We have looked closely at two families of hyperbolic diffeomorphisms: the Smale horseshoe and linear hyperbolic toral automorphisms.

These families have properties in common with expanding maps. In particular they have Markov partitions.

Here is another property of circle maps. Say that $f:\mathbb{R}/\mathbb{Z}\circlearrowleft$ is expanding and g is close to f in the sense that $d(f(p),g(p))<\varepsilon$ and $|f'(p)-g'(p)|<\varepsilon$ for all $p\in\mathbb{R}/\mathbb{Z}$ then f and g are topologically conjugate.

Why? The first condition implies that f and g have the same degree. The second condition implies that g is expanding. We know that two expanding maps of the circle of the same degree are topologically conjugate.

Loose definition. We say that $f, g : \mathbb{T}^2 \circlearrowleft$ are C^1 close if $d(f(p), g(p)) \leq \varepsilon$, $d(f'(p), g'(p)) \leq \varepsilon$ and the derivatives of f and g are close.

Definition. We say $f : \mathbb{T}^2 \circlearrowleft$ is structurally stable if any g which is C^1 close is topologically conjugate to f.

Theorem. If f is a linear hyperbolic diffeo then f is structurally stable.

We don't have all the tools we need to prove this result. In particular we would need to prove that if g is C^1 close to f then g has shadowing. Nevertheless we will prove that shadowing gives rise to semi-conjugacies.

Thm. Let $f = f_A : \mathbb{T}^2 \circlearrowleft$ be a hyperbolic diffeomorphism. There is a $\delta > 0$ so that ~~for~~ for any homeo $g : \mathbb{T}^2 \circlearrowleft$ with $d(f,g) \le \delta$ there is a semi-conjugacy $h : \mathbb{T}^2 \to \mathbb{T}^2$ so that:

$$\begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{g} & \mathbb{T}^2 \\ h\downarrow & & \downarrow h \\ \mathbb{T}^2 & \xrightarrow{f} & \mathbb{T}^2 \end{array}$$

Proof. f is (ε, δ) shadowing for some constants ε and δ. Let $\delta = \delta$. That is to say every δ-pseudo-orbit is ε shadowed by an actual orbit.

Let $p_0 \in \mathbb{T}^2$. Let $p_i = g^i(p_0)$. Then p_i is an δ pseudo-orbit for f:

$$d(f(p_i), p_{i+1}) = d(f(p_i), g(p_i)) \le \delta .$$

According to the shadowing property there is ~~an~~ a unique f-orbit $q_j = f^j(q_0)$ so that $d(p_j, q_j) \le \varepsilon$ for all j.

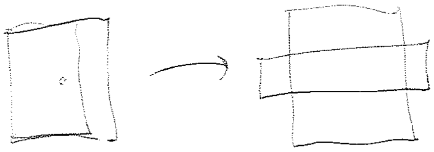
Define $h(p_0) = q_0$.

Claim: $h(g(p_0)) = f(h(p_0))$.

Let $p_i' = p_{i+1}$ and $q_i' = q_{i+1}$. Then p_i' is still a g orbit and q_i' is still an f orbit. q_i' ε-shadows p_i'. By uniqueness of the shadowing orbit q_i' is the unique f orbit that ε-shadows p_i'.
In particular $h(p_0') = q_0'$ or $h(p_1) = q_1$ so $h(g(p_0)) = f(q_0)$
$h(g(p_0)) = f(h(p_0))$.

Claim: h is continuous.

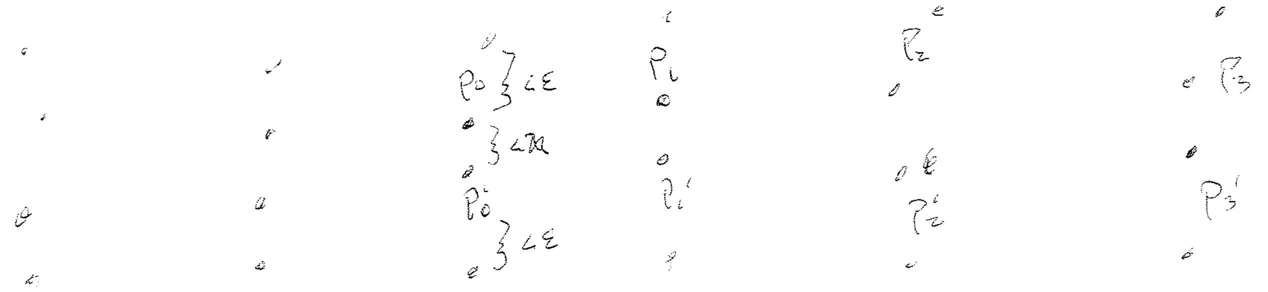
In order to this we need a fact from the proof of shadowing. If $d(f^\ell(q_0), f^\ell(q_0')) \le \varepsilon$ for $-N \le \ell \le N$ then $d(q_0, q_0') \le \frac{\varepsilon}{\lambda^N}$. Apply box construction to $q_\ell = f^\ell(q_0)$.



set $I_{-N} \cap I_N$ is a square with sides $\frac{\varepsilon}{\lambda^N}$.

