MA424 Dynamical Systems
Notes for Term I 2013

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Contents

1. Introduction

These notes are based on lectures given at the Mathematical Institute, University of Warwick in the Autumn 2012. The author is grateful to Sebastian van Strien, Mark Pollicott and Davoud Cheraghi for copies of their own notes and exercises. The lectures follow the traditional syllabus and are aimed at explaining mathematical tools and methods used in Dynamical Systems.

A dynamical system is defined by three objects $(X, \mathbf{T}, f^t)$. The set X is called the phase space and is used to describe the state of the dynamical system. The family of maps $f^t : X \to X$ describes the evolution of the system with time. The variable $t \in \mathbf{T}$ is called the time. In the theory of dynamical systems the time $\mathbf{T}$ is usually one of the following: $\mathbb{Z}, \mathbb{Z}_+, \mathbb{R}, \mathbb{R}_+$. It is natural to distinguish the cases of the discrete and continuous time. In the latter case f^t is called a flow.

The phase space X is usually equipped with some additional structures. For example, X can be a topological space, a metric space, a differentiable manifold, or a measure space. In these lectures we will discuss many examples including $X = [0,1]$, $\mathbb{S}^1$, $\mathbb{R}^2$, $\mathbb{T}^2$ and $\mathbb{S}^2$.

The family of maps $f^t : X \to X$ must satisfy the following properties:

(1) $f^{t_1+t_2} = f^{t_1} \circ f^{t_2}$
(2) $f^0 = \mathrm{id}$
(3) f^t respects structures on X

Among various structures, a measure on X plays a distinguished role. We will not discuss the corresponding class of measure-preserving dynamical systems as this topic will be discussed in Ergodic Theory taught in Term 2.

In Dynamical Systems we will study trajectories of points: Let $x \in X$, then the *trajectory* of x is the set

$$O_x = \{\, f^t(x) : t \in \mathbf{T} \,\}.$$

If the time is discrete, e.g., $\mathbf{T} = \mathbb{Z}$, then

$$O_x = \{\, f^n(x) : n \in \mathbb{Z} \,\},$$

where f^n stands for the composition:

$$f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}} \qquad \text{and} \qquad f^{-n} = \underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{n \text{ times}}$$

and we will speak about a dynamical system generated by iterations of the map $f = f^1$.

CHAPTER 1

One Dimensional Dynamical Systems

1. Circle maps

In this section we study properties of dynamical systems generated by iterations of a circle homeomorphism $f : \mathbb{S} \to \mathbb{S}$. So the phase space $X = \mathbb{S}$ is a circle. The time is discrete, i.e., $t \in \mathbb{Z}$. For any point $x \in \mathbb{S}$, its trajectory is the set

$$O_x = \{ f^n(x) : n \in \mathbb{Z} \},$$

where f^n stands for the composition:

$$f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}} \qquad \text{and} \qquad f^{-n} = \underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{n \text{ times}}.$$

1.1. Preliminary information. We assume that the circle $\mathbb{S} = \mathbb{R}/\mathbb{N}$, i.e. $\mathbb{S}$ is a factor-space of $\mathbb{R}$ with respect to the following equivalence relation: $x \sim y$ iff $x - y$ is integer. Then each point of the circle is considered as an equivalence class $x \pmod 1$ of a point $x \in \mathbb{R}$. We define the natural projection $\pi : \mathbb{R} \to \mathbb{S}$ setting $\pi(x) = x \pmod 1$. From the definition it follows that $\pi(x + j) = \pi(x)$ for any $x \in \mathbb{R}$ and any $j \in \mathbb{Z}$.

The restriction of the projection $\pi : (0, 1) \to \mathbb{S} \setminus \{0\}$ is a bijection, therefore sometimes it is convenient to think about $\mathbb{S}$ as the interval $[0, 1]$ assuming that 0 and 1 represent a single point in $\mathbb{S}$.

We note that the circle has two possible orientations: clockwise and anticlockwise. We say that the anticlockwise direction on the circle is "positive direction".

The circle is a compact metric space. Any two points $x, y \in \mathbb{S}$ with $x \neq y$ define two (closed) arcs which we denote by $[x, y]$ and $[y, x]$ respectively (taking the points in the positive direction on the circle). Then we can define the metric on $\mathbb{S}$ by $\mathrm{dist}(x, y) = \min\Big\{ \big|[x, y]\big|, \big|[y, x]\big| \Big\}$, i.e. the distance equals to the length of the shortest of the two arcs defined by x, y. It is easy to see, that dist satisfies the axioms of a metric (it is non-negative, non-degenerate, symmetric, and satisfies the triangle inequality).

The metric is used to define the notion of a continuous map on the circle.

A homeomorphism $f : \mathbb{S} \to \mathbb{S}$ either preserves or reverses the orientation of the circle. This property can be described in the following way: take any three distinct points $x, y, z \in \mathbb{S}$ located in the positive (anticlockwise) order on the circle (so if you start from x and move anticlockwise you first meet y and then z), then $f(x), f(y), f(z)$ can be either in the original order or not. It can be easily checked that the answer is independent of the choice of the points and, consequently, characterises the homeomorphism f.

1.2. Rotations. Let $\alpha \in \mathbb{R}$. A rotation $R_\alpha : \mathbb{S} \to \mathbb{S}$ is defined by

$$R_\alpha(x) = x + \alpha \pmod 1.$$

Exercise. Show that R_α is a homeomorphism.

Obviously,

$$\begin{aligned} R_\alpha(x) &= x + \alpha \pmod 1 \\ R_\alpha^2(x) &= x + 2\alpha \pmod 1 \\ R_\alpha^m(x) &= x + m\alpha \pmod 1 \qquad \forall m \in \mathbb{Z} \end{aligned}$$

The properties of the trajectory depend on arithmetic properties of α.

Rational Rotations: $\alpha \in \mathbb{Q}$ ($\Leftrightarrow \alpha = \frac{q}{p}$, $q \in \mathbb{Z}, p \in \mathbb{N}$). Since $\alpha = \frac{q}{p}$ we see

$$R_\alpha^p(x) = x + p\alpha = x + q = x \pmod 1$$

So $R_\alpha^p = \mathrm{id}$ and all trajectories are finite:

$$O_x = \{x, x + \alpha, \ldots, x + (p-1)\alpha\} \qquad \forall x \in \mathbb{S}$$

Irrational rotations: $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

PROPOSITION 1. *If α is irrational, then for any x its trajectory O_x is dense in $\mathbb{S}$.*

PROOF. Using the pigeonhole principle: for any $\varepsilon > 0$ there are numbers $0 \le k < l \le 1 + \varepsilon^{-1}$ such that $\mathrm{dist}(R_\alpha^k(x), R_\alpha^l(x)) < \varepsilon$.

Let $m = l - k$. Since rotations preserve distances:

$$\mathrm{dist}(x, R_\alpha^m(x)) = \mathrm{dist}(R_\alpha^k(x), R_\alpha^l(x)) < \varepsilon.$$

Thus R_α^m is a rotation by an angle less than ε.

Then for any $y \in \mathbb{S}$ there is $n \in \mathbb{N}$ such that $\mathrm{dist}(y, R_\alpha^{nm}(x)) < \varepsilon$. Consequently the set of $R_\alpha^j(x)$, $j \in \mathbb{N}$, is dense in $\mathbb{S}$. $\square$

1.3. Lifts.

PROPOSITION 2. *If $f : \mathbb{S} \to \mathbb{S}$ is a homeomorphism, then there is a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ F = f \circ \pi$ where $\pi : \mathbb{R} \to \mathbb{S}$ is the natural projection.*

Proof. Exercise.

Definition. F is called a *lift of f*.

The definition can be illustrated by a commutative diagram: F is a homeomorphism such that the following diagram is commutative

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"] & \mathbb{R} \arrow[d, "\pi"] \\ \mathbb{S} \arrow[r, "f"] & \mathbb{S} \end{tikzcd}$$

The lift has the following properties:

1. F is unique up to adding an integer
2. If f preserves orientation, then F is strictly increasing and

$$F(x+1) = F(x) + 1.$$

(3) If F is a lift of f, then F^n is a lift of f^n for any $n \in \mathbb{Z}$.
(4) If $F : \mathbb{R} \to \mathbb{R}$ is a strictly monotone map such that $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$, then F is a lift of a circle homeomorphism f. Moreover, f preserves orientation.

Example. $F(x) = x + \alpha$ is a lift of the rotation R_α.

1.4. Rotation numbers. Let $f : \mathbb{S} \to \mathbb{S}$ be a homeomorphism and F be a lift of f. Then

$$\rho = \lim_{n\to\infty} \frac{F^n(x)}{n} \pmod 1$$

is called a *rotation number of* f. We will see that the limit exists and depends on the choice of F. Nevertheless, the rotation number ρ is unique as it is defined modulo 1.

The rotation number describes an "average rotation angle" for homeomorphism f.

Example. For any rotation $\rho(R_\alpha) = \alpha \pmod 1$.

Proposition 3. *If $f : \mathbf{S} \to \mathbf{S}$ is an orientation-preserving homeomorphism, F is its lift and $x \in \mathbb{R}$, then the limit*

$$\lim_{n\to\infty} \frac{F^n(x)}{n}$$

exists and is independent of x. Moreover, $\rho \pmod 1$ is independent of the choice of the lift F.

Proof. **(1) Existence of the limit for $x = 0$.**

Let $k_n = [F^n(0)]$ (we use the notation $[x]$ to denote the largest integer k such that $k \le x$), so $k_n \le F^n(0) \le k_n + 1$. Then for any $m \in \mathbb{N}$

$$mk_n \le F^{mn}(0) \le m(k_n + 1)\,.$$

Indeed, the statement is true for $m = 1$. Suppose it is true for some m. Then

$$\begin{aligned} F^{(m+1)n}(0) &= F^{mn}(F^n(0)) \ge F^{nm}(k_n) = F^{nm}(0) + k_n \ge mk_n + k_n \\ F^{mn}(F^n(0)) &\le F^{nm}(k_n + 1) = F^{nm}(0) + k_n + 1 \le m(k_n + 1) + k_n + 1 \end{aligned}$$

i.e., the inequality also true for m replaced by $m + 1$. The induction in m implies the inequality for all $m \in \mathbb{N}$.

Dividing by mn we get

$$\frac{F^{nm}(0)}{nm} \in \left[\frac{k_n}{n}, \frac{k_n + 1}{n}\right]\,.$$

Since the interval is independent of m, we conclude that

$$\left|\frac{F^n(0)}{n} - \frac{F^{nm}(0)}{nm}\right| \le \frac{1}{n}\,.$$

Now for any $n, m \in \mathbb{N}$ we get

$$\left|\frac{F^n(0)}{n} - \frac{F^m(0)}{m}\right| \le \left|\frac{F^n(0)}{n} - \frac{F^{nm}(0)}{nm}\right| + \left|\frac{F^m(0)}{m} - \frac{F^{nm}(0)}{nm}\right| \le \frac{1}{n} + \frac{1}{m}\,.$$

Take any $\varepsilon > 0$. Then for any $m, n > \frac{2}{\varepsilon}$ we get

$$\left| \frac{F^n(0)}{n} - \frac{F^m(0)}{m} \right| < \varepsilon\,.$$

Thus $\frac{F^n(0)}{n}$ is a Cauchy sequence, hence convergent.

(2) The limit exists for all x and is independent of x. Indeed, let $x \in \mathbb{R}$. The function F^n is monotone increasing for any $n \in \mathbb{N}$. Then for any $y \in [0,1]$

$$0 \le F^n(y) - F^n(0) \le F^n(1) - F^n(0) = 1.$$

Then take any $x \in \mathbb{R}$ and let $y = x - [x]$. Since $y \in [0,1)$ we get

$$\left|F^n(x) - F^n(0)\right| = \left|F^n(y) - F^n(0) + [x]\right| \le 1 + \left|[x]\right|.$$

Dividing by n we get

$$\left| \frac{F^n(x)}{n} - \frac{F^n(0)}{n} \right| \le \frac{1 + \left|[x]\right|}{n}.$$

The right hand side converges to 0 when $n \to \infty$. So the limit of $\frac{F^n(x)}{n}$ exists and

$$\lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{F^n(0)}{n}.$$

(3) Dependence of ρ on the lift. Let $\tilde{F}$ be another lift of f. Then there is a number $j \in \mathbb{Z}$ such that $\tilde{F}(x) = F(x) + j$ for all x. Then check by induction that $\tilde{F}^n(x) = F^n(x) + nj$ and consequently

$$\frac{\tilde{F}^n(x)}{n} = \frac{F^n(x) + nj}{n} = \frac{F^n(x)}{n} + j \to \rho + j\,.$$

Consequently, $\rho(\tilde{F}) = \rho(F) + j$ and, in particular, $\rho \mod 1$ is independent of the lift. The proposition is proved. $\square$

1.5. Periodic trajectories. A point $p \in \mathbb{S}$ is called periodic if $p = f^n(p)$ for some $n \in \mathbb{N}$. The smallest n is called *the period*, the trajectory of a periodic point is called a *periodic trajectory.* If $n = 1$, the periodic point is called a *fixed point of f.*

PROPOSITION 4. *Let f be a homeomorphism on the circle. The rotation number of f is integer if and only if it has a fixed point.*

Proof. If f reverses orientation, it has a fixed point and its rotation number is zero (Exercise). So we need to consider the orientation preserving case only. Let f be an orientation-preserving homeomorphism.

$\Leftarrow$). Let $p = f(p)$ and $x \in \mathbb{R}$ be such that $\pi(x) = p$. Let F be a lift of f. Since $f(p) = p$ we have $F(x) = x + m$ for some $m \in \mathbb{Z}$. Then $F^2(x) = F(x+m) = x + 2m$ and by induction $F^n(x) = x + nm$. Using the definition of the rotation number we get

$$\rho(F) := \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{x + nm}{n} = m\,,$$

which is integer.

$\Rightarrow$). Let $\tilde{F}$ be a lift of f and $\rho(\tilde{F}) = m \in \mathbb{Z}$. Then $F = \tilde{F} - m$ is also a lift of f. Moreover, $\rho(F) = \rho(\tilde{F}) - m = 0$ (see the last step in the previous proof).

Consider the iterates of the zero: Since F is increasing, the sequence $F^n(0)$ is monotone (it is increasing if $F(0) > 0$ and decreasing otherwise). Suppose the sequence is unbounded. Then there is $n_0 \in \mathbb{N}$ such that $|F^{n_0}(0)| > 1$. Then $|F^{mn_0}(0)| > m$ for all $m \in \mathbb{N}$. Dividing by mn, we get

$$\frac{|F^{mn_0}(0)|}{mn_0} > \frac{m}{mn_0} = \frac{1}{n_0}\,.$$

This contradicts to $\rho(F) = \lim_{n\to\infty} \frac{F^n(0)}{n} = 0$. Therefore $F^n(0)$ is bounded.

A bounded monotone sequence has a limit. So there is $x_* = \lim_{n\to\infty} F^n(0)$. Since F is continuous we can swap F and lim to get

$$F(x_*) = F\left(\lim_{n\to\infty} F^n(0)\right) = \lim_{n\to\infty} F^{n+1}(0) = x_*\,.$$

Let $p_* = \pi(x_*) = x_* \pmod 1$. It follows $f(p_*) = p_*$, so p_* is a fixed point of f. $\Box$.

Corollary 5. *A circle homeomorphism has a rational rotation number if and only if it has a periodic orbit. More precisely,* $\rho(f) = \frac{m}{n}$ *if and only if there is* $p \in \mathbb{S}$ *such that* $f^n(p) = p$.

Proof. From the definition of the rotation number it follows that $\rho(f^n) = n\rho(f)$ where $n \in \mathbb{N}$. Then note that a periodic point of period n is a fixed point of f^n and apply Proposition 4. $\Box$

Exercises:

(1) Let f be a circle homeomorphism. Prove that if its rotation number $\rho \in \mathbb{Q}$, then any point $x \in \mathbb{S}$ is either periodic or converges to some periodic orbit, i.e. there is a periodic point p such that

$$\lim_{n\to\infty} \operatorname{dist}(f^n(x), f^n(p)) = 0.$$

(2) Let f be an orientation preserving circle homeomorphism. Prove that all its periodic orbits have the same period.
(3) Let f be an orientation reversing circle homeomorphism. We have seen it has exactly two fixed points. Can it have periodic orbits of other periods? Which periods are possible? Find examples.
(4) Given $\rho(f)$. Find $\rho(f^{-1})$. /Hint: F^{-1} is a lift for f^{-1}/

1.6. Topological equivalence. Let $f : X \to X$ and $g : Y \to Y$ be homeomorphisms of some topological spaces X and Y respectively. We say that f and g are *topologically conjugate* if there is a homeomorphism $h : X \to Y$ such that $g = h \circ f \circ h^{-1}$. In other words, the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

The homeomorphism h is called a *topological conjugation* between f and g.

The topological conjugacy introduces an equivalence relation among dynamical systems. This relation is called *topological equivalence.*

We note that h maps trajectories of f into trajectories of g. In particular, fixed points of f are mapped into fixed points of g and periodic trajectories are mapped into periodic trajectories of the same period.

Let $f, g : \mathbb{S} \to \mathbb{S}$ be two circle homeomorphisms. They are topologically conjugate, if there is a homeomorphism $h : \mathbb{S} \to \mathbb{S}$ such that $h \circ f = g \circ h$, i.e., the following diagram is commutative:

$$\begin{tikzcd} \mathbb{S} \arrow[r, "f"] \arrow[d, "h"'] & \mathbb{S} \arrow[d, "h"] \\ \mathbb{S} \arrow[r, "g"] & \mathbb{S} \end{tikzcd}$$

We can say that g is obtained from f by changing a parameterisation (i.e. the angle variable) on the circle.

Proposition 6. *Let circle homeomorphisms f and g be topologically conjugate. If the topological conjugation preserves orientation, then $\rho_f = \rho_g \pmod 1$, otherwise $\rho_f = -\rho_g \pmod 1$.*

Proof. Since f and g are topologically conjugate, there is a homeomorphism h such that $g = h \circ f \circ h^{-1}$. Let F and H be lifts of f and h respectively. Then $G = H \circ F \circ H^{-1}$ is a lift of g. Let $k_n = [F^n(0)]$ and $x = H(0)$. Then

$$\frac{G^n(x)}{n} = \frac{H \circ F^n \circ H^{-1}(x)}{n} = \frac{H \circ F^n(0)}{n} = \frac{H(F^n(0) - k_n) \pm k_n}{n}.$$

The sign is positive if H is increasing and negative if H is decreasing. Taking into account that $|F^n(0) - k_n| \le 1$ we can take the limit $n \to \infty$ and conclude that

$$\rho(G) = \lim_{n\to\infty} \frac{G^n(x)}{n} = \pm \lim_{n\to\infty} \frac{k_n}{n} = \pm \lim_{n\to\infty} \frac{F^n(0)}{n} = \pm\rho(F)\,.$$

$\square$

In particular we get the following corollary: Two rotations R_α and R_β are topologically conjugate if and only if $\alpha = \pm\beta \pmod 1$. Note that the map $x \mapsto -x \pmod 1$ topologically conjugates R_α and $R_{-\alpha}$.

