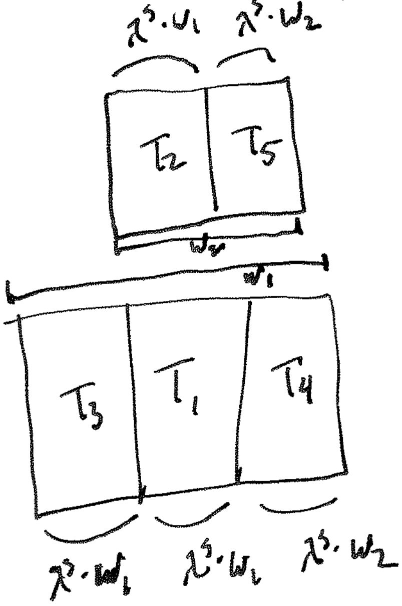
where



~~The scaler~~ w_i goes from R_j to R_k

It is convenient to let the dimensions of Δ_i be $h_j \times w_k$.

We calculate that width of T_j is $\lambda^s\cdot w_2$ and the height of S_j is $\lambda^s\cdot h_j$.



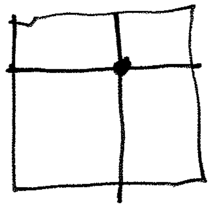
Since $a_j(\Delta_j) = T_j$ we get

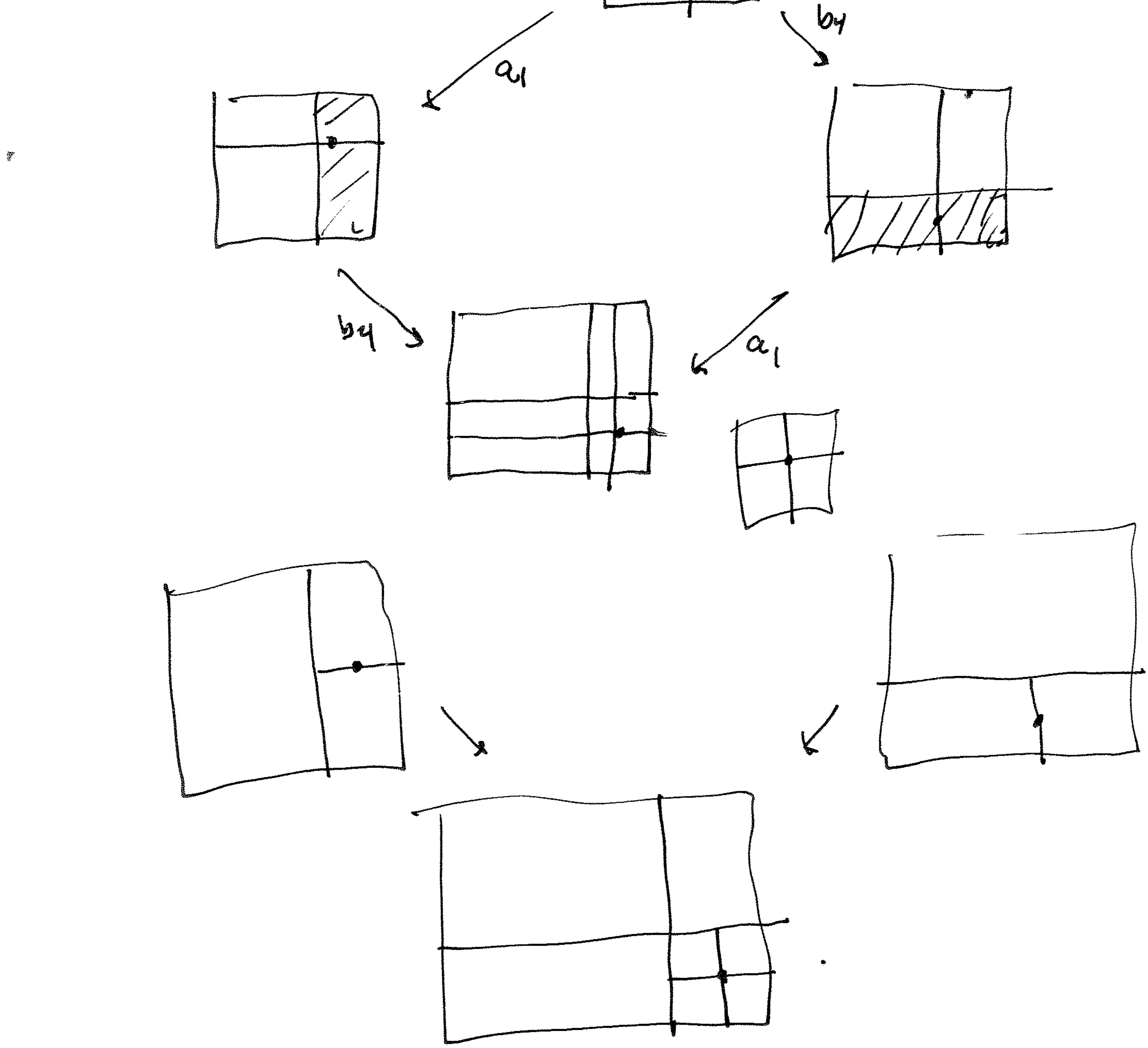
$c_3 = 0,\ c_1 = \lambda^s\cdot w_1,\ c_4 = \lambda^s\cdot w_1 + \lambda^s\cdot w_1.$