Fix $N \ge 1$.
Now since g^ℓ is continuous, for any $\eta > 0$ there is a δ_1 so that $d(p, p') < \delta_N$ implies

$$d(g^\ell(p), g^\ell(p')) \le \eta \text{ for } -N \le \ell \le N.$$



Now show we can ε-shadow p_j and p_j' by f orbits q_j and q_j'.

This gives us 2 f orbits so that

$$d(q_j, q_j') \le d(q_j, p_j) + d(p_j, p_j') + d(p_j', q_j') \le 2\varepsilon + \eta.$$

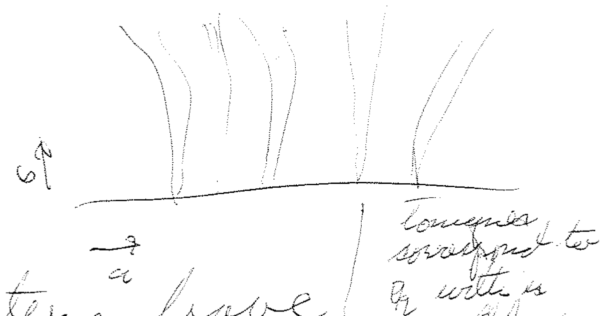
By previous argument

$$d(q_0, q_0') \le \frac{2\varepsilon+\eta}{\lambda^{|N|}}.$$

Now given $\bar{\varepsilon} > 0$ choose $\eta \le 1/4$ and N so that $\frac{2\varepsilon+\eta}{\lambda^{|N|}} \le \bar{\varepsilon}$. Let δ_N be the corresponding constant. Then $d(p, p_0') \le \delta_N \Rightarrow d(h(p), h(p')) = d(q_0, q_0') \le \bar{\varepsilon}$.

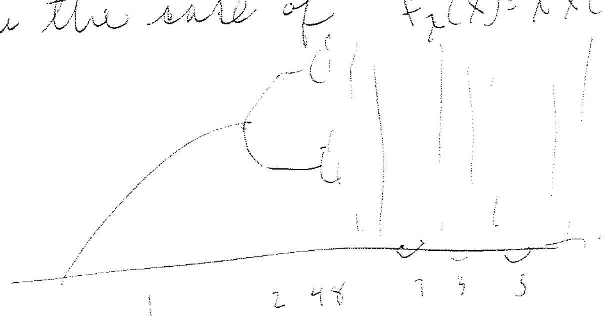
When we look at a family of dynamical systems depending on a parameter it is interesting to look at the role of "structural stability" or instability occurs. In the case of circle maps

$f_c(x) = x + a + b\sin(x/2\pi) \mod 1$



structurally stable parameters have rational rotation numbers and correspond to Arnold tongues. Rotation number was a "Devils Staircase" function: constant on a dense set.

In the case of $f_\lambda(x) = \lambda x(1-x)$ structural stability ~~corresponds~~ apparently corresponds to periodic windows. (Recall Jakobson.)



~~Inside~~ Interesting to ~~look~~ look at this family another way: extend to $\mathbb{C}$. Different but equivalent convention $f_c(z) = z^2 + c$.

A powerful way of looking at this family is to extend to $\mathbb{C}$. Using this idea many important results have been proved in the period 1980-2000.

Intro to complex dynamics.

Let $f_c : \mathbb{C} \to \mathbb{C}$ be defined by $f_c(z) = z^2 + c$.

When $c \in \mathbb{R}$ then $f_c(\mathbb{R}) = \mathbb{R}$ and we can think of f_c as an extension of the real polynomial $f_c(x) = x^2 + c$. (recall that this is conjugate to $x \mapsto \lambda x(1-x)$).

Let $K = \{ z : f_c^n(z) \not\to \infty$ as $n \to \infty \}$.
no expansion on set K.

Let $J = \partial K$
K, J are totally invariant sets for f in the strongest sense: $f(J) = J$, $f^{-1}(J) = J$, $f(K) = K$, $f^{-1}(K) = K$.

Example. $f = f_0(z) = z^2$.

If $|z| > 1$ then $|f^n(z)| = |z|^{2^n} \to \infty$ as $n \to \infty$.

If ~~$|z| < 1$ then~~ $|z| < 1$ then $|f^n(z)| \to 0$ so $f^n(z) \to 0$.

If $|z| = 1$ then $|f^n(z)| = 1$.

Thus $K = \{ z : |z| \le 1 \}$. ← ~~[illegible]~~ stable manifold of 0. Basin of the [illegible]

$J = \{ z : |z| = 1 \}$. ← f is conjugate to $m_2(x) = 2x \bmod 1$. Chaotic behavior.

Example: $f_2(z) = z^2 - 2$.