1.7. Poincaré's Theorem. Definition. A homeomorphism is called *minimal*, if every orbit is dense in the phase space.

Example. Any irrational rotation R_α is minimal.

Let $h : \mathbb{S} \to \mathbb{S}$ be a homeomorphism which preserves orientation. The following commutative diagram defines a circle homeomorphism f:

$$\begin{tikzcd} \mathbb{S} \arrow[r, "f"] \arrow[d, "h"'] & \mathbb{S} \arrow[d, "h"] \\ \mathbb{S} \arrow[r, "R_\alpha"'] & \mathbb{S} \end{tikzcd}$$

In other words, $f := h^{-1} \circ R_\alpha \circ h$ is a circle homeomorphism. Moreover, Proposition 6 implies that $\rho(f) = \rho(R_\alpha) = \alpha \pmod 1$. Moreover, f is minimal as h maps a dense set into a dense set.

THEOREM 7 (Poincaré's Theorem). *Any minimal circle homeomorphism is topologically conjugate to an irrational rotation.*

Before proceeding to the proof of the theorem we state and prove a statement which will be useful more than once (=twice). Let us define the sets $\Lambda_{x_0}, \Omega \subset \mathbb{R}$ by

$$\begin{aligned} \Lambda_{x_0} &= \Big\{F^n(x_0) + m : m, n \in \mathbb{Z}\Big\}, \\ \Omega &= \Big\{n\rho + m : m, n \in \mathbb{Z}\Big\}. \end{aligned}$$

We note that $\Lambda_{x_0} = \pi^{-1}\{f^n(\pi x_0) : n \in \mathbb{Z}\}$ and $\Omega = \pi^{-1}\{R_\rho^n(0) : n \in \mathbb{Z}\}$, where π is the natural projection $\pi : \mathbb{R} \to \mathbb{S}$.

LEMMA 8. *Let f be a circle homeomorphism and $x_0 \in \mathbb{S}$. If the rotation number ρ is irrational, then the map $T : \Lambda_{x_0} \to \Omega$ defined by*

$$T(F^n(x_0) + m) = n\rho + m$$

is a bijection. Moreover, T is strictly increasing,

$$T(x+1) = T(x) + 1 \quad \text{and} \quad T(F(x)) = T(x) + \rho \qquad \text{for all } x \in \Lambda_{x_0}.$$

PROOF OF LEMMA 8. Since ρ is irrational, neither f nor R_ρ have any periodic point. Consequently, the maps $\mathbb{Z}^2 \to \Lambda_{x_0}$ and $\mathbb{Z}^2 \to \Omega$, defined by

$$(m, n) \mapsto F^n(x_0) + m \qquad \text{and} \qquad (m, n) \mapsto n\rho + m$$

respectively, are bijective. Therefore T is a bijection as a composition of two bijections.

Now take any $x_1, x_2 \in \Lambda_{x_0}$ such that $x_1 < x_2$. There are $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ such that $x_j = F^{n_j}(0) + m_j$ for $j = 1, 2$. So we have

$$F^{n_1}(0) + m_1 < F^{n_2}(0) + m_2\,.$$

Let $y = F^{n_2}(0)$. Then

$$F^{n_1 - n_2}(y) < y + m_2 - m_1\,.$$

Exercise: Let $F^n(y) < y + m$ for some $y \in \mathbb{R}$, $n, m \in \mathbb{Z}$. If $n > 0$ then $\rho(F) < \frac{m}{n}$, and if $n < 0$ then $\rho(F) > \frac{m}{n}$.[1]

Consequently, if $n_1 > n_2$ then $\rho < \frac{m_2 - m_1}{n_1 - n_2}$ and if $n_1 < n_2$ then $\rho > \frac{m_2 - m_1}{n_1 - n_2}$. In both cases we obtain

$$n_1\rho + m_1 < n_2\rho + m_2\,.$$

So $T(F^{n_1}(0) + m_1) < T(F^{n_2}(0) + m_2)$ and T is indeed strictly increasing.

[1]Case $n < 0$. Let $z = F^n(y)$. Then $F^n(y) < y + m$ implies $z < F^{|n|}(z) + m$ so $F^{|n|}(z) > z - m$. Applying the inequality several times, we get $F^{k|n|}(z) > z - km$ using induction:

$$F^{k|n|}(z) = F^{(k-1)|n|}(F^{|n|}(z)) > F^{(k-1)|n|}(z - m) = F^{(k-1)|n|}(z) - m > z - km\,.$$

Then $\frac{F^{k|n|}(z)}{k|n|} > \frac{z - km}{k|n|}$, and taking the limit we get $\rho \geq \frac{m}{n}$. An equality is not possible as ρ is irrational.

Let $x = F^n(x_0) + m$ then $T(x) = n\rho + m$. So

$$\begin{aligned} T(x+1) &= T(F^n(x_0)+m+1) = n\rho + m + 1 = T(x) + 1 \\ T \circ F(x) &= T(F^{n+1}(x_0) + m) = (n+1)\rho + m = T(x) + \rho \end{aligned}$$

It follows directly that $T(x+1) = T(x) + 1$ and $T(F(x)) = T(x) + \rho$ for all $x \in \Lambda_{x_0}$. □

Now we are ready to prove Poincaré's theorem.

Proof of Poincaré's Theorem. Since f is minimal, it has no periodic points (periodic orbits are finite sets, hence not dense in $\mathbb{S}$). Consequently, the rotation number ρ is irrational. We construct a lift of a homeomorphism which conjugates f and R_ρ.

Let F be a lift of f, $x_0 \in \mathbb{R}$ and $\Lambda = \Lambda_{x_0}$. Since both $\pi\Omega$ and $\pi\Lambda_{x_0}$ are dense in $\mathbb{S}$ (Λ_{x_0} due to the minimality of f and Ω due to Proposition 1), Ω and Λ_{x_0} are both dense in $\mathbb{R}$. Moreover $T : \Lambda \to \Omega$ is strictly increasing by Lemma 8. Consequently, there is a unique continuous function $H : \mathbb{R} \to \mathbb{R}$ such that $H\big|_{\Lambda_{x_0}} = T$. Moreover, H is strictly increasing, and both H and H^{-1} are continuous.[2]

Finally, the continuity implies that H inherits properties of T:

$$H(x+1) = H(x) + 1 \qquad \text{and} \qquad H \circ F(x) = H(x) + \rho \qquad \text{for all } x \in \mathbb{R}.$$

The first property implies that H is a lift of a circle homeomorphism h and the second one implies the identity $h \circ f = R_\rho \circ h$. □

We just proved that if f is a minimal homeomorphism of the circle, there is a homeomorphism h which solves the equation

$$h(f(x)) = h(x) + \rho \pmod 1 .$$

This is a linear equation on the function h. Its solution is defined up to adding a solution of the corresponding homogeneous equation. Indeed, if $\tilde{h}$ is another solution of the same equation, then $h_0 = h - \tilde{h}$ satisfies the homogeneous equation

$$h_0(f(x)) = h_0(x) \pmod 1.$$

The equation implies that h_0 is constant on a trajectory of x. Since the trajectory is dense and h_0 is continuous, we conclude that h_0 is constant. So a solution of the original equation is unique up to adding a constant. This freedom can be used to chose which point in $\mathbb{S}$ is mapped to zero.

In particular, it follows that the choice of x_0 in the proof defines that map h uniquely.

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[2]Exercise: Let $X, Y \subset \mathbb{R}$ be dense and $\tilde{f} : X \to Y$ be a strictly monotone bijection. Then there is a unique homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f\big|_X = \tilde{f}$.

1.8. Denjoy's Theorem. Before applying Poincaré's theorem to a homeomorphism f, we need to check if f is minimal or not. A direct proof of minimality is not always straightforward. The next theorem shows that the minimality property can be deduced from smoothness properties of the map.

Definition. We say that a continuous function $g : [0,1] \to \mathbb{R}$ has bounded variation if its variation,

$$\mathrm{var}(g) := \sup \left\{ \sum_{i=0}^{n-1} \bigl| g(x_{i+1}) - g(x_i) \bigr| : 0 = x_0 < x_1 < \dots x_n = 1,\ n \in \mathbb{N} \right\},$$

is finite.

Exercise. Show that if $g \in C^1([0,1])$, then $\mathrm{var}(g) \le \max_{x\in[0,1]} |g'(x)|$.

THEOREM 9 (Denjoy's Theorem). *Let f be a circle homeomorphism with an irrational rotation number ρ. If $f \in C^1$ and $w(x) = \log f'(x)$ has bounded variation, then f is topologically conjugate to R_ρ.*

We will need two technical statements before proceeding to the proof of the theorem.

LEMMA 10. *Let f be a circle homeomorphism with an irrational rotation number. There is a sequence $n_k \uparrow \infty$ (i.e.monotone increasing and unbounded) such that for any $x \in \mathbb{S}$ the intervals (x_i, x_{i+n_k}), where $x_i = f^i(x)$ and $0 \le i < n_k$ and (x_i, x_{i+n_k}) is the shortest arc with the ends at x_i and x_{i+n_k}, are disjoint.*

PROOF. Let $y_k = R_\rho^k(0)$, $k \in \mathbb{Z}$, be the trajectory of 0 under R_ρ, an irrational rotation which has the same rotation number as the homeomorphism f.

Let $n_0 = 1$. Then define n_k for $k \ge 1$ recursively:

$$n_k := \min\{\, i \in \mathbb{N} : \mathrm{dist}(y_0, y_i) < \mathrm{dist}(y_0, y_{n_{k-1}})\,\}\,.$$

Since the positive trajectory of an irrational rotation is dense, any interval contains infinitely many y_i with $i > 0$ and, consequently, n_k is defined. The sequence n_k corresponds to times of consecutive closest returns to zero as

$$\mathrm{dist}(y_0, y_i) \ge \mathrm{dist}(y_0, y_{n_{k-1}}) > \mathrm{dist}(y_0, y_{n_k})$$

for $0 < i < n_k$ and, consequently, the sequence is monotone, $n_k > n_{k-1}$.

Now suppose that the intervals (y_i, y_{i+n_k}) with $0 \le i < n_k$ are not disjoin. Then $y_j \in (y_i, y_{i+n_k})$ for some $0 \le i, j < n_k$. Since y_j is inside the interval and the rotation preserves distances:

$$\mathrm{dist}(y_0, y_{i-j}) = \mathrm{dist}(y_j, y_i) < \mathrm{dist}(y_i, y_{i+n_k}) = \mathrm{dist}(y_0, y_{n_k}),$$

$$\mathrm{dist}(y_0, y_{i+n_k-j}) = \mathrm{dist}(y_j, y_{i+n_k}) < \mathrm{dist}(y_i, y_{i+n_k}) = \mathrm{dist}(y_0, y_{n_k}).$$

If $i > j$, the first inequality contradicts the definition of n_k. If $i < j$, the second one contradicts the definition of n_k.

Now let f be a homeomorphism with an irrational rotation number ρ. The monotonicity of the function T of Lemma 8 implies that, for any $i, j, k \in \mathbb{Z}$, $f^i(x) \in [f^j(x), f^k(x)]$ if and only if $R_\rho^i(0) \in [R_\rho^j(0), R_\rho^k(0)]$.

Consequently, the sequence n_k is the same for all homeomorphisms with the rotation number ρ and does not depend on the choice of $x_0 \in \mathbb{S}$. $\square$

LEMMA 11. *Let f be a C^1 diffeomorphism of the circle with an irrational rotation number. If there is a sequence $n_k \in \mathbb{N}$, $n_k \uparrow \infty$ and $C > 0$ such that*

$$\left|(f^{n_k})'(x)\right| \left|(f^{-n_k})'(x)\right| > C \qquad \forall x \in \mathbb{S}, \forall k \in \mathbb{N}$$

then f is minimal.

PROOF. Suppose that there is $x \in \mathbb{S}$ such that its orbit is not dense. Let $Y = \overline{O_x}$. Y is closed so its complement is open and, by our supposition, not empty. Consider an open interval $I \subset \mathbb{S} \setminus Y$ with end points in Y. Let $I_n = f^n(I)$. The intervals I_n are disjoint.[3] Let $|I_n|$ be the length of I_n. Since the interval are disjoint $\sum_n |I_n| \le 1$ (as the length of the circle is equal to 1). So necessarily $|I_n| \to 0$.

On the other hand we can write for $n = n_k$

$$\begin{aligned} |I_n| + |I_{-n}| &= \int_{I_0} \left(|(f^n)'(z)| + |(f^{-n})'(z)|\right) dz \\ &\ge 2\int_{I_0} \sqrt{|(f^n)'(z)||(f^{-n})'(z)|} dz \ge 2\sqrt{C}|I_0|\,. \end{aligned}$$

The last inequality contradicts to $|I_n| \to 0$. Thus f is minimal. $\square$

PROOF OF DENJOY'S THEOREM. Take any $x \in \mathbb{S}$ and let $x_m = f^m(x)$ for $m \in \mathbb{Z}$. Let n_k be the sequence defined by Lemma 10. The inverse function theorem and the chain rule imply that

$$\begin{aligned} \left|\log (f^{n_k})'(x)(f^{-n_k})'(x)\right| &= \left|\log \frac{(f^{n_k})'(x_0)}{(f^{n_k})'(x_{-n_k})}\right| \\ &\le \left|\sum_{i=0}^{n_k-1} \log f'(x_i) - \sum_{i=0}^{n_k-1} \log f'(x_{i-n_k})\right| \\ &\le \sum_{i=0}^{n_k-1} \left|\log f'(x_i) - \log f'(x_{i-n_k})\right| \le \mathrm{Var}(w)\,. \end{aligned}$$

The last inequality uses that the intervals $(x_{i-n_k}, x_i) = f^{-n_k}((x_i, x_{i+n_k}))$ are disjoint due to Lemma 10. Consequently,

$$(f^{n_k})'(x)(f^{-n_k})'(x) \ge \exp\left(-\mathrm{Var}(w)\right)\,.$$

Then Lemma 11 implies that f is minimal. Finally, Poincaré's Theorem implies that f is topologically equivalent to R_ρ. $\square$

---

[3] Indeed, let $I = (a, b)$, $a, b \in Y$. If $f^i(I) \cap f^j(I) \ne \emptyset$, $i \ne j$, then $f^{j-i}(I) \cap I \ne \emptyset$ and either $f^{j-i}(a) \in I$ or $f^{j-i}(b) \in I$. Consider the former case and set $y = f^{j-i}(a)$ (the latter case is similar). Since $a \in Y$ there is a sequence i_k such that $f^{i_k}(x) \to a$. Since f is continuous $f^{j-i+i_k}(x) \to y = f^{j-i}(a)$ which implies that $y \in Y$ but $y \in I \subset \mathbb{S} \setminus Y$. Contradiction implies that the intervals are disjoint.

1.9. Wandering intervals. Definition. We say that an interval $I \subset \mathbb{S}$ is wandering, if its iterates $I, f(I), f^2(I), \dots$ are pairewise disjoint and I is not attracted by a periodic point.

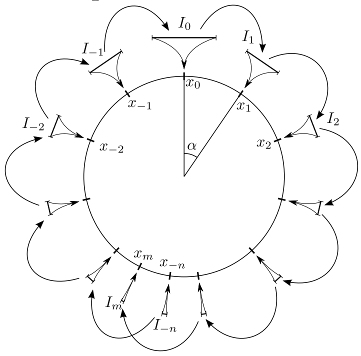
If f does not have any periodic point, the second part of the definition is automatically satisfied. We only note that this requirement his introduced to ensure that a circle homeomorphism with a rational rotation number does not have wandering intervals: in that case every orbit is either periodic or converges to a periodic one. On the other hand, in the case of an irrational rotation number the second requirement of the definition does not add any restriction (independently of its precise meaning) as there are no periodic points.

We already proved that an irrational rotation does not have any wandering interval (as every orbit is dense). Denjoy's theorem states that a C^2 diffeomorphism of $\mathbb{S}$ is topologically conjugate to an irrational rotation, so it is minimal and does not have any wandering interval.

The following construction provides an example of a C^1 diffeomorphism f which has a wandering interval. Therefore, f is not minimal and, consequently, is not topologically conjugate to an irrational rotation.

1.10. Denjoy's example. *For every irrational* $\alpha \in (0,1)$, *there is a diffeomorphism* $f : \mathbb{S} \to \mathbb{S}$ *with rotation number* α (mod 1) *which has wandering intervals.*

The idea of Denjoy's example is illustrated on the figure below:



We start with an irrational rotation setting $x_n = n\alpha$ (mod 1). At each point x_n, we cut the circle and glue in an interval I_n of length ℓ_n. Then we construct a diffeomorphism, such that $f(I_n) = I_{n+1}$ and on $\mathbb{S} \setminus \cup_{n\in\mathbb{Z}} I_n$ the map is left "unchanged". (The illustration from Wikipedia)

Now let us provide an accurate description of the example.

(1) Take an irrational number $\alpha \in (0,1)$.
(2) Take a sequence of positive numbers $(\ell_n)_{n=-\infty}^{+\infty}$ such that

$$L = \sum_{n\in\mathbb{Z}} \ell_n < 1, \qquad \frac{1}{2} < \frac{\ell_{n+1}}{\ell_n} < 2, \qquad \text{and} \qquad \frac{\ell_{n+1}}{\ell_n} \underset{n\to\pm\infty}{\longrightarrow} 1\,.$$

Any sequence which satisfies these properties can be used.[4]

(3) Let $x_n = n\alpha - [n\alpha]$ (in the essence, this is the trajectory of $x_0 = 0$ of the rotation by α).

Properties: $x_0 = 0$, $x_1 = \alpha$, $x_n \in [0, 1)$, $(x_n)_{n\in\mathbb{Z}}$ is dense in $[0, 1]$.

(4) Define a sequence $(a_n)_{n\in\mathbb{Z}}$ by

$$a_0 = 0 \qquad \text{and} \qquad a_n = (1 - L)x_n + \sum_{k\,:\,x_k \in [x_0, x_n)} \ell_k\,.$$

Properties: the sum is convergent, $a_n \in [0, 1)$, the map $x_n \mapsto a_n$ is monotone increasing.

Proof: Since the sequence $(x_k)_{k\in\mathbb{Z}}$ is dense in the interval $[0, 1]$, any interval contains infinitely many x_k. Thus the sum contains infinitely many terms. Since all terms are positive and any partial sum (a sum other any finite subset of indeces) does not exceed L, the total sum is independent of the order of terms and does not exceed L. It follows $0 \le a_n < 1$.

Monotonicity of the map $x_n \mapsto a_n$ is obvious.