implicitly this is an eigenvalue equation

~~If $p \in \mathrm{im}(\alpha\beta)$ then $p = ($~~

$$a_k b_j \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda^s x + c_k \\ \lambda^s y + d_j \end{pmatrix}$$





The strategy for the construction of the semi-conjugacy h is the same as that for the horseshoe. We will show that ~~as~~ $i \to \infty, j \to -\infty$ ~~the set $B(\omega)$~~ $B(\bar{\omega})$ is a rectangle and as $i \to \infty$, $j \to -\infty$ the height and width of $B(\bar{\omega})$ goes to 0. We then define $h(\bar{\omega})$ as an intersection of a nested sequence of rectangles.

Let $\vec{\omega} = \vec{\omega_j} \cdots \omega_{-1} . \omega_0 \cdots \vec{\omega_k}$ $j \le 0,\ k \ge 0$

be a word. Note that the ω_0 location is always included, $|j|$ is the number of specified positions to the left of 0, k is the number of specified positions to the right of 0.

Let h_j be the width of R_j and w_j be the height of B_j

Claim. $B(\vec{\omega})$ is a rectangle with height

$$\frac{h_{\omega_j}}{\lambda^{|j|}} \text{ and width } \frac{w_{\omega_k}}{\lambda^k}.$$

Note that the height and width depend only on the ω's not the α's.

Note that the claim implies that if $j=0$ then $B(\bar{\omega})$ has full height and if $k=0$ then $B(\bar{\omega})$ has full width.

$S_2 = \cancel{A}B(\vec{2}.1)$ $k=0$ $j=-1$ S_2 has full width

$T_2 = \cancel{B}A(.2\vec{1})$ $j=0$ $k=1$ T_2 has full height.

12.1

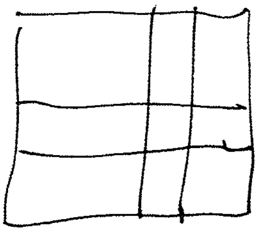
To find the width of T_2 we can apply f. f multiplies widths by λ^u. $f(T_2) = S_2$ has width w_1. So T_2 has width $\frac{w_1}{\lambda}$ as was predicted by the claim.

Proof of the claim by induction on the length of the word. If the word has length 1 then the claim is true.

If the word ~~is~~ has length 2 say $.\overset{2}{\vec{2}}1$ then it has full ~~[illegible]~~ height.^{h_2} If we shift left this we get $\overset{2}{\vec{2}}.1$ $j=-1$ corresponds to shrinking vertically by λ so the height is $\frac{h_2}{\lambda}$ as predicted.

$$\frac{h_2}{\lambda^{|j|}} = \frac{h_2}{\lambda}$$

If the assertion is true for words of length $n-1$ and we have a word $w_j \ldots w_1 . w_0 \ldots w_k$ with $i \leq n-1$ $i \geq 1$ then we consider the words $w_{-j} \ldots w_{-1} . w_0$ and $. w_0 \ldots w_k$. These words are shorter. One corresponds to a rectangle of full width, the other to a rectangle of full height.