There is a semi-conjugacy h from f_0 to f_2

where $h(z) = z + z^{-1}$. $\qquad f_2(z+z^{-1}) = z^2 + 2 + z^{-2} - 2 = z^2 + z^{-2}$

$$\begin{tikzcd} z \arrow[r, "f_0"] \arrow[d, "h"'] & z^2 \arrow[d, "h"] \\ z+z^{-1} \arrow[r, "f_2"] & z^2+z^{-2} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}-\{0\} \arrow[r, "f_0"] \arrow[d, "h"'] & \mathbb{C}-\{0\} \arrow[d, "h"] \\ \mathbb{C} \arrow[r, "f_2"] & \mathbb{C} \end{tikzcd}$$

Unit circle satisfies $z^{-1} = \bar{z}$

$(x+iy)(x-iy) = x^2+y^2 = 1$

for $|z|=1 \Rightarrow h(z) = (x+iy)+(x-iy) = 2x = 2\,\mathrm{Re}(z)$.

Image of the unit circle is the real interval $[-2,2]$. $\quad \{x+iy \in \mathbb{C} : -2 \le x \le 2,\ y=0\}$.

h is surjective: Consider the equation $h(z)=w$

$z+z^{-1}=w$ or $z^2+1=wz$ or $z^2-wz+1=0$ roots: $\dfrac{w \pm \sqrt{w^2-4}}{2}$.

Every w has 2 inverse images other than ± 2.

~~One of those~~ ~~The z~~ $h(z) = h(z^{-1})$.

One of the inverse images has modulus greater than 1.

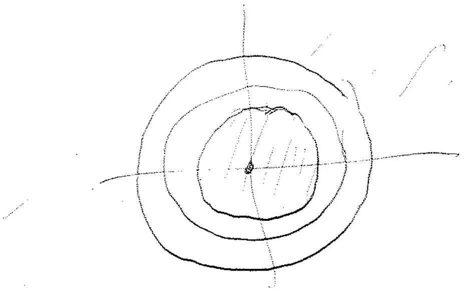
Say $h(z_0) = w \qquad |z_0| > 1$.

Now $f_0^n(z_0) \to \infty$. $\qquad h f_0^n(z_0) = f_2^n(h z_0) = f_2^n(w)$

If $|z|$ is large then $h(z) \sim z$ so $|h(z)|$ is large.

Thus $|f_2^n(w)| \to \infty$

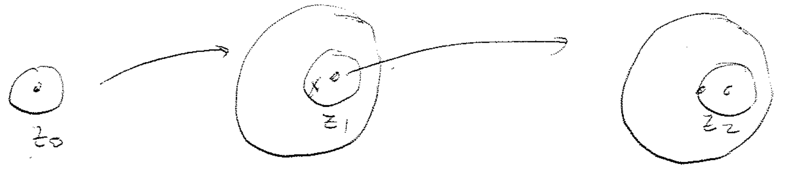
Prop. If $|c|$ is small then the Julia set of f_c is a circle ~~topologically conjugate to~~ and $f_c|J_c$ is top. conjugate to the doubling map.



$|f_c'(z)| \geq \lambda > 1$ on this annulus.

f is nearly linear on a small scale,

Both f_0 and f_c have shadowing for orbits that stay in the annulus.



Use this to construct semi-conjugacies in both directions.

~~In fact~~

~~Reader~~ Interesting property of complex dynamics is that we have a neat criterion for hyperbolicity.

Pinching occurs at certain pairs of rational rays

Curious that it can be shown that structural stability is dense but it is not known that hyperbolicity is dense. Shown to be dense on the real axis.

Much of our analysis depends on the property of being expanding.

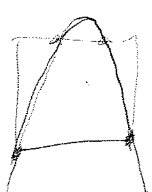
Def. Let $f: X \circlearrowleft$ where X is a compact metric space. f is expanding if there are constants $\varepsilon > 0$ and $\lambda > 1$ so that $d(f(p), f(q)) \geq \lambda d(p, q)$ when $d(p, q) \leq \varepsilon$.

The property of being expanding depends on the choice of a metric. For the standard metric on $\mathbb{R}$ expanding corresponds to $|f'| \geq \lambda > 1$.

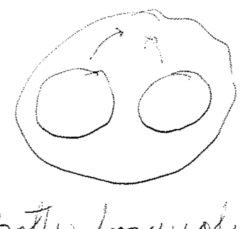
Interesting feature of complex analysis is the existence of natural metrics such as the metric $ds^2 = \dfrac{dx^2 + dy^2}{(1-|z|^2)^2}$ on the disk.

We have criteria for expansion and contraction such as the Ahlfors-Pick theorem and Montel's Thm.

Example: f_c is expanding on its Julia set if c is real and $c < -2$.



Corresponds to $r \mapsto \lambda r(1-r)$
$\lambda > 4$.
f^{-1} consists of 2 branches.



f^{-1} ~~[illegible]~~ Both branches of f^{-1} are contracting.