(5) Define intervals $I_n \subset (0, 1)$ by

$$I_n = (a_n, b_n), \qquad b_n = a_n + \ell_n\,.$$

Properties: $I_n \subset (0, 1)$. The intervals I_n are pairwise disjoint.

Proof: Since α is irrational $x_n \ne x_m$ for all $n \ne m$. Suppose $x_n < x_m$, then

$$a_m - a_n = (1 - L)(x_m - x_n) + \sum_{k\,:\,x_k \in [x_n, x_m)} \ell_k \;\ge\; \ell_n\,.$$

So $b_n = a_n + \ell_n < a_m$ and $I_n \cap I_m = [a_n, b_n] \cap [a_m, b_m] = \emptyset$.

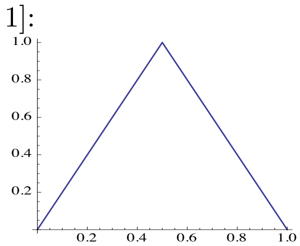
(6) Define a sequence $(C_n)_{n\in\mathbb{Z}}$ by

$$C_n = 2\left(\frac{\ell_{n+1}}{\ell_n} - 1\right).$$

Properties: $-1 < C_n < 2$,

$$\lim_{n\to\pm\infty} C_n = 0\,, \quad \text{and} \quad \int_{a_n}^{b_n} C_n \chi\left(\frac{x - a_n}{\ell_n}\right) = \ell_{n+1} - \ell_n,$$

where $\chi(x) = 1 - |1 - 2x|$. For your information: this is the graph of $\chi : [0, 1] \to [0, 1]$:



Proof: Exercise /easy/.

[4] For example, $\ell_n = \frac{1}{n^2+25}$. We note that

$$L = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + c^2} = \frac{\pi \coth c\pi}{c} < 1 \qquad \text{for } c \ge 4,$$

other properties are easy to check. I use $c = 5$ to produce illustrations.

(7) Define a function F' for $x \in [0,1]$ by

$$F'(x) = \begin{cases} 1 + C_n\chi(\frac{x-a_n}{\ell_n}) & \text{if } \exists n \text{ such that } x \in I_n, \\ 1 & x \in \mathcal{C} := [0,1] \setminus \cup_{n\in\mathbb{Z}} I_n, \end{cases}$$

Define $F'(x) = F'(x - [x])$ for all other $x \in \mathbb{R}$.

Properties: $F' : \mathbb{R} \to \mathbb{R}$ is continuous, periodic, $F'(x) > 0$ for all $x \in \mathbb{R}$ and

$$\int_0^1 F'(x)dx = 1\,.$$

Proof: Exercise.

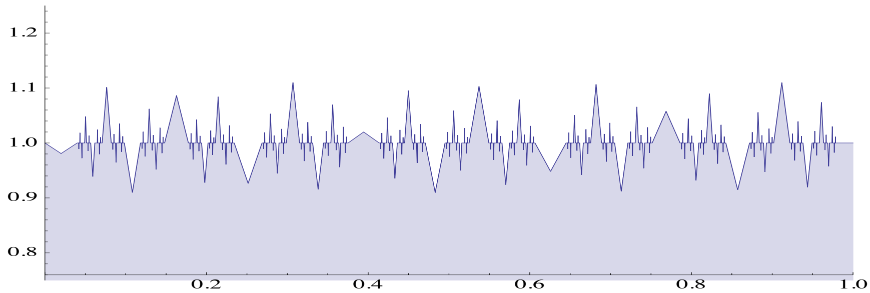


FIGURE 1. Graph of the function F' constructed numerically for $\alpha = \frac{\sqrt{5}-1}{2}$ and $\ell_n = \frac{1}{n^2+25}$. This function is equal to 1 on the set $\mathcal{C} = [0,1] \setminus \cup_n I_n$.

(8) Define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = a_1 + \int_0^x F'(t)dt\,.$$

Properties: F is a diffeomorphism, $\frac{dF}{dx}(x) = F'(x)$, $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$.

Proof: Exercise /easy/.

(9) Corollary: there is a diffeomorphism $f : \mathbb{S} \to \mathbb{S}$, such that F is a lift of f.

PROPOSITION 12. *The diffeomorphism f has the following properties:*

A. *πI_0 is a wandering interval for f, where $\pi : \mathbb{R} \to \mathbb{S}$ is the natural projection $x \mapsto x \pmod 1$.*

B. *$\rho(f) = \alpha \pmod 1$.*

Proof. First we take any $n \in \mathbb{Z}$ and show that $F(a_n) = a_{n+1}$ if $x_n \in [0, 1-\alpha)$ and $F(a_n) = a_{n+1} + 1$ if $x_n \in [1-\alpha, 1)$.

Indeed, the definition of F (item 8 in the list above) and the properties of C_n (item 6) imply

$$\begin{aligned} F(a_n) &= a_1 + \int_0^{a_n} F'(x)dx = a_1 + a_n + \sum_{k: I_k \in [0,a_n)} \int_{I_k} C_k \chi\left(\frac{x - a_k}{\ell_k}\right) dx \\ &= a_1 + a_n + \sum_{k\,:\,a_k \in [0,a_n)} \ell_{k+1} - \sum_{k\,:\,a_k \in [0,a_n)} \ell_k\,. \end{aligned}$$

Using that $x_1 = \alpha$,

$$a_1 = (1-L)\alpha + \sum_{x_k \in [0,\alpha)} \ell_k \qquad \text{and} \qquad a_n = (1-L)x_n + \sum_{x_k \in [0,x_n)} \ell_k,$$

we get

$$F(a_n) \;=\; (1-L)(\alpha + x_n) + \sum_{k\,:\,x_k \in [0,x_n)} \ell_{k+1} + \sum_{x_k \in [0,\alpha)} \ell_k\,.$$

The definition $x_n = n\alpha - [n\alpha]$ implies that

$$x_{n+1} = \begin{cases} x_n + \alpha, & \text{if } x_n + \alpha < 1, \\ x_n + \alpha - 1, & \text{if } x_n + \alpha \geq 1. \end{cases}$$

In the first case, $x_k \in [0, x_n)$ iff $x_{k+1} \in [\alpha, x_{n+1})$. Indeed $x_{k+1} = x_k + \alpha < 1$. Then

$$\begin{aligned} F(a_n) &= (1-L)x_{n+1} + \sum_{k\,:\,x_k \in [0,x_n)} \ell_{k+1} + \sum_{x_k \in [0,\alpha)} \ell_k \\ &= (1-L)x_{n+1} + \sum_{k\,:\,x_k \in [\alpha,x_{n+1})} \ell_k + \sum_{x_k \in [0,\alpha)} \ell_k = a_{n+1}\,. \end{aligned}$$

In the second case, $x_k \in [0, x_n)$ iff $x_{k+1} \in [0, x_{n+1}) \cup [\alpha, 1)$. Indeed, $x_{k+1} = x_k + \alpha$ if $x_k \in [0, 1-\alpha)$ and $x_{k+1} = x_k + \alpha - 1$ if $x_k \in [1-\alpha, x_n)$. Then

$$\begin{aligned} F(a_n) &= (1-L)(x_{n+1}+1) + \sum_{k\,:\,x_k \in [0,x_n)} \ell_{k+1} + \sum_{x_k \in [0,\alpha)} \ell_k \\ &= (1-L)x_{n+1} + 1 - L + \sum_{k\,:\,x_k \in [0,x_{n+1}) \cup [\alpha,1)} \ell_k + \sum_{x_k \in [0,\alpha)} \ell_k \\ &= (1-L)x_{n+1} + 1 - L + \sum_{k\,:\,x_k \in [0,x_{n+1})} \ell_k + \sum_{x_k \in [0,1)} \ell_k \\ &= a_{n+1} + 1\,. \end{aligned}$$

We have just proved that

$$\boxed{F(a_n) = \begin{cases} a_{n+1}, & \text{if } x_n + \alpha < 1, \\ a_{n+1} + 1, & \text{if } x_n + \alpha \geq 1. \end{cases}}$$

Since

$$F(b_n) - F(a_n) \;=\; \int_{I_n} F'(x)dx = \ell_n + \int_{a_n}^{a_n+\ell_n} C_n \chi\left(\frac{x-a_n}{\ell_n}\right) dx = \ell_{n+1},$$

we see that the interval $I_n = (a_n, b_n)$ is mapped either to (a_{n+1}, b_{n+1}) or to $(a_{n+1}+1, b_{n+1}+1)$. Let $J_n = \pi I_n$. Then $f(J_n) = J_{n+1}$. The intervals J_n are disjoint as $I_n \subset (0,1)$ are disjoint. So J_n are wandering.

Part A of the proposition is proved.

Now we find the rotation number of f. Let $k_n = [n\alpha]$ for $n \in \mathbb{Z}$. Let us prove that

$$F^n(0) = k_n + a_n.$$

Proof. The statement is true for $n = 1$. Indeed, $x_1 = \alpha \in (0,1)$ implies $k_1 = 0$. On the other hand, $F^1(0) = a_1$. Continuing by induction, we suppose that $F^n(0) = k_n + a_n$ for some n. Then

$$F^{n+1}(0) = F(F^n(0)) = F(a_n + k_n) = F(a_n) + k_n = a_{n+1} + k_{n+1}\,.$$

To check the last equality two cases are to be considered: $x_n + \alpha < 1$ and $x_n + \alpha \geq 1$.

The induction implies $F^n(0) = k_n + a_n$ for all $n \in \mathbb{N}$. Similar arguments are used to prove the statement for negative n $\boxed{\cdot}$

Finally we can find the rotation number: Since $F^n(0) = k_n + a_n$ and $n\alpha = k_n + x_n$, we get

$$\left|\frac{F^n(0)}{n} - \alpha\right| = \left|\frac{F^n(0) - n\alpha}{n}\right| = \left|\frac{a_n - x_n}{n}\right| \leq \frac{1}{n}.$$

Taking the limit $n \to \infty$:

$$\lim_{n\to\infty} \frac{F^n(0)}{n} = \alpha.$$

Since F is a lift of f, we get $\rho(f) = \alpha \pmod 1$. $\square$

Remark: The set $\mathcal{C} = [0,1] \setminus \cup_{n\in\mathbb{Z}} I_n$ is closed and every point of $\mathcal{C}$ is a boundary point, the Lebesgue measure of $\mathcal{C}$ is positive (equals to $1 - L \approx 0.38$) but C does not contain any interval. So $\mathcal{C}$ is a Cantor set of positive Lebesgue measure.

Remark: There is a unique continuous function $h : \mathbb{S} \to \mathbb{S}$ such that for every n the interval J_n is mapped to the single point x_n (mod 1). Obviously, h is not invertible. Since the sequence $(x_n)_{n\in\mathbb{Z}}$ is dense in $[0,1]$, the map h is surjective. Moreover, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{f} & \mathbb{S} \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \mathbb{S} & \xrightarrow[R_\alpha]{} & \mathbb{S} \end{array}$$

We say that f is *semi-conjugate* to the rotation R_α.

Definition. We say that $g : Y \to Y$ is *topologically semiconjugate to* $f : X \to X$, if there exists a continuous surjection $h : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

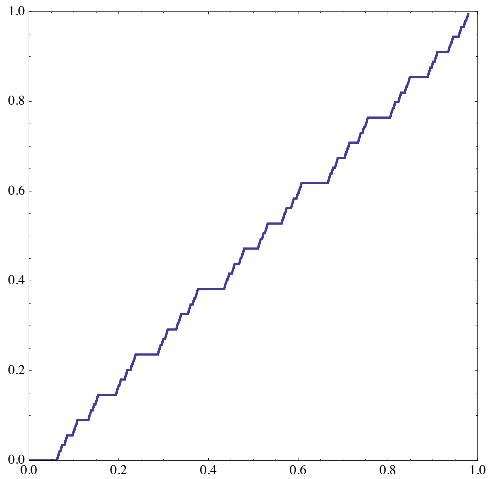


FIGURE 2. Graph of the function h constructed numerically for Denjoy's example with $\alpha = \frac{\sqrt{5}-1}{2}$. This function simiconjugates f and R_α.

1.11. Families of circle maps. Example: Consider the map $f : \mathbb{S} \to \mathbb{S}$ defined via its lift:

$$F(x) = x + \alpha + \varepsilon \sin(2\pi x)$$

where ε and α are constant.

If $|\varepsilon| < \frac{1}{2\pi}$, then $F'(x) > 0$. Consequently, F is strictly monotone. The inverse function theorem implies that F^{-1} is continuously differentiable. Moreover, $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$, hence F is a lift of a circle diffeomorphism f.

For $\varepsilon = \frac{1}{2\pi}$ the map F is strictly monotone, but $F'(\frac{1}{2}) = 0$ so the inverse map F^{-1} is not differentiable at the point $F(\frac{1}{2})$. Thus $F : \mathbb{R} \to \mathbb{R}$ is a homeomorphism only.

This map depends on two parameters α and ε. Let $\rho = \rho(\alpha, \varepsilon)$ be the rotation number of f. If $\varepsilon = 0$, then f is obviously linear rotation and consequently $\rho(\alpha, 0) = \alpha \pmod 1$. Let us fix $\varepsilon \neq 0$. A typical graph of $\rho(\alpha, \varepsilon)$ looks like this one:

The function $\rho(\alpha, \varepsilon)$ has the following properties. The rotation number is non-decreasing function of α. Indeed, if $\alpha_1 < \alpha_2$, then $F(x, \alpha_1) < F(x, \alpha_2)$ for all $x \in \mathbb{R}$ and consequently $\rho(\alpha_1, \varepsilon) \le \rho(\alpha_2, \varepsilon)$.

It can proved (we are not proving it now) that the function $\rho(\alpha, \varepsilon)$ is continuos. Moreover, it is locally constant at every rational value. /Indeed, ρ is rational if and only if f has a periodic point. If $f^n(p) = p$ for some α_0, ε_0 and $(f^n)'(p) \neq 1$, then there is a solution of the equation $f^n(p) = p$ for all α, ε in a small neighbourhood of α_0, ε_0 – use implicit function theorem/.

If we paint yellow points on the plane (α, ε) which correspond to rational $\rho(\alpha, \varepsilon)$ we will see Arnold's tongues:
There is a "tongue" rising from every rational point on the α-axis.

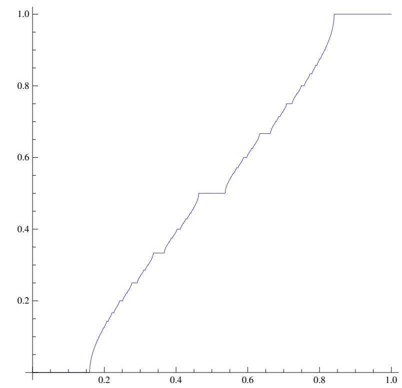


FIGURE 3. "Devil's staircase": Graph of $\rho(\alpha)$ for the map $f_\alpha(x) = x + \alpha + \frac{1}{2\pi}\sin 2\pi x \pmod 1$.

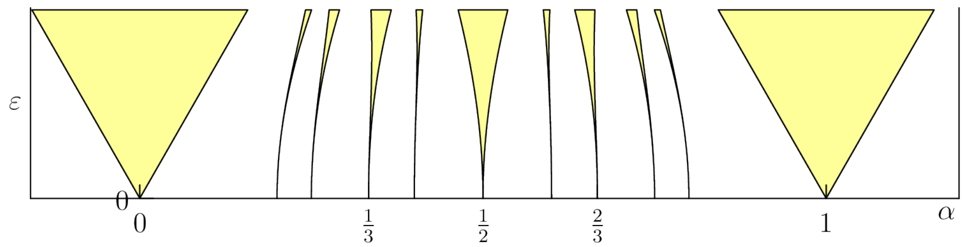


FIGURE 4. Arnold's tongues (taken from Wikipedia: `http://en.wikipedia.org/wiki/Arnold_tongue`)

2. Expanding maps of the circle

Definition. A continuously differentiable map $f : \mathbb{S} \to \mathbb{S}$ is called *expanding* if $|f'(x)| > 1$ $\forall x \in \mathbb{S}$.

Remarks:

(i) Since $\mathbb{S}$ is compact and f' continuous there is a constant K such that $|f'(x)| > K > 1$ $\forall x \in \mathbb{S}$.
(ii) f is not a homeomorphism (it is not invertible).
(iii) If $f'(x) > 1$, then f is orientation-preserving. Otherwise it is orientation reversing.
(iv) We will mainly consider the orientation preserving case.
(v) If f is expanding then f^2 is also expanding. Moreover, f^2 is orientation preserving. Indeed, the chain rule implies that $(f^2)'(x) = f'(f(x))f'(x) > 1$.

Examples.

(1) $f(x) = mx \pmod 1$, $m \in \mathbb{N}$, $m > 1$.
(2) $f(z) = z^m$, $|z| = 1$, $z \in \mathbb{C}$. (Note: set $z = e^{2\pi i x}$, you'll get the previous example).

Let f be expanding, then f is a local diffeomorphism, i.e., for each x there are open intervals $U, V \subset \mathbb{S}$ such that $x \in U$ and $f : U \to V$ is a diffeomorphism. Since $\mathbb{S}$ is compact, for any $y \in \mathbb{S}$ the number of preimages is finite and independent of y.

Definition. Let f be an expanding map of the circle and $y \in \mathbb{S}$. The number of preimages of y is called *the degree of f*.

Example. Let $f(x) = mx \pmod 1$, $m \geq 2$, integer. Then $\deg(f) = m$.

Definition. A function $F : \mathbb{R} \to \mathbb{R}$ is called *the lift of f* if $\pi \circ F = f \circ \pi$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathbb{S} & \xrightarrow{f} & \mathbb{S} \end{array}$$

The lift has the following properties:

(1) F is unique up to adding an integer
(2) If f preserves orientation, then F is strictly increasing and
$$F(x+1) = F(x) + d$$
where d is the degree of f.
(3) If f reverses orientation, then F is strictly decreasing and
$$F(x+1) = F(x) - d.$$

Example. Let $f(x) = mx \pmod 1$, $m \geq 2$, integer. Then $F(x) = mx$.

Proposition 13. *If $f, g : \mathbb{S} \to \mathbb{S}$ are both expanding, then $\deg(f \circ g) = \deg(f)\deg(g)$. In particular, $\deg(f^n) = (\deg(f))^n$ for all $n \in \mathbb{N}$.*

Proof. Count the number of preimages. □

Recall, that $x \in \mathbb{S}$ is called a periodic point of period n if $x = f^n(x)$. The least $n > 1$ is called a period of p.

Proposition 14. *If $f : \mathbb{S} \to \mathbb{S}$ is expanding, orientation preserving and $d = \deg(f)$, then there are exactly $d^n - 1$ points $p \in \mathbb{S}$ such that $f^n(p) = p$.*

Proof. Let $n = 1$. The number of solutions of the equation $x = f(x)$, $x \in \mathbb{S}$ equals to the number of solutions of the equation $x = F(x) \pmod 1$, $x \in [0, 1)$. The latter coincides with the number of integer values of the function $g(x) := F(x) - x$ when $x \in [0, 1)$. The function g is monotone increasing and
$$g(1) - g(0) = F(1) - 1 - F(0) = d - 1.$$
So the number of fixed points is exactly $d - 1$.