The intersection is again a rectangle. ~~Its height and width~~ Its height corresponds to the word $w_{-j} \ldots w_0$. Its width is the width of $. w_0 \ldots w_k$.

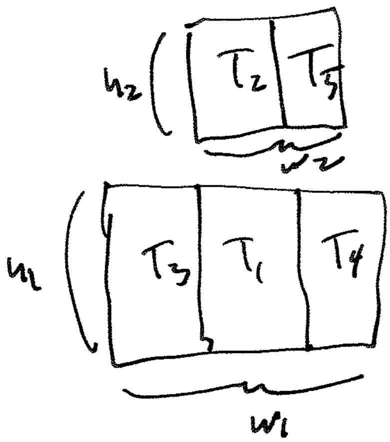
(25) (28)

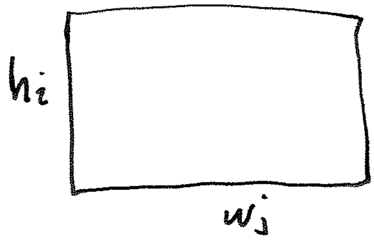
If we have a word with say $k=0$

$w_{-j} \ldots w_{-1} \overset{\alpha_{-1}}{\rightarrow} w_0$ then it is contained in a rectangle $S_{\alpha_{-1}}$. If we apply f^{-1} we shift the word right. ~~and we~~ $f^{-1}|S$ multiplies widths by λ and contracts heights by λ. Furthermore we can analyze this set by means of the previous argument.

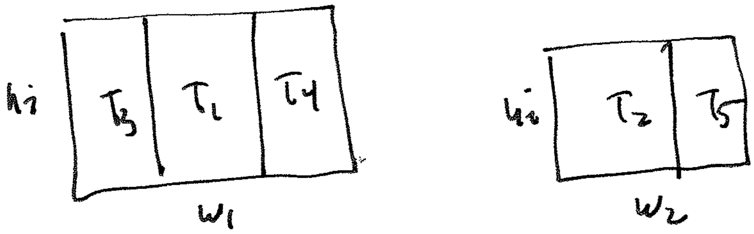
$R_{i,j}$ is $\{(x,y) : 0 \le x \le w_j,\ 0 \le y \le h_i\}$

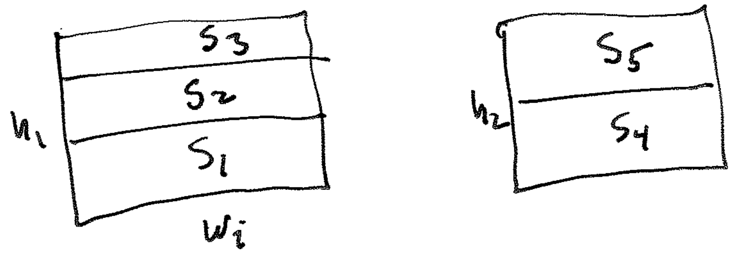
$R_i = R_{i,i}$





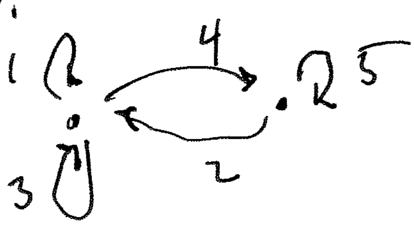
Define S and T decompositions of $R_{i,j}$.





On the bottom row of the triangle all rectangles are $R_{i,i} = R_i$.

On the second row we have rectangles of possibly 4 types. On all higher rows we have 4 types.

