MA424 Example Sheet 2

13 October 2015

Let $\mathbb{S}$ denote the circle $\mathbb{R}/\mathbb{Z}$.

1. Let f be the diffeomorphism of the circle defined by $f(x) = x + 1/4 + 1/10\sin(8\pi x) \bmod 1$. Find the periodic points, the rotation number and the forward time and backward time behaviour of a typical point. Do the same for $f(x) = x + 1/2 + 1/10\sin(8\pi x) \bmod 1$.

2. Identify $\mathbb{S}$ with $\{\mathbb{R}^2 - 0\}/\mathbb{R}^+$ where $\mathbb{R}^+$ is the set of positive numbers and $\mathbb{R}^+$ acts by scalar multiplication. An invertible linear map on $\mathbb{R}^2$ gives a homeomorphism of $\mathbb{S}$. Find the rotation numbers the homeomorphisms given by the following linear transformations. Describe the forward time behaviour of a typical point.
$$\begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 & 0 \\ 0 & -1/2 \end{bmatrix}.$$

3. Let $\mathcal{O}$ be the set distinct triples in $\mathbb{S} \times \mathbb{S} \times \mathbb{S}$. Show that $\mathcal{O}$ has exactly two components. If $f : \mathbb{S} \to \mathbb{S}$ is a homeomorphism then f induces a map from $\mathcal{O}$ to itself by taking (p, q, r) to $(f(p), f(q), f(r))$. Say that f is orientation preserving if it takes each component of $\mathcal{O}$ to itself and orientation preserving otherwise. Show that f is orientation preserving if a lift is monotone increasing and orientation reversing if a lift is monotone decreasing.

4. Let $f : \mathbb{S} \to \mathbb{S}$ be orientation reversing. Prove that the equation $f(x) = x$ has exactly two solutions. (In other words show that f has exactly two fixed points).

5. Let f be an orientation preserving circle homeomorphism. Prove that all its periodic orbits (if any) have the same period.