For $n > 1$, the proposition follows immediately as $\deg(f^n) = \deg(f)^n$. □

It follows, that the number of periodic points of the prime period n equals to $d^n - d^{n-1}$.

Since a periodic orbit consists of a finite number of points, it is not dense. Thus an expanding maps is not minimal.

Example (Symbolic dynamics). Consider the linear doubling map $f(x) = 2x \pmod 1$.

We can use binary numeral system: any $x \in [0,1]$ can be written in the form
$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \dots$$
where $a_k \in \{0,1\}$. Then
$$f(x) = \frac{a_2}{2} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \dots$$
So we see that any $x \in \mathbb{S}$ can be represented by a sequence $(a_k)_{k=1}^{\infty}$ and the map f acts on these sequences as a shift, which deletes the first elements and then shifts every element of the sequence to the left:
$$\sigma : (a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, \dots).$$
The map $\sigma : \Sigma \to \Sigma$ where
$$\Sigma = \Big\{ (a_1, a_2, a_3, \dots) : a_k \in \{0,1\} \Big\}$$
is called a *shift map*. This representation simplifies the study of the dynamics. For example, we can easily deduce that there is a dense orbit. Indeed, let
$$\Sigma_f = \Big\{ (a_1, \dots, a_n) : n \in \mathbb{N}, a_k \in \{0,1\} \Big\}$$
be the set of all finite sequences. Obviously, the set
$$\left\{ x = \sum_{k=1}^{n} \frac{a_k}{2^k} : n \in \mathbb{N}, (a_k) \in \Sigma_f \right\}$$
is dense in $[0,1]$. The set Σ_f is countable, so we can concatenate all its elements in a single infinite sequence and take $x_0 \in (0,1)$ which has this sequence as its binary representation. Its orbit, $\{f^n(x_0) : n \in \mathbb{N}\}$, is dense (as every finite binary number appears at the beginning of the binary expansion of $f^n(x_0)$ for some n).

2.1. Symbolic Dynamics for an expanding map on the circle. Let $f : \mathbb{S} \to \mathbb{S}$ be expanding of degree 2. In the previous lecture we proved that there is a unique fixed point $p \in \mathbb{S}$, $f(p) = p$. Moreover, since p has exactly two preimages, there is a unique point $q \neq p$ such that $f(q) = p$. These two points define two intervals $A_0, A_1 \subset \mathbb{S}$: $A_0 = [p,q]$ and $A_1 = [q,p]$. Note that the intervals are chosen to be closed. Obviously,
$$\mathbb{S} = A_0 \cup A_1 \qquad \text{and} \qquad A_0 \cap A_1 = \{p, q\}.$$
For any $x \in \mathbb{S}$ we define a sequence $(\omega_k)_{k=0}^{\infty}$ such that
$$\omega_k = \begin{cases} 0 & \text{if } f^k(x) \in A_0 \\ 1 & \text{if } f^k(x) \in A_1 \end{cases}$$
Since $A_0 \cup A_1 = \mathbb{S}$ such a sequence exists for all $x \in \mathbb{S}$. The sequence is not necessarily unique as $A_0 \cap A_1 \neq \emptyset$. The definition is obviously ambiguous for the points p and q. Moreover, if $f^{k_0}(x) \in \{p,q\}$ for some k_0, then $f^k(x) = p$ for all $k > k_0$ and the definition is ambiguous for all ω_k with $k \geq k_0$. We note that the number of these exceptional points is countable because the set $f^{-k}\{p\}$ consists of 2^k points as $\deg(f^k) = 2^k$.

For each of these points we assign two sequences instead of one using the following additional rules. The sequences

$$(0,0,0,0,\dots) \qquad \text{and} \qquad (1,1,1,1,\dots)$$

are assign to the fixed point p. The sequences

$$(1,0,0,0,0,\dots) \qquad \text{and} \qquad (0,1,1,1,1,\dots)$$

are assign to the point q.

If $f^{k_0}(x) = q$ for some $k_0 \geq 0$, then $f^k(x) \neq p, q$ for $0 \leq k < k_0$ and consequently $\omega_0, \dots, \omega_{k_0-1}$ are defined uniquely by the general rule. Then we assign two different sequences to x

$$(\omega_0, \dots \omega_{k_0-1}, 0, 1, 1, 1, 1 \dots) \qquad \text{and} \qquad (\omega_0, \dots \omega_{k_0-1}, 1, 0, 0, 0, 0 \dots)\,.$$

These definitions are designed with the following property in mind: if a sequence $(\omega_0, \omega_1, \omega_2, \dots)$ represents a point x, then the shifted sequence $(\omega_1, \omega_2, \dots)$ represents $f(x)$.

Let Σ be the shift space:

$$\Sigma = \left\{ (\omega_k)_{k=0}^{\infty} : \omega_k \in \{0,1\} \right\}$$

and $\sigma : \Sigma \to \Sigma$,

$$\sigma(\omega_0, \omega_1, \omega_2, \dots) = (\omega_1, \omega_2, \dots),$$

be the left shift.

Let d be a metric on Σ defined by the following rules: (a) if $\omega, \omega' \in \Sigma$ and $\omega \neq \omega'$, then

$$d(\omega, \omega') = 2^{-\min\{k \,:\, \omega_k \neq \omega'_k\}},$$

and (b) $d(\omega, \omega) = 0$.

Exercise: Check that (Σ, d) is a complete metric space. Show that Σ is compact.

THEOREM 15. *If $f : \mathbb{S} \to \mathbb{S}$ is an expanding map of the circle, f preserves orientation and* $\deg f = 2$, *then there is a continuous surjective map $h : \Sigma \to \mathbb{S}$ such that $h \circ \sigma = f \circ h$, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \Big\downarrow h & & \Big\downarrow h \\ \mathbb{S} & \xrightarrow{f} & \mathbb{S} \end{array}$$

Proof. Take any $n \in \mathbb{N}$. Then $\deg f^n = (\deg f)^n = 2^n$, so there are exactly 2^n preimages of p under f^n. We denote these points by p_j starting from $p_0 = p$ and numbering them consecutively following the anticlockwise direction on the circle

$$f^n(p_j) = p \qquad \text{for } 0 \leq j \leq 2^n - 1.$$

It is convenient to set $p_{2^n} = p_0$. These points define 2^n intervals which we denote by $A_{\omega_0 \dots \omega_{n-1}} = [p_j, p_{j+1}]$, where $(\omega_0, \dots, \omega_{n-1})$ is the binary representation of j:

$$j = 2^{n-1}\omega_0 + 2^{n-2}\omega_1 + \dots + 2^0\omega_{n-1}\,, \qquad \omega_k \in \{0,1\}.$$

(1) $f^n(A_{\omega_0\dots\omega_{n-1}}) = \mathbb{S}\setminus\{p\}$. The map f^n maps the end points of $A_{\omega_0\dots\omega_{n-1}}$ to p, and none of the internal of $A_{\omega_0\dots\omega_{n-1}}$ is mapped to p.

(2) $A_{\omega_0\dots\omega_{n-1}}$ is a closed interval of the length

$$|A_{\omega_0\dots\omega_{n-1}}| < K^{-n}.$$

Proof: Let p_j, p_{j+1} be the end points of $A_{\omega_0\dots\omega_{n-1}}$. Property 1 implies that $f^n(A_{\omega_0\dots\omega_{n-1}})$ is of the unit length. So

$$\int_{p_j}^{p_{j+1}} (f^n)'(x)dx = 1\,.$$

The chain rule implies that

$$(f^n)'(x) = f'(x)\, f'(f(x))\, f'(f^2(x))\, \dots\, f'(f^{n-1}(x)) > K^n$$

as every factor is larger than K. So

$$1 = \int_{p_j}^{p_{j+1}} (f^n)'(x)dx \geq K^n|A_{\omega_0\dots\omega_{n-1}}|$$

and the estimate follows directly.

(3) $A_{\omega_0\dots\omega_{n-1}\omega_n} \subset A_{\omega_0\dots\omega_{n-1}}$

Proof: Property 1 implies that in every interval $A_{\omega_0\dots\omega_{n-1}}$ there is a unique point q_j such that $f^n(q_j) = q$. Since $f^{n+1}(q_j) = f(f^n(q_j)) = f(q) = p$ and $f^{n+1}(p_j) = f(f^n(p_j)) = f(p) = p$, the points q_j and p_j are end points for the next generation of intervals. Since q_j divides $[p_j, p_{j+1}]$ in two parts, the rule for numbering of the intervals implies

$$A_{\omega_0,\dots,\omega_{n-1},0} = [p_j, q_j] \qquad \text{and} \qquad A_{\omega_0,\dots,\omega_{n-1},1} = [q_j, p_{j+1}]$$

(4) $f^n(A_{\omega_0\dots\omega_n}) = A_{\omega_n}$

Proof: Image of an interval is another interval and end-points are mapped to end-points, so

$$f^n(A_{\omega_0,\dots,\omega_{n-1},0}) = f^n([p_j, q_j]) = [p, q] = A_0$$
$$f^n(A_{\omega_0,\dots,\omega_{n-1},1}) = f^n([q_j, p_{j+1}]) = [q, p] = A_1$$

(5) $f(A_{\omega_0,\omega_1,\dots,\omega_n}) = A_{\omega_1,\dots,\omega_n}$.

Proof: For $n = 1$ this property follows from the previous one. Suppose the property holds for some $n-1$:

$$f(A_{\omega_0,\omega_1,\dots,\omega_{n-1}}) = A_{\omega_1,\dots,\omega_{n-1}}\,.$$

There is j, $0 \leq j < 2^n$, such that $A_{\omega_0,\omega_1,\dots,\omega_{n-1}} = [p_j, p_{j+1}]$. We already proved that there is a unique $q_j \in (p_j, p_{j+1})$ such that $f^n(q_j) = q$. So we get $f^{n-1}(f(q_j)) = q$. It follows:

$$\begin{aligned}
A_{\omega_0,\omega_1,\dots,\omega_{n-1}} &= [p_j, p_{j+1}] \\
A_{\omega_0,\omega_1,\dots,\omega_{n-1},0} &= [p_j, q_j] \\
A_{\omega_0,\omega_1,\dots,\omega_{n-1},1} &= [q_j, p_{j+1}] \\
A_{\omega_1,\dots,\omega_{n-1}} &= [f(p_j), f(p_{j+1})] \\
A_{\omega_1,\dots,\omega_{n-1},0} &= [f(p_j), f(q_j)] \\
A_{\omega_1,\dots,\omega_{n-1},1} &= [f(q_j), f(p_{j+1})]
\end{aligned}$$

So we see that the end points of $A_{\omega_0,\omega_1,\dots,\omega_{n-1},\omega_n}$ are mapped by f to the end points of $A_{\omega_1,\dots,\omega_n}$ and Property 5 follows.

Now we are ready to define the function $h : \Sigma \to \mathbb{S}$. Take any $\omega = (\omega_k)_{k=0}^{\infty} \in \Sigma$. Let $B_n(\omega) = A_{\omega_0\dots\omega_{n-1}}$. The properties above imply that $B_{n+1}(\omega) \subset B_n(\omega)$ for all n, B_n are closed, and $|B_n| < K^{-n}$. Consequently the intersections of all $B_n(\omega)$ is not empty, and consists of exactly one point. Thus there is a unique $x \in \mathbb{S}$ such that

$$\{x\} = \bigcap_{n\in\mathbb{N}} B_n(\omega)$$

Let $h(\omega) = x$.

The function h is continuous. Take any $\varepsilon > 0$. Then there is $n \in \mathbb{N}$ such that $K^{-n} < \varepsilon$. Let $\delta = 2^{-n}$. Then the definition of the metric on Σ implies that if $\omega' \in \Sigma$ and $d(\omega', \omega) < \delta$, then $\omega'_j = \omega_j$ for $0 \le j \le n$. Then the definition of B_n implies that $B_n(\omega) = B_n(\omega') = A_{\omega_0\dots\omega_{n-1}}$. Since $h(\omega) \in B_n(\omega)$ and $h(\omega') \in B_n(\omega')$ we conclude that

$$\operatorname{dist}(h(\omega), h(\omega')) \le |A_{\omega_0\dots\omega_{n-1}}| < K^{-n} < \varepsilon.$$

Thus h is continuous.

The function h is surjective. Take any $x \in \mathbb{S}$. We construct the corresponding sequence $\omega \in \Sigma$ inductively: Let $\omega_0 = 0$ if $x \in [p, q)$ and $\omega_0 = 1$ otherwise.

Then suppose that for some $n \ge 1$ there is a sequence $\omega_0, \dots, \omega_{n-1}$ such that $x \in [p_j, p_{j+1})$ where j equals to the binary number $\omega_0 \dots \omega_{n-1}$ and p_j are defined at the beginning of the proof. Then let $\omega_n = 0$ if $x \in [p_j, q_j)$ and $\omega_n = 1$ if $x \in [q_j, p_{j+1})$.

Then $x \in B_n(\omega)$ for all n and therefore $h(\omega) = x$.

The diagram is commutative. Let $x = h(\omega)$. Then $x \in B_n(\omega)$ for all n. According to the construction, $B_{n+1}(\omega) = A_{\omega_0\dots\omega_n}$. Then Property 5 implies $f(B_{n+1}(\omega)) = A_{\omega_1\dots\omega_n} = B_n(\sigma(\omega))$. Consequently $f(x) \in \bigcap_{n\in\mathbb{Z}} B_n(\sigma(\omega))$ and

$$h(\sigma(\omega)) = f(x) = f(h(\omega)).$$

Thus the diagram is commutative. □

Theorem 16. *Any orientation preserving expanding map of the circle of degree 2 is topologically conjugate to the linear expanding map $f(x) = 2x$ (mod 1).*

Proof. Take $x \in \mathbb{S}$. Then there is $\omega \in \Sigma$ such that $h(\omega) = x$. Then let $y = \frac{\omega_0}{2} + \frac{\omega_1}{2^2} + \frac{\omega_3}{2^3} + \dots$ The map $g : x \mapsto y$ defines a continuous map of the circle which conjugates f and $2x \pmod 1$. □

Remark: the statements remain true if we replace the maps of degree two by maps of degree m. E.g., any expanding orientation-preserving map of degree m is topologically conjugate to $f(x) = mx \pmod 1$.

Corollaries.

(1) All orientation preserving expanding map of the circle of the same degree are topologically equivalent.

(2) Periodic orbits are dense
(3) There is a dense trajectory
(4) f is topologically mixing: For any two open sets $U, V \subset \mathbb{S}$ there is $n \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$ for all $k \geq n$.

3. Interval maps

3.1. Sharkovskii's Theorem. Let $I = [a, b] \subset \mathbb{R}$ be an interval and $f : I \to I$ a continuous map. The iterates of f can be represented in a graphical form (see the figure).

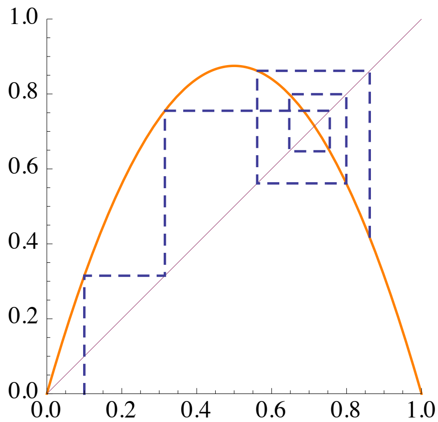


FIGURE 5. Iterates of an interval map $f : [0, 1] \to [0.1]$.

Fixed points. A fixed point $x = f(x)$ is the intersection of the graph $y = f(x)$ with the diagonal $y = x$. Suppose that there is a close interval $J \subseteq I$ such that $f(J) \subseteq J$. Then there is $x \in J$ such that $x = f(x)$. Indeed, let $J = [c, d]$. Then $f(c) \geq c$ and $f(d) \leq d$, so the intermediate value theorem implies that the graph of f intersects the diagonal, i.e., there is x such that $f(x) = x$, $c \leq x \leq d$.

PROPOSITION 17. *Let $f : I \to I$ be continuous. If $J \subseteq I$ is a closed interval such that $J \subseteq f(J)$, then there is $x \in J$ such that $x = f(x)$.*

PROOF. Take m and M be the minimum and maximum f on J respectively. Since f is continuous it attains its extremal values, so there are $x_1, x_2 \in J$ such that $f(x_1) = m$ and $f(x_2) = M$. Since $J \subset f(J)$, $m \leq x_1, x_2 \leq M$. Consequently, $f(x_1) - x_1 \leq 0$ and $f(x_2) - x_2 \geq 0$. Therefore there is $x \in [x_1, x_2] \subset J$ such that $f(x) = x$. □

Intervals. The image of a closed interval is a closed interval.

LEMMA 18. *Let $f : I \to I$ be continuous. If $J_1, J_2 \subseteq I$ are closed intervals such that $J_2 \subseteq f(J_1)$ then there is a closed interval $J_0 \subseteq J_1$ such that $J_2 = f(J_0)$.*

PROOF. Let $J_2 = [y_1, y_2]$. Since $J_2 \subset f(J_1)$, then there is $x_1' \in J_1$ such that $f(x_1') = y_1$. Let $x_2 \in J_2$ be the nearest point to x_1' in the preimage of y_2. If necessary, replace x_1' by the nearest preimage of y_1 inside $[x_1', x_2]$. Then J_0 is defined by its endpoints in x_1, x_2. □

COROLLARY 19. *If $J \subset I$ is a closed interval such that $J \subseteq f(J)$ then there is an infinite sequence of closed intervals I_n such that $I_0 = J$, $I_{n+1} \subseteq I_n$ and $f(I_{n+1}) = I_n$ for all $n \geq 0$.*

PROOF. Let $I_0 = J$. Since $I_0 \subseteq f(I_0)$, the lemma implies there is $I_1 \subset I_0$ such that $f(I_1) = I_0$. Now continue by induction, suppose that there are intervals $I_0, \ldots, I_n$ such that $I_{k+1} \subseteq I_k$ and $f(I_{k+1}) = I_k$ for $0 \leq k < n-1$. In particular, $I_n \subseteq I_{n-1}$ and $f(I_n) = I_{n-1}$. So $I_n \subseteq f(I_n)$. Consequently, there is $I_{n+1} \subseteq I_n$ such that $f(I_{n+1}) = I_n$. The corollary follows by induction in n. □

Periodic points. Recall, $x \in I$ is a periodic point of period $n \in \mathbb{N}$, if $x = f^n(x)$. If taken literally, this definition does not define the period uniquely, as $x = f^n(x)$ implies that $x = f^{nm}(x)$ for all $m \in \mathbb{N}$. We say that the period n is *prime*, if the points $x, f(x), f^2(x), \ldots, f^{n-1}(x)$ are distinct. Equivalently, n is a prime period if $x \neq f^k(x)$ for $1 < k < n$ but $x = f^n(x)$.