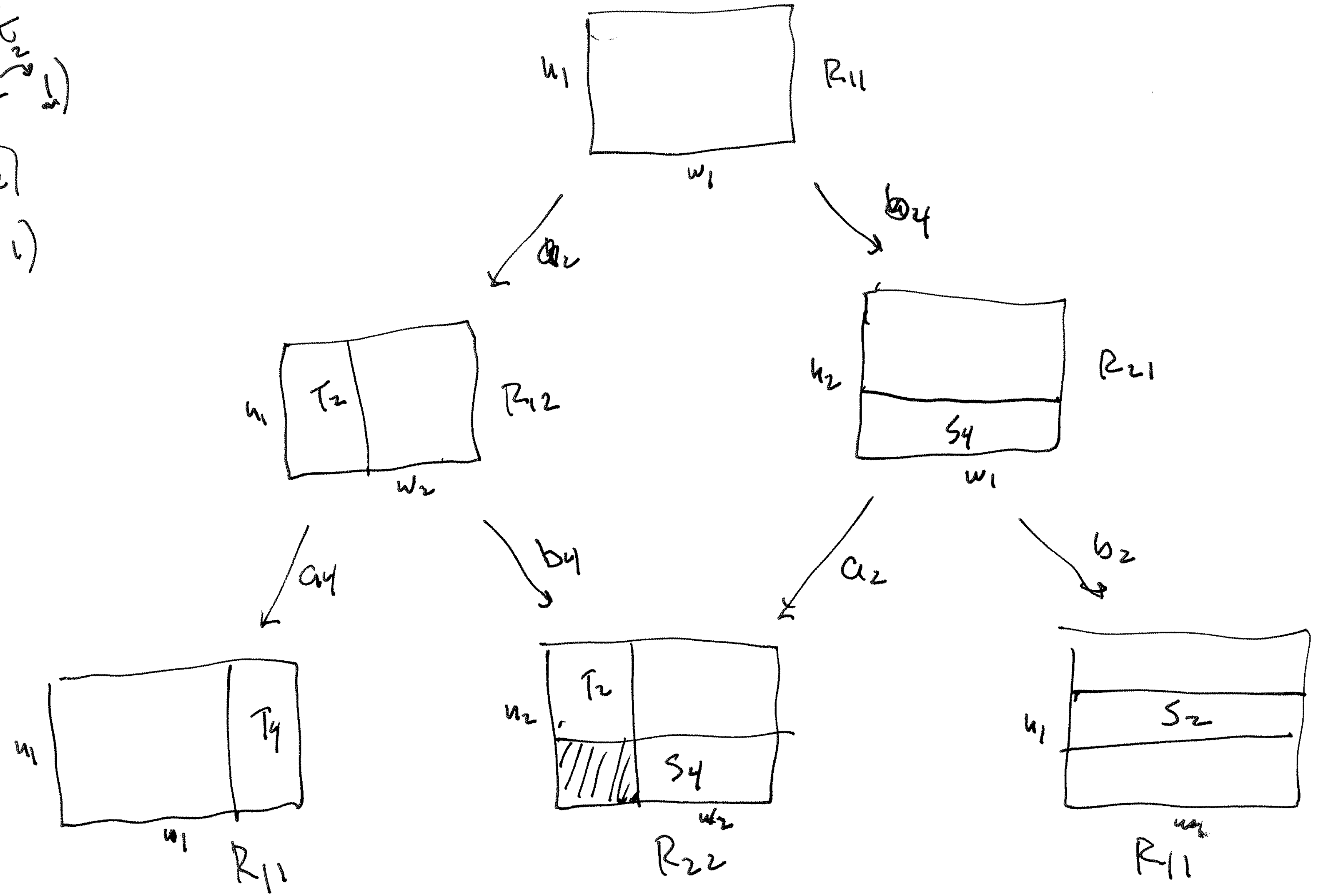


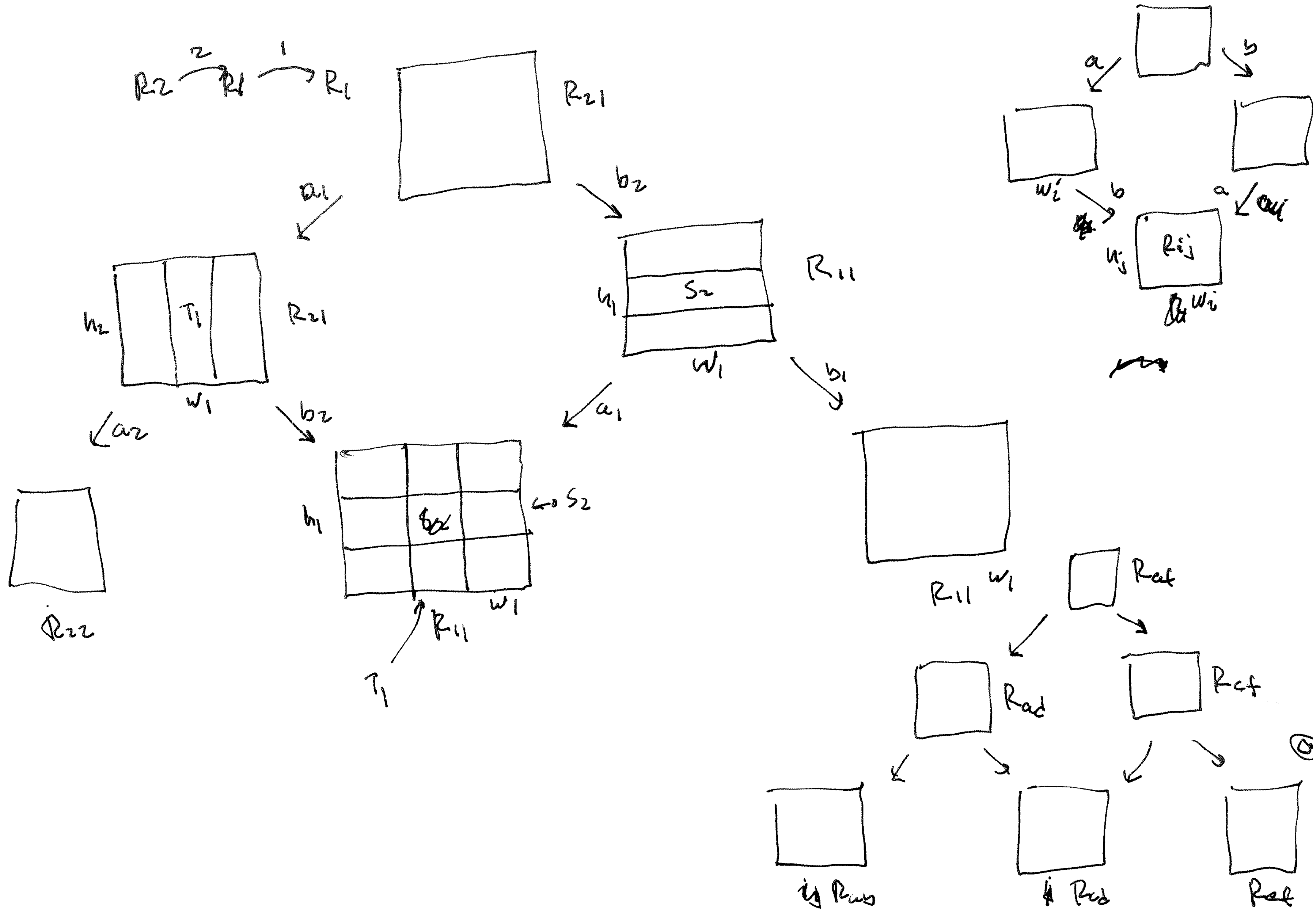
18 & 19

Construct $A(1 \xrightarrow{4} 2 \xrightarrow{2} 1)$

from $A(1 \xrightarrow{4} 2)$

and $A(2 \xrightarrow{2} 1)$





R2
R1
R21
a1
b2
T1
h2
w1
a2
R22
S2
h1
R11
b1
a
b
wi
Rij
Raf
Rad
Rcf

Lemma. Let $\omega = \omega_{-j} \ldots \omega_{-1} . \omega_0 \ldots \omega_k$ be a word in $\Sigma^c_{\mathcal{B}}$. ~~Then $A(\omega)$ is a rectangle with height h~~ where ω_{-j} starts at $R_{\alpha_{-j}}$ and ω_k ends at R_{α_k}. Then $A(\omega)$ is a rectangle with height $h_{\alpha_{-j}} \cdot (\lambda^s)^j$ and width $W_{\alpha_k} \cdot (\lambda^s)^k$.

Proof. This is true

Define a semi-conjugacy $h : \Sigma_g^e \to \mathbb{T}^2$.

$$h(\omega) = \bigcap_n A(\omega_{-n} \cdots \omega_{-1} . \omega_0 \cdots \omega_{n-1}).$$

As before the decrease in the diameter of the sets $A(\omega_{-n} \cdots \omega_{n-1})$ means that $h(\omega)$ is well defined. It also implies continuity.

The semi-conjugacy property follows from the way that f acts on words.

Cor. Periodic points of f_A are dense in $\mathbb{T}^2$ and there are ~~dense~~ dense orbits.