Corollary. A circle homeomorphism has a rational ~~k~~rotation number iff it has a periodic orbit.

Proof. If f has a periodic orbit then $f^n(p)=p$ so $\rho(f^n)=0$. So $n\rho(f)=\rho(f^n)=0 \mod 1$. Thus $n\rho(f)=m$ and $\rho(f)=\frac{m}{n}$.

If $\rho(f)=\frac{m}{n}$ then $\rho(f^n)=0 \mod 1$ so f^n has a fixed point and $f^n(p)=p$ for some p.

What does it mean to say that two dynamical systems are the same?

Definition. Let $f^t : X \circlearrowleft$ and $g^t : Y \circlearrowleft$ be dynamical systems with $t \in T$ $(= \mathbb{Z}, \mathbb{Z}_+, \mathbb{R})$.
Then f and g are topologically conjugate if there is a homeomorphism $h : X \to Y$ so that $g^t \circ h = h \circ f^t$ and h is invertible. ~~[illegible]~~ for all $t \in T$, i.e. the following square commutes:

$$\begin{tikzcd} X \arrow[r, "f^t"] \arrow[d, "h"] & X \arrow[d, "h"] \\ Y \arrow[r, "g^t"] & Y \end{tikzcd}$$

The notion of topological conjugacy defines an equivalence relation on dynamical systems.

If h is invertible we also have $g^t = h \circ f^t \circ h^{-1}$.

If h is not invertible then we say h is a semi-conjugacy.

~~If our set of time~~ If time is discrete then

$f^n = f \circ \ldots \circ f$ and it suffices to check that

$$h \circ f = g \circ h, \qquad \begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

since
$$\begin{aligned} h \circ f^n &= h \circ f \circ f \ldots \circ f \\ &= g \circ h \circ f \ldots \circ f \\ &= g \circ g \circ h \circ \ldots \circ f \\ &= g \circ \ldots \quad g^n. \end{aligned}$$

A conjugacy takes orbits for f to orbits for g.

If $\mathcal{O}_f(x) = \{f^t(x)\}$ then $h(\mathcal{O}_f(x)) = \{hf^t(x)\} = \{g^t h(x)\}$

$$= \mathcal{O}_g(h(x)).$$

A conjugacy takes closed orbits for f to closed orbits for g; If $f^p(x) = x$ then $g^p(h(x)) = h f^p(x) = h(x)$

so $h(x)$ is ~~a clos~~ has period p for g.

A conjugacy takes dense orbits to dense orbits so if f is minimal then g is minimal.

We don't have any good examples of
$h(x) = -x$ R_α is conjugate to $R_{-\alpha}$.
conjugacies yet. We can construct a "false" example. Let h be a homeomorphism of $\mathbb{R}/\mathbb{Z}$. Then R_α and ~~the~~ $hR_\alpha h^{-1}$ are conjugate. Note that $hR_\alpha h^{-1}$ need not be a rotation.

Question. Given $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ when is f conjugate to a rotation?

If f^t and g^t are conjugate then if f has a periodic point of period p then g has a periodic point of period p.

If f is minimal then g is minimal.

It is interesting to know when two circle homeomorphisms are topologically conjugate or not.

Proposition. If circle homeomorphisms f and g are topologically conjugate via a circle homeomorphism h then $\rho(f) = \rho(g)$ if h is orientation preserving and $\rho(f) = -\rho(g)$ if h is orientation reversing.

Example. Say $h(x) = -x$ then $R_\alpha = h R_{-\alpha} h^{-1}$ since $h R_{-\alpha} h^{-1}(x) = h R_{-\alpha}(-x) = h(-x-\alpha) = x+\alpha = R_\alpha(x)$.

~~Proposition~~ Proof. Say that f and g are orientation preserving homeomorphisms of $\mathbb{R}/\mathbb{Z}$ and $g = h \circ f \circ h^{-1}$. Let F and H be lifts of f and h. Then $G = H \circ F \circ H^{-1}$ is a lift of g. Let $x = H(0)$ then

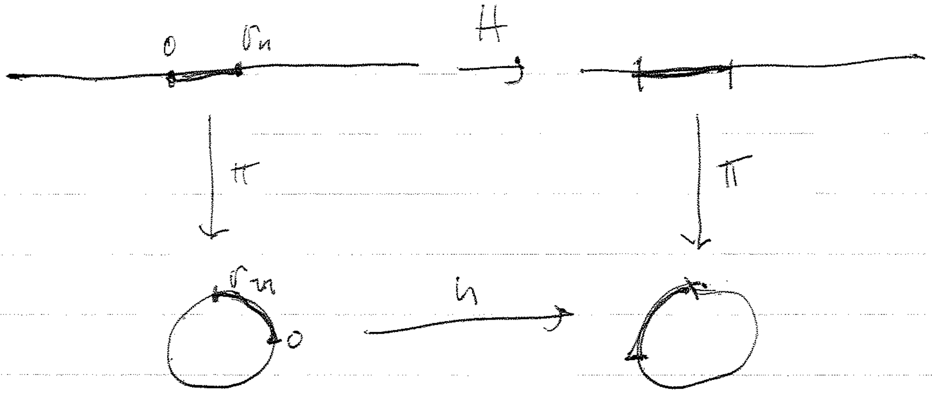
$$\frac{G^n(x)}{n} = \frac{H \circ F^n \circ H^{-1}(x)}{n} = \frac{H \circ F^n(0)}{n}$$

Say $F^n(0) = k_n + v_n$ with $|v_n| < 1$. $H(F^n(0)) = H(k_n + v_n)$.

Recall $H(x+1) = H(x) \pm 1$ with a $+$ if H is or. pres. and a $-$ if H is or. reversing.

So $H(k_n + v_n) = H(v_n) \pm k_n$.

Claim: $|H(v_n)| < 1$.



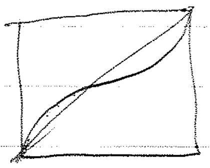
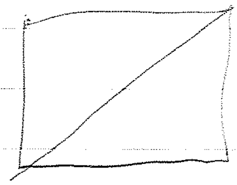
Thus $\dfrac{H(F^n(0))}{n} = \dfrac{H(k_n + r_n)}{n} = \dfrac{H(r_n) \pm k_n}{n}$.

$$\rho(G) = \lim_{n \to \infty} \frac{G^n(x)}{n} = \pm \lim_{n \to \infty} \frac{k_n}{n} = \pm \rho(F).$$

Corollary. R_α and R_β are topologically conjugate if and only if $\alpha = \pm\beta$.

Does something similar hold for circle diffeomorphisms?

If $\rho(f) = \rho(g) = 0$ then it need not be the case that f and g are conjugate.

graph of f  $g = \text{id}$. 

Theorem (Poincaré) A minimal homeomorphism is topologically conjugate to an irrational rotation.

We begin by proving a Lemma.

Let F be a lift of f where $\rho(F) = \rho_0$ is irrational. Let $\Lambda_1 = \{ F^n(x_0) + m : m, n \in \mathbb{Z} \}$

$$\Lambda_2 = \{ n\rho + m : m, n \in \mathbb{Z} \}.$$

Define $\tau : \Lambda_1 \to \Lambda_2$ by $\tau(F^n(x_0) + m) = n\rho + m$.

Then τ is strictly increasing and

$$\tau(x+1) = \tau(x) + 1 \quad , \quad \tau(F(x)) = \tau(x) + \rho \quad \text{for all } x \in \Lambda_1$$