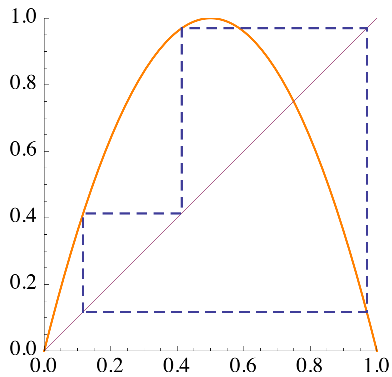


FIGURE 6. Example of a map with an orbit of prime period 3

THEOREM 20 (Sharkovskii). *If $f : I \to I$ is continuous and there is a point of the prime period 3 then for each $n \in \mathbb{N}$ there is a periodic point of prime period n.*

PROOF. We have to show that if there is $x \in I$ such that $f(x) \neq x$, $f^2(x) \neq x$ and $f^3(x) = x$, then for every $n \in \mathbb{N}$ there is $z \in I$ such that $z \neq f^k(z)$, $1 \leq k < n$ and $z = f^n(z)$.

Since $f(I) \subseteq I$, there is a fixed point of f, so there is a periodic point of the prime period $n = 1$.

Consider the period-3 point. Its orbit consists of 3 points. Let x be the smallest one. Then either $x < f(x) < f^2(x)$ or $x < f^2(x) < f(x)$. For definiteness, consider the first case (the second case can be treated in a similar way). Define $J_1 = [x, f(x)]$ and $J_2 = [f(x), f^2(x)]$. Looking on the positions of the images of the endpoints we conclude that

$$J_2 \subseteq f(J_1) \qquad J_1, J_2 \subset f(J_2)\,.$$

Take any integer $n_0 \geq 0$. Since $J_2 \subset f(J_2)$, Corollary 19 implies that there are intervals

$$I_{n_0} \subseteq I_{n_0-1} \subseteq \cdots \subseteq I_0 = J_2$$

such that $f(I_k) = I_{k-1}$. In particular, $f^{n_0}(I_{n_0}) = I_0$. Applying f once, we get $f^{n_0+1}(I_{n_0}) = f(I_0) = f(J_2) \supset J_1$. Then Lemma 18 implies that there is $I' \subset I_{n_0}$ such that $f^{n_0+1}(I') = J_1$. Apply f one more time: $f^{n_0+2}(I') = f(J_1) \supseteq J_2$. Since $I' \subseteq J_2$ we conclude that $I' \subseteq f^{n_0+2}(I')$ and, consequently, there is $z \in I'$ such that $z = f^{n_0+2}(z)$.

In order to complete the proof of Sharkovskii theorem we need to check that $n = n_0 + 2$ is the prime period for z. For $n_0 = 0$ we have by our construction $z \in J_2$ and $f(z) \in J_1$, therefore z is not a fixed point since $J_1 \cap J_2 = \{f(x)\}$.

For $n_0 \geq 1$, suppose that the period is not prime. Then there is k, $1 \leq k \leq n-1$, such that $z = f^k(z)$. Since $z \in I'$, we get $f^{n_0+1}(z) \in f^{n_0+1}(I') = J_1$. On the other hand,

$$f^{n_0+1}(z) = f^{n_0+1-k}(z) \in f^{n_0+1-k}(I') \subset f^{n_0+1-k}(I_{n_0}) = I_{n_0-k+1} \subseteq J_2.$$

Consequently, $f^{n_0+1}(z) \in J_1 \cap J_2 = \{f(x)\}$. So $f^{n_0+1}(z) = f(x)$. Since z is n_0+2 periodic and $f(x)$ has period 3, we conclude $z = f^2(x)$ and $f(z) = x$. Finally we note that $x \notin J_2$ but $f(z) \in f(I') \subseteq f(I_{n_0}) = I_{n_0-1} \subseteq J_2$. This is a contradiction, which implies that the period is prime. $\square$

Example. The tent map $T : [0,1] \to [0,1]$ is defined by

$$T(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}], \\ 2(1-x) & x \in [\frac{1}{2}, 1] \end{cases}$$

has an orbit of period 3. Indeed,

$$\frac{2}{9} \xmapsto{T} \frac{4}{9} \xmapsto{T} \frac{8}{9} \xmapsto{T} \frac{2}{9}$$

Sharkovskii's theorem implies that T has periodic orbits of all periods.

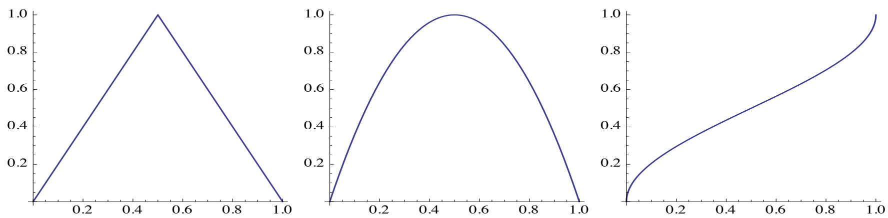


FIGURE 7. The tent map, logistic map and topological conjugacy.

Example. Quadratic map $f : [0,1] \to [0,1]$, $f(x) = 4x(1-x)$. The function $h : [0,1] \to [0,1]$, $h(x) = \frac{2}{\pi} \arcsin \sqrt{x}$, topologically conjugates f and the tent map T /Exercise/. Consequently, f has periodic orbits of all prime periods.

Note: we can explicitly find all trajectories of f:

$$x_n = \sin^2(2^n \pi \theta)$$

where $\theta = \frac{1}{\pi} \arcsin \sqrt{x}$. In fact, the quadratic map is topologically semiconjugate to the angle doubling map $g : \mathbb{S} \to \mathbb{S}$, $g(\theta) = 2\theta \pmod 1$. Indeed, let

$h : \mathbb{S} \to [0.1]$ be defined $h(\theta) = \sin^2(\pi\theta)$. Then we can check it directly:

$$\begin{aligned} f(h(\theta)) &= 4\sin^2(\pi\theta)\big(1 - \sin^2(\pi\theta)\big) \\ &= 4\sin^2(\pi\theta)\cos^2(\pi\theta) = \sin^2(2\pi\theta) = h(2\theta) = h(g(\theta))\,. \end{aligned}$$

Therefore the following diagram is commutative:

$$\begin{tikzcd} \mathbb{S} \arrow[r, "g"] \arrow[d, "h"'] & \mathbb{S} \arrow[d, "h"] \\ {[0,1]} \arrow[r, "f"'] & {[0,1]} \end{tikzcd}$$

We note that h is not a homeomorphism (it is smooth and even differentiable but it is not invertible). Recall, that a circle and an interval are not homeomorphic, i.e., there is no homeomorphism between these two sets.

Remark: Sharkovskii's ordering.

CHAPTER 2

Topological Dynamical Systems

1. Topological transitivity and mixing

1.1. Definitions. A topological space is a set X together with a collection of subsets of X, called open sets, which satisfy the following axioms:

(1) The empty set and X itself are open.
(2) Any union of open sets is open.
(3) The intersection of a finite number of open sets is open.

The topology is used to define notions of convergence and continuity. In particular, a map is continuous if the preimage of every open set is open.

Examples.

(a) Let X be a set. Let X and $\emptyset$ be its only open subsets. Then X is a topological space. In this topology any map $f : X \to X$ is continuous.
(b) Let X be a set. Let any subset of X be open. Then X is a topological space. In this topology any map $f : X \to X$ is continuous.
(c) Let X be a metric space. Let

$$B_r(x) = \{y \in X : \operatorname{dist}(x, y) < r\}.$$

A set $U \subset X$ is open if for any $x \in U$ there is an open ball $B \subset U$ such that $x \in B$. In this topology the continuity coincides with the usual ε-δ definition.

Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}, \mathbb{R}_+, \mathbb{Z}_+\}$.

A family of maps $f^t : X \to X$ is a *topological dynamical system* if

(1) f^t is continuous for all $t \in \mathbb{T}$
(2) $f^0 = \mathrm{Id}$, $f^{t+\tau} = f^t \circ f^\tau$ $\forall t, \tau \in \mathbb{T}$.

In the case of a flow, we require the map $F : X \times \mathbb{T} \to X$, $F(x, t) := f^t(x)$, to be continuous.

We say that two topological dynamical systems $f^t : X \to X$ and $g^t : Y \to Y$ are topologically conjugate if there is a homeomorphism $h : X \to Y$ such that the following diagram commutes for all $t \in \mathbb{T}$:

$$\begin{tikzcd} X \arrow[r, "f^t"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g^t"'] & Y \end{tikzcd}$$

In the case of the discrete time ($\mathbb{T} = \mathbb{Z}, \mathbb{Z}_+$) it is sufficient to require the commutativity for $t = 1$ only. The commutativity for all times follows automatically.

A topological conjugacy maps an orbit of f^t to an orbit of g^t. In particular, a fixed point of one system is mapped to a fixed point of another one, a periodic orbit is mapped to a periodic orbit of the same period, a dense orbit is mapped to a dense orbit, and so on. So two topologically conjugate dynamical systems have similar orbit structure.

1.2. Invariant sets. We say that a set $A \subset X$ is

(1) positively invariant if $f^t(A) \subset A$ $\forall t \geq 0$;
(2) negatively invariant if $f^{-t}(A) \subset A$ $\forall t \geq 0$; [1]
(3) invariant if it is both positively and negatively invariant.

PROPOSITION 21. *If $A \subset X$ is positively invariant, then $X \setminus A$ is negatively invariant.*

Examples:

(1) Let $x \in X$. The trajectory $O_x = \{f^t(x) : t \in \mathbb{T}\}$ is positively invariant
(2) The set of all periodic points is invariant. Let $\tau > 0$. The sets
$$\mathrm{Per}_\tau = \left\{ x \in X : f^\tau(x) = x \right\} \qquad \text{and} \qquad \mathrm{Per} = \bigcup_{\tau>0} \mathrm{Per}_\tau$$
are positively invariant.
(3) **Omega-limit set.** Let $x \in X$. We say that $y \in \omega(x)$ if there is $t_k \to +\infty$ such that $f^{t_k}(x) \to y$. This set is called an ω-limit set. The ω-limit set is positively invariant.

Exercise: $\omega(x) = \bigcap_{\tau>0} \overline{O^+(f^\tau(x))}$, where $O^+(x) = \{ f^t(x) : t \geq 0 \}$ is a positive semi-trajectory.

1.3. Topological transitivity. Let X be a topological space and $f : X \to X$ be a continuous map. Then iterates of f define a topological dynamical system with discrete time, $\mathbb{T} = \mathbb{Z}_+$.

Topological dynamical system $f^t : X \to X$ is called *topologically transitive*, if for any two non-empty open sets $U, V \subset X$ there is $t > 0$ such that $f^t(U) \cap V \neq \emptyset$.

Exercise. Topological dynamical system $f^t : X \to X$ is topologically transitive, if for any two non-empty open sets $U, V \subset X$ there is $t > 0$ such that $U \cap f^{-t}(V) \neq \emptyset$.

A point $x \in X$ is called *topologically transitive*, if $\overline{O_x} = X$.

Exercise. A point is topologically transitive if its orbit visits every non-empty open subset of X, i.e. for any $U \subset X$ open, non-empty, there is $t \in \mathbb{T}$ such that $f^t(x) \in U$.

Examples:

(a) An irrational rotation $f : x \mapsto x + \alpha \pmod 1$ is topologically transitive. All points of $\mathbb{S}$ are topologically transitive.

[1] $f^{-t}(A) = \{ x \in X : f^t(x) \in A \}$ and is defined even if the map f^t is not invertible.

(b) An expanding map of the circle (e.g. $f(x) = 2x \pmod 1$) is topologically transitive. There are topologically transitive points. Some points are not topologically transitive (e.g. periodic points).

(c) $X = \mathbb{Z}_p = \{0, 1, \dots, p-1\}$, $f(x) = x + 1 \pmod p$. Topology on X is discrete (any subset is open). This dynamical system consists of a single trajectory. So any point in X is topologically transitive. The dynamical system is topologically transitive.

(d) Let $X = \{0, 1\}$ with discrete topology. $f : X \to X$ is defined by $f(0) = f(1) = 0$. Then $O_1 = X$, so 1 is a topologically transitive point. But f^t is not topologically transitive. Indeed, let $U = \{0\}$ and $V = \{1\}$, then $f^n U = U$ and $f^n U \cap V = \emptyset$ for all $n > 0$.

(e) Let $X \subset [0, 1]$ be the set of all real numbers in $[0, 1]$ which can be represented by a finite binary fraction:

$$X = \left\{x : x = a_1 2^{-1} + a_2 2^{-2} + \cdots + a_n 2^{-n}, \quad n \in \mathbb{N}, a_k \in \{0, 1\}\right\} .$$

Let $f : X \to X$ be defined by $f(x) = 2x \pmod 1$. Then any orbit is finite (indeed, for any $x \in X$ there is $n \in \mathbb{N}$ such that $f^n(x) = 0$). As X is not finite, there is no dense orbit. On the other hand, f^t is topologically transitive.

In order to discuss relations between topological transitivity and topologically transitive points we need some definitions from topology.

A residual set = a countable intersection of open dense sets.[2] A topological space X is a *Baire space* if any residual set in X is dense in X.

Any complete metric space is a Baire space.

The following theorems can be naturally stated for a Baire space. In order to avoid writing proofs using the terminology from the topology, we restrict our discussion to complete separable metric spaces.

A topological space is called *separable* if it contains a countable dense set.

Example: $\mathbb{R}^n$, $\mathbb{S}$, $C^0[0, 1]$ are separable. Any compact metric space is separable.

PROPOSITION 22. *Let X be a complete separable metric space and $f : X \to X$ be continuous. The topological dynamical system defined by the map is topologically transitive if and only if topologically transitive points form a residual set*

PROOF. Let B_i, $i \in \mathbb{N}$, be a countable collection of open balls such that any non-empty open set in X contains a ball from this collection.[3]

---

[2]**Example**: Irrational numbers form a residual set in $\mathbb{R}$.

[3]This collection is called a *base of topology.* Let $(a_j)_{j \in \mathbb{N}} \subset X$ be dense in X. Take any non-empty open set $U \subset X$. Since $(a_j)_{j \geq 1}$ is dense in X, there is $a_j \in U$. Since U is open, there is k such that $B_{1/k}(a_j) \subset U$. The set of all balls $B_{1/k}(a_j)$ is countable, then counting the balls we obtain the sequence B_i.

Then we note that

$$\begin{aligned}
x \in X &\text{ is topologically transitive} \\
&\iff \forall i \in \mathbb{N}\ \exists t \in \mathbb{T} : f^t(x) \in B_i \\
&\iff \forall i \in \mathbb{N}\ x \in A_i := \bigcup_{t\in\mathbb{T}} f^{-t}(B_i) \\
&\iff x \in A := \bigcap_{i\in\mathbb{N}} \bigcup_{t\in\mathbb{T}} f^{-t}(B_i).
\end{aligned}$$

So A is the set of all topologically transitive points.

The set $A_i := \bigcup_{t\in\mathbb{T}} f^{-t}(B_i)$ is open. Indeed, f^t is continuous for any $t \in \mathbb{T}$, and B_i is open, so A_i is a union of open sets, and hence open.

Suppose that f is topologically transitive. Since B_i is open and non-empty, the set $\cup_{t>0} f^{-t}(B_i)$ has a non-empty intersection with every open set. Consequently, A_i is open and dense. Then the set $A = \bigcap_{i\in\mathbb{N}} A_i$ is residual.

Now suppose A is residual. Since X is a Baire space, A is dense. Since $A_i \supseteq A$ for every i, A_i is also dense. Take any $U, V \subset X$ open and non-empty. Then there is B_i such that $B_i \subset U$. Since A_i is dense, $V \cap A_i \neq \emptyset$ and consequently there is $t \in \mathbb{T}$ such that $f^{-t}(B_i) \cap V \neq \emptyset$. So f^t is topologically transitive. □

A point $x \in X$ is isolated if $\{x\}$ is open. Let X be a metric space. A point $x \in X$ is isolated if there is $r > 0$ such that the open ball $B_r(x) = \{x\}$.

Proposition 23. *Let X be a metric space without isolated points and $f : X \to X$ be continuous. If the topological dynamics system defined by f has a transitive point, then it is topologically transitive.*

Proof. Let x be a topologically transitive point. Let $U \subset X$ be open and non-empty. Then define the "hitting set" by

$$H(U) = \{n \in \mathbb{T} : f^n(x) \in U\}.$$

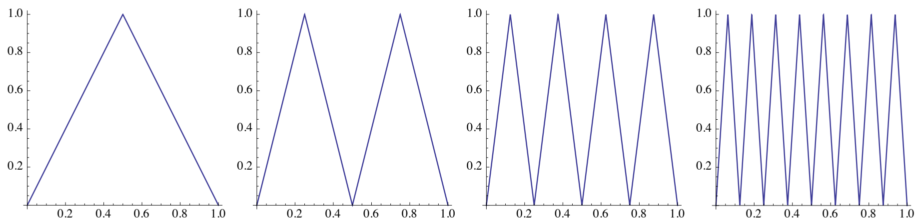
The set $H(U)$ is infinite. Indeed, suppose $H(U)$ is finite, then there are finitely many n such that $f^n(x) \in U$ so the set $U \cap O_x$ is finite. On the other hand, the absence of isolated points implies that U contains infinitely many points (otherwise we could find a ball which consists of a single point taking as a radius the least distance between points in U). Then the set $U \setminus O_x$ is non-empty and open. This implies that O_x is not dense. The contradiction implies $H(U)$ is infinite.

Let $U, V \subset X$ be open and non-empty. The hitting sets $H(U)$ and $H(V)$ are infinite. Since $\mathbb{T} = \mathbb{Z}_+$, the unboundedness of $H(V)$ implies that for any $k \in H(U)$ there is $m \in H(V)$ such that $m > k$. Let $n = m - k$ and $y = f^k(x)$. Then $y \in U$ and $f^n(y) = f^m(x) \in V$. So we found $n > 0$ such that $f^n(U) \cap V \neq \emptyset$ and f is topologically transitive. □

Examples: all system in the list below are topologically transitive.

(1) an elementary periodic cascade: $X = \{0, 1, 2, \ldots, p-1\} = \mathbb{Z}_p$, $t \in \mathbb{Z}$, $f^t(x) = x + t \pmod p$. For any $x \in X$, $O_x = X$. So all points are topologically transitive.

(2) an elementary periodic flow: $\omega \in \mathbb{R}$ is fixed. $X = \{|z| = 1\} \subset \mathbb{C}$, $t \in \mathbb{R}$, $f^t(z) = e^{it\omega}z$. For any $x \in X$, $O_x = X$. So all points are topologically transitive.
(3) irrational rotations: $\alpha = \frac{\omega}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$. $X = \{|z| = 1\} \subset \mathbb{C}$, $t \in \mathbb{Z}$, $f^t(z) = e^{it\omega}z$. We already checked that all orbits are topologically transitive. The circle is a compact metric space and R_α is invertible, so f^t is topologically transitive.
(4) Expanding maps of the circle, e.g., $X = \{z \in \mathbb{C} : |z| = 1\}$, $t \in \mathbb{Z}_+$, $f(z) = z^m$. We already established existence of a dense orbit. Moreover, any open set contains an interval. Since the map is continuous and expanding the image of the interval is a longer interval (m times longer in the example above, in general use the mean value theorem for derivatives). Therefore after a finite number of iterates, the image of the interval will be longer than 1, so it will intersect all subsets of $\mathbb{S}$. So f^t is transitive.
(5) the tent map $f : [0, 1] \to [0, 1]$, $f(x) = 1 - |2x - 1|$. The graphs of $f(x), f^2(x), f^3(x), f^4(x)$ are shown below:



It is easy to see that the graph of f^n consists of 2^{n-1} "tents". So for any two intervals $I, J \subset [0, 1]$ there is n such that $f^{-n}(I) \cap J \neq \emptyset$. Therefore f^t is topologically transitive.

Remark: let us replace the phase space X by $\tilde{X} = [0, 1] \cap \mathbb{Q}$ and consider the tent map $\tilde{f} : \tilde{X} \to \tilde{X}$ defined by the same formula. Note that the tent map maps a rational point into a rational one.

For any $x \in \tilde{X}$, O_x is a finite set. Indeed, if $x = \frac{q}{p}$ with $p, q \in \mathbb{N}$, then $f(x) = \frac{q'}{p}$ with some $q' \in \mathbb{N}$ and the same p. So the number of points in O_x is not larger than p. Therefore there are no topologically transitive points in $\tilde{X}$. Nevertheless, the arguments above can be used to show that $\tilde{f}$ is topologically transitive.
(6) a topological shift map. $X = \Sigma = \{0, \ldots, p-1\}^{\mathbb{N}}$, $t \in \mathbb{Z}_+$, $f^t = \sigma^t$, where $\sigma : \Sigma \to \Sigma$ is the shift map:
$$\sigma(\omega_0, \omega_1, \omega_2, \ldots) = (\omega_1, \omega_2, \ldots)$$
We postpone the proof of topological transitivity for σ till after discussing the topology on Σ.

1.4. Topological mixing. $f^t : X \to X$ is topologically mixing if $\forall U, V \subset X$ open, non-empty, there is $t_0 \in \mathbb{T}$ such that $f^tU \cap V \neq \emptyset$ for all $t > t_0$.

Proposition 24. *If f^t is topologically mixing, then f^t is topologically transitive.*

PROOF. Follows directly from the definitions. □

Examples:

(1) an elementary periodic cascade is not mixing. Hint: $U = \{0\}$, $V = \{1\}$.
(2) an elementary periodic flow is not mixing. Hint: U, V are two small intervals.
(3) irrational rotations are not mixing. Hint: U, V are two small intervals.
(4) expanding maps of the circle are mixing.
(5) the tent map is mixing.
(6) a topological shift map is mixing /to be proved later/.

2. Shift maps

2.1. Symbolic dynamics. Let X be a topological space and $S_k \subset X$ closed subsets such that $\bigcup_{k=1}^{N} S_k = X$ and the interiors of S_k are disjoint (i.e. an intersection $S_{k_1} \cap S_{k_2}$ with $k_1 \neq k_2$ does not contain any open subset).

Let $f : X \to X$ be a continuous map. Take a point $x \in X$ and set $x_k = f^k(x)$. For every k there is $\omega_k \in \{1, \ldots, N\}$ such that $x_k \in S_{\omega_k}$. Therefore we can associate with the point x a sequence $\omega = (\omega_k)_{k=0}^{\infty}$.

If the trajectory of x does not intersect the boundaries of S_k, the sequence ω is defined uniquely.

If the map f is invertible, we can define a be-infinite sequence $(\omega_k)_{k=-\infty}^{\infty}$ in a similar way.

For any sequence ω the set $\bigcap_k f^{-k}(S_{\omega_k})$ consists of all points which follow the itinerary prescribed by the sequence, i.e. $f^k(x) \in S_{\omega_k}$. The set is closed (a countable intersection of closed sets) and can be empty.

The definition implies directly that if x corresponds to a sequence ω, then $f(x)$ corresponds to the shifted sequence $\omega' = \sigma(\omega)$, i.e., $\omega'_k = \omega_{k+1}$.

We have already seen that an expanding map of the circle is topologically semiconjugate to a shift map. Topological conjugacy preserves many important features of the dynamics such as existence and density of periodic orbits, topological transitivity and mixing. Usually it is much easier to establish these properties for the shift maps.

In this section we will study the shift map in more detail.

2.2. Shift spaces. Let $N \geq 2$ be integer and define the shift space

$$\begin{aligned} \Sigma &= \left\{1, 2, \ldots, N\right\}^{\mathbb{N}} \\ &= \left\{(\omega_n)_{n=0}^{\infty} : \forall n \ \omega_n \in \{1, \ldots, N\}\right\}. \end{aligned}$$

The shift space Σ is the set of all sequences in $\{1, \ldots, N\}$.

Let $\lambda > 1$. For any two sequences $\omega, \omega' \in \Sigma$ let

$$d_\lambda(\omega, \omega') = \max_{n \geq 0} \left\{ \frac{e(\omega_n, \omega'_n)}{\lambda^n} \right\},$$

where

$$e(i, j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}.$$

Remark: $e(i,j) = 1 - \delta_{ij}$ where δ_{ij} is the Kronecker symbol.

Exercise: Check that d_λ is a metric on Σ.

The metric defines a topology on Σ in the traditional way: First, an open ball of a radius $r > 0$ centred at $\omega \in \Sigma$ is given by

$$B_r(\omega) := \left\{\omega' \in \Sigma : d_\lambda(\omega, \omega') < r\right\} = \left\{\omega' \in \Sigma : \omega_k = \omega'_k \quad \text{if } \lambda^{-k} > r\right\}.$$

Then a subset $U \subset \Sigma$ is open if for every point $\omega \in U$ there is $r > 0$ such that the ball $B_r(\omega) \subset U$.

It is easy to check directly from this definition that any open ball is indeed an open set.

Definition. Let $\omega_0, \dots, \omega_n \in \{1, \dots, N\}$. A *cylinder set* is a subset of Σ

$$C_{\omega_0,\dots,\omega_n} = \left\{(\omega'_k)_{k=0}^\infty : \omega'_k = \omega_k \text{ for } 0 \le k \le n\right\}.$$

Proposition 25. *If $\omega \in \Sigma$, then for any integer $n \ge 0$*

$$C_{\omega_0,\dots,\omega_n} = B_{\lambda^{-n}}(\omega).$$

Proof. The proposition follows directly from the definitions of the cylinder, the ball and the metric d_λ. Indeed, $\omega' \in B_{\lambda^{-n}}(\omega)$ iff $d_\lambda(\omega, \omega') < \lambda^{-n}$. Equivalently, $\omega_k = \omega'_k$ for all k such that $\lambda^{-k} \ge \lambda^{-n}$, i.e., for $k \le n$. The last property is true iff $\omega' \in C_{\omega_0,\dots,\omega_n}$. ☐

Proposition 26. *A cylinder set is both open and closed.*

Proof. According to the previous proposition any cylinder is an open ball, hence open.

In order to show that the cylinder set is closed, consider its compliment. If $\omega'' \notin C_{\omega_0,\dots,\omega_n}$, then $\omega''_k \ne \omega_k$ for some k, $0 \le k \le n$. If $\omega' \in C_{\omega_0,\dots,\omega_n}$, then $\omega'_k = \omega_k$. So $d_\lambda(\omega'', \omega') \ge \lambda^{-k}$. Then $B_r(\omega'') \cap C_{\omega_0,\dots,\omega_n} = \emptyset$ for all $r < \lambda^{-k}$. Consequently, the complement to the cylinder is open, and the cylinder itself is closed. ☐

Proposition 27. *The space Σ is compact.*

Proof. Take a sequence $\omega^{(j)} \in \Sigma$. Then construct a convergent subsequence inductively. The space Σ contains exactly N cylinders C_j, $j = 1, \dots, N$.

One of those contains infinitely many elements of the sequence. Denote it by B_1. Take j_1 to be the smallest index such that $\omega^{(j_1)} \in B_1$.

Then repeat inductively, the cylinder B_{n-1} of length $n-1$ contains exactly N sub-cylinders of length n. One of those, say $B_n \subset B_{n-1}$, contains infinitely many elements. Take j_n to be the smallest index such that $\omega^{(j_n)} \in B_n$ and $j_n > j_{n-1}$.

Since $\operatorname{diam}(B_n) = \lambda^{-n} \to 0$, the subsequence $\omega^{(j_n)}$ converges. ☐

2.3. Shifts. The shift map $\sigma : \Sigma \to \Sigma$ is defined by

$$(\sigma(\omega))_k = \omega_{k+1} \quad \text{for all } k \ge 0.$$

Every point $\omega \in \Sigma$ has exactly N preimages.

Proposition 28. *Periodic points of σ are dense in Σ.*

PROOF. A sequence $\omega \in \Sigma$ is a periodic point of period n for the shift map, i.e., $\sigma^n(\omega) = \omega$, iff the sequence ω is periodic:

$$\omega = (\underbrace{\omega_0, \ldots, \omega_{n-1}}_{n}, \underbrace{\omega_0, \ldots, \omega_{n-1}}_{n}, \underbrace{\omega_0, \ldots, \omega_{n-1}}_{n}, \ldots)\,.$$

Periodic sequences are obviously dense in Σ as every cylinder set contains a periodic sequence. □

Exercise: Find the number of periodic orbits of period n /without assuming the period is prime/.

PROPOSITION 29. *$\sigma : \Sigma \to \Sigma$ is topologically mixing.*

PROOF. Take any two non-empty open sets $U, V \subset \Sigma$. Since U, V are open, then for any $\alpha \in U$ and any $\beta \in V$ there is $n \in \mathbb{N}$ such that cylinder sets

$$\begin{aligned} C_{\alpha_0,\ldots,\alpha_{n-1}} &\subset U\,, \\ C_{\beta_0,\ldots,\beta_{n-1}} &\subset V\,. \end{aligned}$$

Take any integer $m > n$. Consider the sequence

$$\omega = (\underbrace{\alpha_0, \ldots, \alpha_{m-1}}_{m}, \underbrace{\beta_0, \ldots, \beta_{n-1}}_{n}, \ldots)\,.$$

Obviously, $\omega \in U$ and $\sigma^m(\omega) \in V$. Consequently, $U \cap \sigma^{-m}(V) \neq \emptyset$ for all $m > n$ and therefore σ is mixing. □

2.4. Two sided shifts. Let $\Omega = \{1, 2, \ldots, N\}$ where $N \geq 2$. Consider the space of all bi-infinite sequences:

$$\Sigma = \Omega^{\mathbb{Z}} = \left\{ (\omega_n)_{n=-\infty}^{+\infty} : \omega_n \in \Omega \right\}\,.$$

Let $\lambda > 1$ and define a metric on Σ: For any $\omega, \omega' \in \Sigma$ let

$$d_\lambda(\omega, \omega') = \max_{n \in \mathbb{Z}} \frac{e(\omega_n, \omega'_n)}{\lambda^{|n|}}\,,$$

where

$$e(i, j) = 1 - \delta_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

The definition implies that ω' is close to ω if both sequences share a large common block centred around the zero position.

PROPOSITION 30. *Σ is a compact metric space.*

PROOF. The proof is similar to the case of a single sided shifts /Exercise/. □

The shift map $\sigma : \Sigma \to \Sigma$ shifts a sequence by one position to the left, i.e., $\omega' = \sigma(\omega)$ if and only if $\omega'_n = \omega_{n+1}$ for all $n \in \mathbb{Z}$.

PROPOSITION 31. *$\sigma : \Sigma \to \Sigma$ is a homeomorphism.*

PROPOSITION 32. *Periodic points of $\sigma : \Sigma \to \Sigma$ are dense in Σ. The map $\sigma : \Sigma \to \Sigma$ is a topological mixing.*

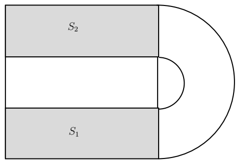
PROOF. The proof is similar to the case of a single sided shifts /Exercise/. □

3. Smale horseshoe

In this section we will study an example of a diffeomorphism with the following remarkable property: there is a closed invariant subset Λ such that the restriction of the diffeomorphism on Λ is topologically conjugate to the shift map.

We will define this map explicitly, but similar constructions are of much more general interest.

Linear horseshoe map. Let $\Delta = [0,1]\times[0,1]$. Consider a diffeomorphism $f:\mathbb{R}^2 \to \mathbb{R}^2$ such that the image $f(\Delta)$ has the form of a horseshoe as shown on the figure.



Let us define the restriction of the map f on Δ more precisely. Consider two closed rectangles $L, R \subset \Delta$:

$$L = [0, \tfrac{1}{3}] \times [0,1] \qquad \text{and} \qquad R = [\tfrac{2}{3}, 1] \times [0,1]\,.$$

Let f be any diffeomorphism such that $f^{-1}(\Delta)\cap\Delta = L \cup R$ and on these rectangles f is linear:

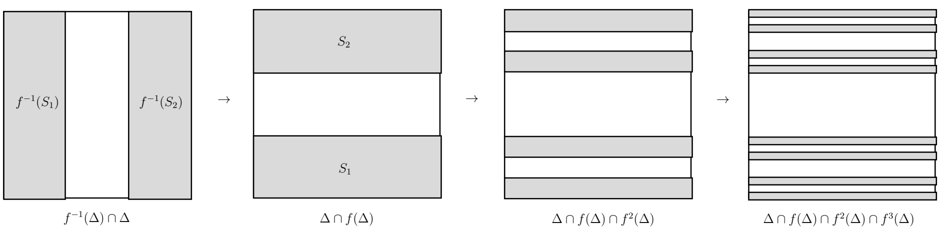
$$f(x,y) = \begin{cases} (3x, \frac{1}{3}y), & (x,y)\in L\,,\\ (3-3x, 1-\frac{1}{3}y), & (x,y)\in R\,. \end{cases}$$

Moreover, any point from $\Delta\setminus(L\cup R)$ has an image outside Δ.

This definition implies that $F(L) = S_1$ and $F(R) = S_2$ where

$$S_1 = [0,1]\times[0,\tfrac{1}{3}] \quad \text{and} \quad S_2 = [0,1]\times[\tfrac{2}{3},1]\,.$$

Moreover, $f(\Delta)\cap\Delta = S_1\cup S_2$. Then iterating the map one more time, we note that $\Delta\cap f(\Delta)\cap f^2(\Delta)$ consists of 4 horizontal rectangular strips. Then $\Delta\cap f(\Delta)\cap f^2(\Delta)\cap f^3(\Delta)$ consists of 8 horizontal rectangular strips, and $\bigcap_{k=0}^{n} f^k(\Delta)$ consists of 2^n horizontal strips. This construction resembles the definition of the Cantor set as on each step the middle third is removed.



Using similar argument, show that $\displaystyle\bigcap_{k=0}^{n} f^{-k}(\Delta)$ consists of 2^n vertical strips.

Proposition 33. *The set* Λ *defined by*

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^{-n}(\Delta)$$

is closed, non-empty and invariant. Moreover, it is a product of two middle-third Cantor sets.

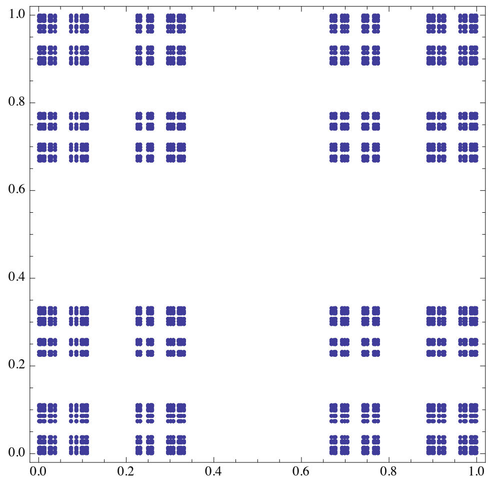


Figure 1. The set Λ

Proposition 34. *The restriction of* f *onto* Λ *(i.e.* $f : \Lambda \to \Lambda$*) is topologically conjugate to* $\sigma : \Sigma \to \Sigma$*, a left shift on the space of two-sided sequences of two symbols.*

Proof. Let $\Sigma = \{1, 2\}^{\mathbb{Z}}$ and $\sigma : \Sigma \to \Sigma$ be the left shift. Take any $\omega \in \Sigma$ and any integer $r \geq 1$, then define the set

$$C_r(\omega) = \bigcap_{-r\leq n<r} f^{-n}(S_{\omega_n})\,.$$

It has the following properties

- $C_r(\omega)$ is closed and non-empty;
- $C_r(\omega)$ is a square;
- The length of its side is 3^{-r};
- $C_{r+1}(\omega) \subset C_r(\omega)$.

It follows that the intersection $\bigcap_{r\geq 0} C_r(\omega)$ consists of exactly one point, denote it by $\mathbf{x}$. Then define a map $h : \Sigma \to \Lambda$ by setting $h(\omega) = \mathbf{x}$.

It follows that

$$h(\omega) \in \bigcap_{n\in\mathbb{Z}} f^{-n}(S_{\omega_n})$$

and $f^n(\mathbf{x}) \in S_{\omega_n}$ for all $n \in \mathbb{Z}$.

h is continuous: Let $\lambda > 1$ and d_λ be the metric on Σ. Take any r and any $\omega' \in \Sigma$ such that $d_\lambda(\omega, \omega') < \lambda^{-r}$. Then $\omega'_n = \omega_n$ for all $|n| < r$. So $C_r(\omega') = C_r(\omega)$. We note $h(\omega) \in C_r(\omega)$ and $h(\omega') \in C_r(\omega')$. Then taking into account the size of the square $C_r(\omega)$ we conclude that $\|h(\omega) - h(\omega')\| < 2\cdot 3^{-r}$. The continuity of h follows from the usual ϵ-δ definition of continuity.

h is injective: Suppose that for some $\mathbf{x} \in \Lambda$ we can find two sequences such that $h(\omega) = h(\omega') = \mathbf{x}$. Then the definition of h implies that $f^n(x) \in S_{\omega_n}$ and $f^n(x) \in S_{\omega'_n}$ for all $n \in \mathbb{Z}$. Since $S_1 \cap S_2 = \emptyset$, we conclude that $\omega_n = \omega'_n$ for all n.

h is surjective: Let $\mathbf{x} \in \Lambda$. Then, for any $n \in \mathbb{Z}$, $f^n(\mathbf{x}) \in \Delta \cap f(\Delta) = S_1 \cup S_2$. Let $\omega_n = 1$ if $f^n(\mathbf{x}) \in S_1$ and $\omega_n = 2$ if $f^n(\mathbf{x}) \in S_2$. So $\mathbf{x} \in \bigcap_{n\in\mathbb{Z}} f^{-n}(S_{\omega_n})$ and $h(\omega) = \mathbf{x}$.

h is a homeomorphism: Since Σ and Λ are compact and h is a continuous bijection, the inverse map h^{-1} is also continuous.

h is a topological conjugacy: For any $\omega \in \Sigma$ we have established that $h(\omega) = \mathbf{x}$ iff $f^n(\mathbf{x}) \in S_{\omega_n}$. Let $\mathbf{y} = f(\mathbf{x})$. Obviously, $f^n(\mathbf{y}) = f^{n+1}(\mathbf{x}) \in S_{\omega_{n+1}}$. Consequently, $h(\sigma(\omega)) = f(h(\omega))$, and the following diagram commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \Lambda & \xrightarrow[f]{} & \Lambda \end{array}$$

$\square$

4. Topological entropy

Topological entropy measures complexity of trajectories of a topological dynamical system. It has an additional important property: the topological entropy is invariant under topological conjugacy.

For the sake of simplicity we define the topological entropy only for a dynamical system on a compact metric space (X, d). The metric d will be used in the definition, but later we will check that the entropy depends on the topology only as equivalent metrics lead to the same value of the topological entropy.

4.1. Topological entropy.

4.1. Topological entropy. Let (X, d) be a compact metric space and $f : X \to X$ be continuous. The topological entropy of f is defined using the following construction.

Let $n \in \mathbb{N}$. Then the equation

$$d_n(x, y) = \max_{0\le i\le n} d(f^i(x), f^i(y))$$

defines a metric on X (Exercise: check the axioms of the metric are satisfied). In particular, $d_0(x, y) = d(x, y)$. Then let

$$B(x, n, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}$$

be a ball with respect to the metric d_n.

A set $E \subset X$ is called an (n, ε)-spanning set if $X \subseteq \bigcup_{x\in E} B(x, n, \varepsilon)$.

Note: Since $B(x, n, \varepsilon) \subset X$, we can also write $X = \bigcup_{x\in E} B(x, n, \varepsilon)$.

Let $S(n, \varepsilon)$ denote the least cardinality of an (n, ε)-spanning set. Since X is compact, any open cover of X contains a finite subcover, so $S(n, \varepsilon)$ is a positive integer.

Then we define an exponential growth rate:

$$h(f,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log S(n,\varepsilon).$$

We note that the upper limit always exists (but can be $+\infty$). The function $h(f,\varepsilon)$ is monotone: if $0 < \varepsilon' < \varepsilon$ then $h(f,\varepsilon') \geq h(f,\varepsilon)$. This property follows from the observation that $S(n,\varepsilon') \geq S(n,\varepsilon)$ as an (n,ε')-spanning set is also (n,ε)-spanning.

Finally, the topological entropy of f is defined by

$$h(f) = \lim_{\varepsilon\to+0} h(f,\varepsilon)\,.$$

The monotonicity implies that the limit exists (but can be $+\infty$). Since $S(n,\varepsilon) \geq 1$ we conclude that $h(f,\varepsilon) \geq 0$ and $h(f) \geq 0$.

4.2. Equivalent metrics and the topological entropy. We say that two metrics d and d' are equivalent on X is they define the same topology, i.e., the collection of open sets are the same in (X,d) and (X,d'). It is easy to see that two metrics are equivalent if and only if the identity map $I : (X,d) \to (X,d')$, $I(x) = x$, is a homeomorphism. /Hint: a map is continuous iff preimage of every open set is open./

Proposition 35. *Let (X,d) be a compact metric space, d' a metric on X equivalent to d and $f : X \to X$ continuous. Then $h_{d'}(f) = h_d(f)$.*

Proof. We recall that a continuous function on a compact metric space is uniformly continuous. The uniform continuity of the identity map $I : (X,d) \to (X,d')$ implies that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for any $x,y \in X$

$$d(x,y) < \delta \qquad \Longrightarrow \qquad d'(x,y) < \varepsilon\,.$$

Consequently, $B_d(x,n,\delta) \subseteq B_{d'}(x,n,\varepsilon)$. If E is (n,δ)-spanning with respect to d, then it is also (n,ε)-spanning with respect to d'. So $S_d(n,\delta) \geq S_{d'}(n,\varepsilon)$. We conclude $h_d(f,\delta) \geq h_{d'}(f,\varepsilon)$ and, taking the limit, $h_d(f) \geq h_{d'}(f)$.

Swapping d and d' we also get $h_{d'}(f) \geq h_d(f)$. So $h_{d'}(f) = h_d(f)$. □

4.3. Examples.

(1) Let $\alpha \in \mathbb{R}$ and $R_\alpha(x) = x + \alpha \pmod 1$. Then $h(R_\alpha) = 0$.

Proof: Note that the linear rotation preserves distances, so $d_n(x,y) = \max_{0\leq i\leq n} d(f^i(x), f^i(y)) = d(x,y)$. Consequently, $B(x,n,\varepsilon) = B(x,\varepsilon)$ for all n. Consequently, $S(n,\varepsilon)$ is independent of n. It follows immediately that

$$h(f,\varepsilon) = \limsup_{n\to\infty} \frac{\log S(n,\varepsilon)}{n} = 0$$

and

$$h(f) = \lim_{\varepsilon\to+0} h(f,\varepsilon) = 0\,.$$

(2) If $f : X \to X$ is an isometry and X is compact, then $h(\mathrm{f}) = 0$.

The proof is essentially the same as in the case of the rotation.

(3) $f : \mathbb{S} \to \mathbb{S}$ is a C^2 diffeomorphism with an irrational rotation number, $h(f) = 0$.

This property follows from the fact that f is topologically conjugate to a rotation (Denjoy theorem).

Remark: If $f : \mathbb{S} \to \mathbb{S}$ is a homeomorphism, then $h(f) = 0$. /No proof is given in the lectures/

(4) Let $f : \mathbb{S} \to \mathbb{S}$ be defined by $f(x) = mx \pmod 1$ where the integer $m \geq 2$ is the degree of f. Let us show that $h(f) = \log m$.

Indeed, take any $\varepsilon \in (0, \frac{1}{2})$. Since the map is linear, for any $n \in \mathbb{N}$ we get
$$B(x, n, \varepsilon) = B(x, \tfrac{\varepsilon}{m^n})$$
where $B(x, r)$ is the ball in the usual metric on $\mathbb{S}$.

As the circle has a unit length, the smallest (n, ε)-spanning set consists of
$$S(n, \varepsilon) = \left[\frac{m^n}{\varepsilon}\right] + 1$$
elements. Then
$$\begin{aligned} h(f, \varepsilon) &= \limsup_{n\to\infty} \frac{1}{n} \log S(n, \varepsilon) = \lim_{n\to\infty} \frac{1}{n} \log\left(\left[\frac{m^n}{\varepsilon}\right] + 1\right) \\ &= \lim_{n\to\infty} \frac{\log m^n}{n} + \lim_{n\to\infty} \frac{1}{n} \log\left(m^{-n}\left(\left[\frac{m^n}{\varepsilon}\right] + 1\right)\right) \\ &= \log m\,. \end{aligned}$$
Finally,
$$h(f) = \lim_{\varepsilon\to+0} h(f, \varepsilon) = \log m.$$

4.4. Entropy and topological semiconjugacy.

PROPOSITION 36. *Let (X, d_X) and (Y, d_Y) be two compact metric spaces, $f : X \to X$ and $g : Y \to Y$ continuous, and $\pi : X \to Y$ be a continuous surjective map such that the following diagram commutes:*
$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{g} & Y \end{array}$$
Then $h(g) \leq h(f)$.

PROOF. Since X is compact, π is uniformly continuous, i.e.,
$$\forall\varepsilon > 0\ \exists\delta > 0 : d_X(x_1, x_2) < \delta \implies d_Y(y_1, y_2) < \varepsilon$$
where $y_1 = \pi(x_1)$ and $y_2 = \pi(x_2)$. Therefore
$$\pi(B_X(x, n, \delta)) \subseteq B_Y(\pi(x), n, \varepsilon).$$
As π is surjective, we conclude that $S_X(n, \delta) \geq S_Y(n, \varepsilon)$, which implies $h(f, \delta) \geq h(g, \varepsilon)$. The proposition follows from taking the limit $\varepsilon \to 0$. $\square$

Swapping f and g we obtain the following corollary.

PROPOSITION 37. *Let (X, d_X) and (Y, d_Y) be two compact metric spaces, $f : X \to X$ and $g : Y \to Y$ continuous, and $\pi : X \to Y$ be a homeomorphism such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{g} & Y \end{array}$$

Then $h(f) = h(g)$.

Since any expanding map of the circle is topologically conjugate to the linear expanding map of the same degree, we conclude $h(f) = \log m$ where m is the degree of f. (orientation preserving case is assumed)

4.5. (n, ε)-separated sets. We give an alternative description of the topological entropy. Let (X, d) be a compact metric space and $f : X \to X$ be continuous. Let $n \in \mathbb{N}$ and

$$d_n(x, y) = \max_{0 \le i \le n} d(f^i(x), f^i(y)).$$

Let $\varepsilon > 0$. A set $E \subset X$ is called (n, ε)-*separated* if $d_n(x, y) \ge \varepsilon$ $\forall x, y \in E$ such that $x \ne y$.

Since X is compact, the set E is finite (Exercise).

Let $N(n, \varepsilon)$ be the maximum number of elements in an (n, ε)-separated set.

PROPOSITION 38. *Let (X, d) be a compact metric space and $f : X \to X$ continuous. Then*

$$h(f) = \lim_{\varepsilon \to +0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon)\,.$$

PROOF. Step I: we show that $S(n, \varepsilon) \le N(n, \varepsilon) \le S(\frac{\varepsilon}{2}, n)$.

Let E be an (n, ε)-separated set of cardinality $N(n, \varepsilon)$ (the maximal possible cardinality). Then

$$\bigcup_{x \in E} B(x, n, \varepsilon) = X.$$

Indeed, if there is $y \in X \setminus \bigcup_{x \in E} B(x, n, \varepsilon)$, then $y \notin B(x, n, \varepsilon)$ for all $x \in E$. Thus $d_n(y, x) \ge \varepsilon$ for all $x \in E$ and, consequently, $E \cup \{y\}$ is also (n, ε)-separated, which is not possible as the set E already has the largest possible number of elements.

Consequently, E is (n, ε)-spanning. So $N(n, \varepsilon) \ge S(n, \varepsilon)$.

Now let E' be an (ε', n)-spanning set of cardinality $S(\varepsilon', n)$. Then

$$\bigcup_{x \in E'} B(x, n, \varepsilon') = X\,.$$

Let $\varepsilon' = \frac{\varepsilon}{2}$ and $x \in E'$. For any two points $y, y' \in B(x, n, \varepsilon')$, the triangle inequality implies $d_n(y, y') \le d_n(y, x) + d_n(y', x) < \varepsilon$. Consequently, $B(x, n, \varepsilon') \cap E$ contains at most one point. So $N(n, \varepsilon) \le S(\frac{\varepsilon}{2}, n)$.

Step II. The inequalities of Step I state that

$$S(n, \varepsilon) \le N(n, \varepsilon) \le S(n, \tfrac{\varepsilon}{2}).$$

Since log is monotone increasing:

$$\frac{1}{n}\log S(n,\varepsilon) \le \frac{1}{n}\log N(n,\varepsilon) \le \frac{1}{n}\log S(n,\tfrac{\varepsilon}{2}).$$

Taking lim sup we get

$$h(f,\varepsilon) \le \limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon) \le h(f,\tfrac{\varepsilon}{2})$$

Taking the limit $\varepsilon \to +0$ we get

$$h(f) \le \lim_{\varepsilon\to+0}\limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon) \le h(f)$$

which completes the proof. □

5. Topological Markov chains

An $N \times N$ matrix is called a *transition matrix* if all entries A_{ij}, $1 \le i,j \le N$, are either 0 or 1.

The matrix A is used to indicate which pairs of consecutive symbols are admissible in a symbolic sequence: ω_i can be followed by ω_j if and only if $\mathrm{A}_{ij} = 1$. We denote the space of all admissible sequences by

$$\Sigma_{\mathrm{A}} = \left\{(\omega_n)_{n=0}^{\infty} : \omega_n \in \{1,\ldots,N\},\ \mathrm{A}_{\omega_n\omega_{n+1}} = 1 \quad \forall n \ge 0\right\}$$

The restriction of the shift map $\sigma_{\mathrm{A}} = \sigma|_{\Sigma_{\mathrm{A}}}$ is called a *topological Markov chain* associated with the matrix A.

Examples.

1) $\mathrm{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The shift space Σ_{A} consists of all sequences which do not contain the word “22”.

2) $\mathrm{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The shift space $\Sigma_{\mathrm{A}} = \Sigma$.

Remark. The space Σ_{A} consists of all sequences which do not include blocks of two symbols $\omega_i\omega_j$ such that $\mathrm{A}_{\omega_i\omega_j} = 0$. This space is an important case of a more general notion of a *subshift of finite type*, where the shift space is defined by excluding all sequences which contain a finite word from a prescribed finite list. Naturally, such shift spaces are invariant under the shift map.

Remark. A similar definition can be stated for bi-infinitive sequences.

Graphs. We can associate a directed graph with a shift space: vertices of the graph are labeled $1,\ldots,N$; vertex i is connected to vertex j iff $\mathrm{A}_{ij} = 1$. Then every trajectory of the shift map can be considered as an infinite path on the graph.

Let N_{ij}^m be the number of paths from i to j which consist of m edges.

Lemma 39. $N_{ij}^m = (\mathrm{A}^m)_{ij}$.

PROOF. If $m = 1$, then $N_{ij}^m = A_{ij}$. We continue by induction:

$$N_{ij}^{m+1} = \sum_{k=1}^{N} N_{ik}^m A_{kj} = \sum_{k=1}^{N} (\mathrm{A}^m)_{ik} A_{kj} = (\mathrm{A}^{m+1})_{ij} .$$

□

Taking into account that periodic points of the shift map correspond to periodic sequences we arrive to the following proposition.

COROLLARY 40. *Number of period m points of σ_{A} is equal to* $\mathrm{trace}(\mathrm{A}^m)$.

A transition matrix A is called irreducible if for any i, j there is n such that $(\mathrm{A}^n)_{ij} > 0$.

A transition matrix A is called aperiodic if there is n such that $(\mathrm{A}^n)_{ij} > 0$ for all i, j (all elements are positive).

THEOREM 41. *If transition matrix* A *is aperiodic, then the corresponding topological Markov chain $\sigma_A : \Sigma_A \to \Sigma_A$ is topologically mixing and periodic orbits of σ_A are dense in Σ_A.*

PROOF. The proof is similar to the previously considered case of the full shift. □

Examples.

(1) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is aperiodic.

(2) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not aperiodic.

(3) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not aperiodic.

5.1. Spectral radius (from linear algebra). In order to find the topological entropy for a topological Markov chain we need to recall some information from Linear Algebra.

Let A be an $N \times N$ matrix. It has at most N eigenvalues $\lambda_i \in \mathbb{C}$: $Av_i = \lambda_i v_i$ for some $v_i \neq 0$, $v_i \in \mathbb{C}^N$. The number

$$\rho(A) = \max_i |\lambda_i|$$

is called the *spectral radius of A.* Let $\|A\|$ be a norm of A, then Gelfand's formula says:

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$$

If $\rho(A) \neq 0$, then the continuity of the logarithm implies that

$$\log \rho(A) = \lim_{k\to\infty} \frac{\log \|A^k\|}{k} .$$

In the next section we will use the following norm for the matrix

$$\|\mathrm{A}\| = \sum_{i,j=1}^{N} |A_{ij}| .$$

5.2. Entropy of a subshift of finite type. Let A be a transition matrix, i.e., $A_{ij} \in \{0,1\}$ for all $i, j \in \{1, \dots, N\}$. The corresponding shift space is

$$\Sigma_A = \left\{ (\omega_n)_{n=0}^{\infty} : \omega_n \in \{1, \dots, N\},\ A_{\omega_n \omega_{n+1}} = 1 \right\}.$$

The shift space is a compact metric space. Let $\sigma_A : \Sigma_A \to \Sigma_A$ be the restriction of the shift map on Σ_A.

The definition of the metric on Σ_A involves the constant $\lambda > 1$. The metrics with different values of λ are equivalent (as the open balls are the same). Consequently, the topological entropy of a shift map is independent of the choice of the constant λ.

PROPOSITION 42. *If the transition matrix A is aperiodic, then the topological entropy*

$$h(\sigma_{\mathrm{A}}) = \log \rho(A),$$

where $\rho(A)$ is the spectreal radius of the matrix A.

PROOF. Let $\lambda > 1$ be the constant from the definition of the metric d on Σ_A. The metric d_n from the definition of the topological entropy takes the form

$$d_n(\omega, \omega') = \max_{0 \le i \le n} d(\sigma^i(\omega), \sigma^i(\omega')).$$

Since σ_A shifts the sequence to the left, we conclude that the ball with respect to the metric d_n takes the form:

$$B(\alpha, n, \lambda^{-k}) = C_{\alpha_0, \dots, \alpha_{n+k}} = \{\omega \in \Sigma_A : \omega_i = \alpha_i \quad \text{for } 0 \le i \le n+k\}$$

Now we can find the entropy of the subshift constructing an (n, λ^{-k})-separated set of the largest cardinality. Let $W_m(A)$ be the set of all words on length m which are compatible with the transition matrix A:

$$W_m(A) = \{(\alpha_0, \dots, \alpha_{m-1}) : A_{\alpha_i \alpha_{i+1}} = 1\}.$$

Since A is aperiodic, there is n_0 such that $(A^{n_0})_{ij} > 0$ for all i, j. Then a cylinder $C_{\alpha_0, \dots, \alpha_{n+k}}$ with $(\alpha_0, \dots, \alpha_{n+k}) \in W_{n+k+1}(A)$ is non-empty. Moreover such cylinders (of the same length) are pairwise disjoint and the union of all these cylinders covers the whole shift space Σ_A. Let $E \subset \Sigma_A$ contain exactly one point from each of these cylinders. The set E is (n, λ^{-k})-separated. Taking into account that any (n, λ^{-k})-separated set can have no more than one point in each of the cylinders, we conclude that the maximal cardinality of an (n, λ^{-k})-separated set equals to the number of cylinders:

$$N(n, \lambda^{-k}) = \#W_{n+k+1}(A).$$

Then we use Proposition 38 to find the topological entropy. Let

$$\begin{aligned} \tilde{h}(\sigma_A, \lambda^{-k}) \ &:= \ \limsup_{n\to\infty} \frac{\log(N(n, \lambda^{-k}))}{n} = \limsup_{n\to\infty} \frac{\log(\#W_{n+k+1}(A))}{n} \\ &= \ \limsup_{n\to\infty} \frac{\log(\#W_{n+1}(A))}{n} \end{aligned}$$

Since the right hand side is independent of k we get

$$h(\sigma_A) = \lim_{k\to\infty} \tilde{h}(\sigma_A, \lambda^{-k}) = \limsup_{n\to\infty} \frac{\log(\#W_{n+1}(A))}{n}.$$

In order to complete the proof we find the limit. Lemma 39 implies that

$$\#W_{n+1}(A) = \sum_{i,j=1}^{N} (A^n)_{ij} = \|A^n\| \,.$$

Taking the limit and using the corollary of Gelfand's formula, we get

$$\lim_{n\to\infty} \frac{\log(\#W_{n+1}(A))}{n} = \lim_{n\to\infty} \frac{\log \|A^n\|}{n} = \log \rho(A) \,.$$

If a sequence converges, $\limsup$ is equal to lim, so we get

$$h(\sigma_A) = \log \rho(A)$$

The proof is complete. □

Additionally we proved that the topological entropy equals to the exponential growth rate of the number of admissible words:

$$h(\sigma_A) = \lim_{n\to\infty} \frac{\log(\#W_n(A))}{n} \,.$$

This formula is interesting as it is more general and provides the topological entropy not only for the topological Markov chain but also for any subshift of finite type.

CHAPTER 3

Smooth Dynamical Systems

1. Hyperbolic automorphisms of the torus

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be a torus and $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ the natural projection, which assigns an equivalence class to a point. The metric on $\mathbb{T}^2$ is defined by

$$d(p, q) = \min_{\pi(x)=p, \pi(y)=q} \|x - y\|.$$

Note that $\mathbb{T}^2 = \mathbb{S} \times \mathbb{S}$.

Consider a matrix

$$A \in \mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

PROPOSITION 43. *Let $A \in \mathrm{SL}(2, \mathbb{Z})$. The map $f(x, y) = (ax+by, cx+dy)$ (mod 1) is a homeomorphism of the torus.*

PROOF. It is easy to check that the linear map $x \mapsto Ax$ maps an equivalence class to an equivalence class (i.e. if $x \sim \tilde{x}$ then $Ax \sim A\tilde{x}$), so the equality $y = Ax \pmod 1$ defines a map from the torus to the torus.

The definition of the metric on the torus implies

$$d(f(p), f(q)) \le \max\{|a|, |b|, |c|, |d|\} d(p, q)$$

for any $p, q \in \mathbb{T}^2$. Thus f is continuous.

The inverse matrix is given by

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So $A^{-1} \in SL(2, \mathbb{Z})$ and also defines a continuous map $\mathbb{T}^2 \to \mathbb{T}^2$. Obviously, this map defined by A^{-1} is the inverse to f. So $f^{-1}(x, y) = (dx - by, -cx + ay)$ (mod 1) and f is a homeomorphism. □

We say that f is *hyperbolic* if the matrix A does not have eigenvalues with absolute value equal to 1. We also say that A is a hyperbolic matrix. The corresponding automorphism f is called a hyperbolic automorphism of the torus.

Exercises. Show that $A \in \mathrm{SL}(2, \mathbb{Z})$ is hyperbolic iff $|\mathrm{Tr}(A)| > 2$. (Recall $\mathrm{Tr}(A) = a + d$). Show that eigenvectors of a hyperbolic A are real. Show that if A is hyperbolic, then A^n is hyperbolic for all $n \in \mathbb{Z} \setminus \{0\}$.

Example: The map $f : \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$f(x, y) = (2x + y, x + y) \pmod 1$$

is a hyperbolic automorphism of the torus.

Let

$$\mathrm{Per}_n(f) = \left\{p \in \mathbb{T}^2 : f^n(p) = p\right\}, \qquad \mathrm{Per}(f) = \bigcup_{n\in\mathbb{N}} \mathrm{Per}_n(f).$$

PROPOSITION 44. *Let f be a hyperbolic automorphism of the torus associated to a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$. Then*

(1) $\mathrm{Per}(f) = \mathbb{Q}^2/\mathbb{Z}^2$ *(periodic points=points with rational coordinates)*
(2) $\#\mathrm{Per}_n(f) = |\mathrm{Tr}(A^n) - 2|$
(3) *f is topologically mixing*

PROOF. (1a) any point with rational coordinates is periodic. Indeed, take any point with rational coordinates. It can be written as $x = (\frac{m_1}{l}, \frac{m_2}{l})$ with $m_1, m_2 \in \mathbb{Z}$ and $l \in \mathbb{N}$. Let $m = (m_1, m_2)$. Then $A^k x = \frac{1}{l}A^k m$ for all $k \in \mathbb{Z}$. There are $k_1 < k_2$ such that $A^{k_1}m = A^{k_2}m \pmod{l}$ (as there is only l^2 different values for the reminders when we divide by l). Then let $n = k_2 - k_1$. So $A^n m = m \pmod{l}$ and $A^n x = x \pmod{1}$. Thus $x \in \mathrm{Per}_n$.

(1b) any periodic point has rational coordinates. Indeed, let $p \in \mathrm{Per}_n$. There is a unique $x \in \Delta = [0,1) \times [0,1) \subset \mathbb{R}^2$ such that $p = \pi(x)$. Then $x = A^n x \pmod{1}$. So there is $j \in \mathbb{Z}^2$ such that $x - A^n x = j$. The coefficients of the matrix $(I - A^n)$ are integer. Moreover $\det(I - A^n) \neq 0$ (otherwise 1 would be an eigenvalue of A^n). So the coefficients of $(I - A^n)^{-1}$ are rational. Hence $x = (I - A^n)^{-1} j \in \mathbb{Q}^2$.

(2) counting periodic orbits. We note that the number of periodic orbits in Per_n coincides with the number of $x \in \Delta$ such that $(I - A^n)x \in \mathbb{Z}^2$. The number of different $j \in \mathbb{Z}^2$ which belong to $(I - A^n)(\Delta)$ equals to the area of the parallelogram $(I - A^n)(\Delta)$. Since Δ has the unit area, the area of $(I - A^n)(\Delta)$ equals to $|\det(I - A^n)|$. Then

$$|\det(I - A^n)| = |(1 - \lambda_+^n)(1 - \lambda_-^n)| = |2 - (\lambda_+ + \lambda_-)| = \mathrm{Tr}(A^n) - 2$$

where $\lambda_\pm$ are eigenvalues of A.

(3) topological mixing (sketch). Take $U, V \subset \mathbb{T}^2$ open, nonempty and find N such that for $f^n U \cap V \neq \emptyset$ for any $n > N$. Let λ_+ and λ_- be eigenvalues of A, and v_+, v_- be the corresponding eigenvectors. Consider the straight lines $\ell_\pm = \{tv_\pm : t \in \mathbb{R}\}$. Each of these lines has an irrational slope and consequently its projection on the torus, $W_\pm = \pi(\ell_\pm)$, is dense.

Take two small parallelograms with sides parallel to the eigenvectors, the first one $U' \subset U$, the second one $V' \subset V$. Note that the density implies that $W_- \cap U' \neq \emptyset$ and $W_+ \cap V' \neq \emptyset$. Then check that there is N such that $f^n(U') \cap V'$ for all $n > N$ (the sides are parallel to eigenvectors, so f^n maps a parallelogram to a parallelogram. Draw a picture for the map A.).

□

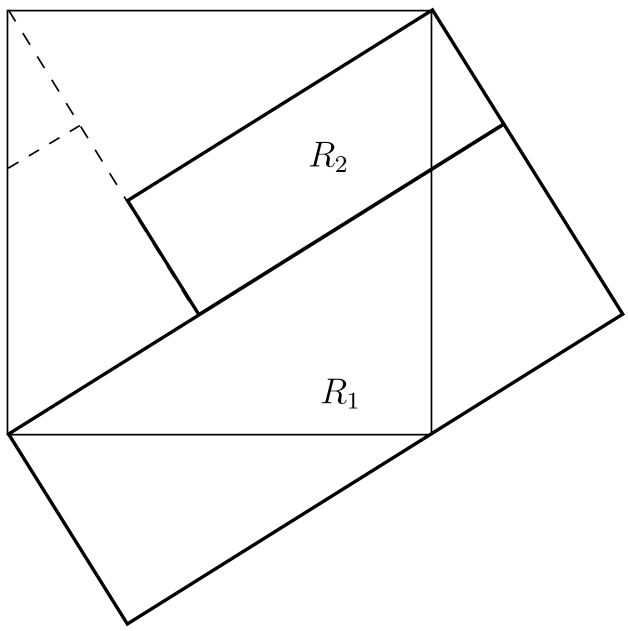
2. Symbolic dynamics for a linear automorphism of the torus

THEOREM 45. *The map* $f : (x, y) \mapsto (2x + y, x + y) \pmod 1$ *is topologically semiconjugate to the topological Markov chain* $\sigma_B : \Sigma_B \to \Sigma_B$ *where* σ_B *is a shift map on the space* Σ_B *of bi-infinite sequences compatible with the transition matrix*

$$\mathrm{B} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

REMARK 46. *Since* $\Sigma_B \subset \Sigma$ *and* Σ *is homeomorhic to a Cantor set,* Σ_B *is not homeomorphic to the torus. Consequently,* f *is not topologically conjugate to any subshift, only a semiconjugacy is possible.*

PROOF. Let R_1 and R_2 be two (closed) rectangles with the sides parallel to the eigenvectors of the matrix A shown on the following illustration:



Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the natural projection. Check (look for equal triangles on the picture) that $\pi(R_1 \cap R_2) = \mathbb{T}^2$.

Intersections $f(\pi(R_i)) \cap \pi(R_j)$ with $j = 1, 2$ define rectangles S_k with $k = 1, \ldots, 5$, which looks like strips with sides parallel to the eigenvectors of A (see the next pictures), in particular $R_1 = S_1 \cup S_2 \cup S_3$ and $R_2 = S_4 \cup S_5$. In order to shorten the notation we introduce $S'_k = \pi(S_k)$ and $R'_k = \pi(R_k)$. Then

$$R'_1 = S'_1 \cup S'_2 \cup S'_3, \qquad R'_2 = S'_4 \cup S'_5,$$
$$f(R'_1) = S'_1 \cup S'_3 \cup S'_4, \qquad f(R'_2) = S'_2 \cup S'_5.$$

We conclude that $f(S'_j) \cap S'_k = \emptyset$ if $B_{jk} = 0$, and $f(S'_j) \cap S'_k \neq \emptyset$ if $B_{jk} = 1$. (where B is the transition matrix)

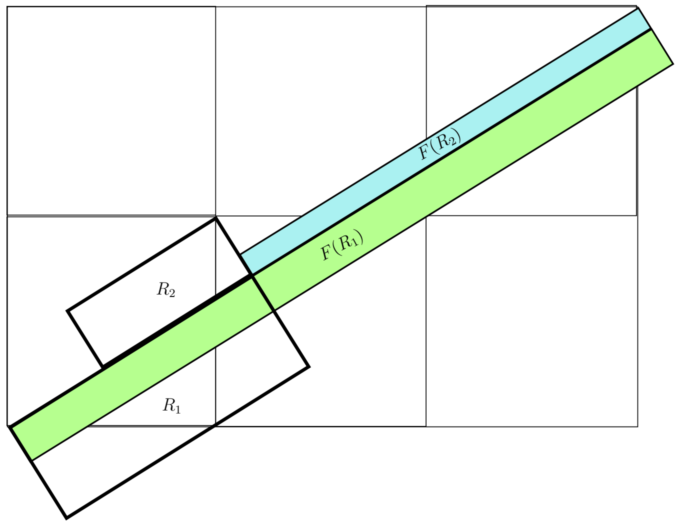


FIGURE 1. Rectangles R_1 and R_2 and their images (bold boundary) under the linear map $F : (x, y) \mapsto (2x + y, x + y)$.

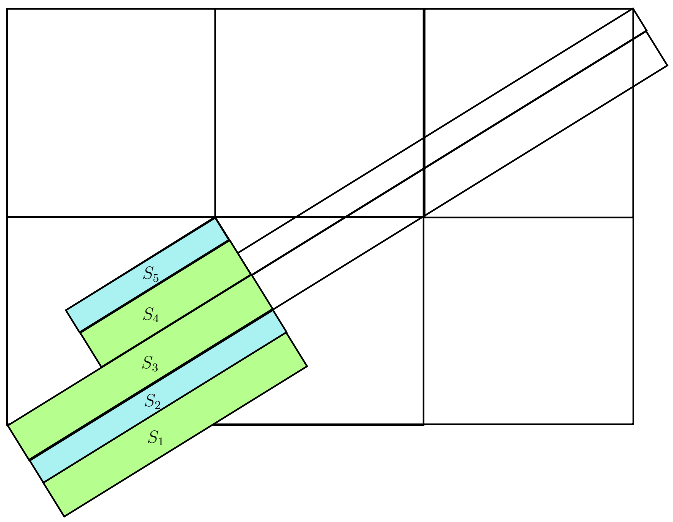


FIGURE 2. Images of R_1 and R_2 are split into smaller rectangles and translated by integer vectors back to R_1 and R_2.

For any sequence $(\omega_k)_{k\in\mathbb{Z}}$, $\omega_k \in \{1, 2, 3, 4, 5\}$, the set

$$\bigcap_{k\in\mathbb{Z}} f^{-k}(S'_{\omega_k})$$

either consists of a single point which we denote $h(\omega)$ (provided $\omega \in \Sigma_B$) or is empty (otherwise). This can be shown by arguments very similar to the ones used in the analysis of the Smale horseshoe. $\square$

3. Shadowing

Quite often scientists explore trajectories of a dynamical system using computer simulations. A computer usually uses only finite precision and introduces a (small??) error at each time step. The errors can accumulate very fast. For example, if you iterate the tent map $f(x) = 1 - |1 - 2x|$ for $x \in (0, 1)$, the computer will typically loose one binary digit per each iterate as every multiplication by 2 doubles not only x but also any inaccuracy in its value. Taking into account that modern computers usually use less than 64 binary digits to store a number, all precision will be lost after around 60 iterates.

We note that sometimes it is possible to prove that the computations follow some true trajectory for long times.

Definition. Let (X, d) be a metric space and $f : X \to X$ be a function. We say that $(x_i)_{i\in\mathbb{Z}} \subset X$ is *an ε-pseudo orbit* if

$$d(x_i, f(x_{i-1})) < \varepsilon \quad \forall i \in \mathbb{Z}\,.$$

Theorem 47 (Shadowing property). *Let $\mathrm{A} \in \mathrm{SL}(2, \mathbb{Z})$ be hyperbolic and $f : \mathbb{T}^2 \to \mathbb{T}^2$ associated hyperbolic toral automorphism. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $(x_i)_{i\in\mathbb{Z}} \subset \mathbb{T}^2$ there is $p \in \mathbb{T}^2$ such that*

$$d(x_i, f^i(p)) < \varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

If ε is sufficiently small, the point p with these properties is unique.

Definition. The trajectory of p is called a *shadow orbit*.

The proof of the theorem relies on the following lemma which describes bounded solutions of an infinite system of linear equations.

Lemma 48. *Let $\mathrm{A} \in \mathrm{SL}(2, \mathbb{Z})$ be hyperbolic. For any bounded sequence $(u_i)_{i\in\mathbb{Z}} \subset \mathbb{R}^2$, there is a unique bounded sequence $(w_i)_{i\in\mathbb{Z}} \subset \mathbb{R}^2$, such that*

$$w_i - \mathrm{A}w_{i-1} = u_i \quad \forall i \in \mathbb{Z}\,. \tag{1}$$

Moreover, there is $C_0 > 0$, which depends on the matrix A only, such that

$$\sup_i \|w_i\| \le C_0 \sup_i \|u_i\|.$$

Proof. Let $\lambda_\pm$ be eigenvalues of A. Since A is hyperbolic we can assume $|\lambda_+| > 1$ and $|\lambda_-| < 1$. Then there is a matrix B such that $\det \mathrm{B} = 1$ and $A = \mathrm{B}^{-1}\Lambda\mathrm{B}$ where Λ is a diagonal matrix: $\Lambda = \mathrm{diag}(\lambda_+, \lambda_-)$. Then the equation (1) is equivalent to

$$\mathrm{B}w_i - \Lambda\mathrm{B}w_{i-1} = \mathrm{B}u_i.$$

Let

$$\begin{pmatrix} \alpha_i^+ \\ \alpha_i^- \end{pmatrix} = \mathrm{B}w_i, \qquad \begin{pmatrix} \beta_i^+ \\ \beta_i^- \end{pmatrix} = \mathrm{B}u_i$$

and plug these into the equation. We get the system

$$\alpha_i^+ - \lambda_+\alpha_{i-1}^+ = \beta_i^+ \quad \text{and} \quad \alpha_i^- - \lambda_-\alpha_{i-1}^- = \beta_i^-\,.$$

These equations have an explicit solution given by

$$\alpha_i^+ = -\sum_{k=i+1}^{\infty} \lambda_+^{i-k}\beta_k^+ \quad \text{and} \quad \alpha_i^- = \sum_{k=-\infty}^{i} \lambda_-^{i-k}\beta_k^-$$

as we can easily check (we demonstrate convergence of the series a bit later):

$$\begin{aligned}
\alpha_i^+ - \lambda_+\alpha_{i-1}^+ &= -\sum_{k=i+1}^{\infty} \lambda_+^{i-k}\beta_k^+ + \lambda_+\sum_{k=i}^{\infty}\lambda_+^{i-1-k}\beta_k^+ = \lambda_+^i\beta_i^+ \,, \\
\alpha_i^- - \lambda_-\alpha_{i-1}^- &= \sum_{k=-\infty}^{i} \lambda_-^{i-k}\beta_k^- - \lambda_- \sum_{k=-\infty}^{i-1} \lambda_-^{i-1-k}\beta_k^- = \lambda_-^i\beta_i^- \,.
\end{aligned}$$

Moreover, the series converge (using a convergent dominating geometric series):

$$\begin{aligned}
|\alpha_i^+| &\le \sum_{k=i+1}^{\infty} |\lambda_+|^{i-k}|\beta_k^+| \le \sup_k |\beta_k^+| \sum_{k=i+1}^{\infty} |\lambda_+|^{i-k} \\
&= \sup_k |\beta_k^+| \frac{|\lambda_+|^{-1}}{1-|\lambda_+|^{-1}} \le \frac{\sup_k |\beta_k^+|}{1-|\lambda_-|} \\
|\alpha_i^-| &\le \sum_{k=-\infty}^{i} |\lambda_-|^{i-k}|\beta_i^-| \le \sup_k |\beta_k^-| \sum_{k=-\infty}^{i} |\lambda_-|^{i-k} \\
&= \frac{\sup_k |\beta_k^-|}{1-|\lambda_-|}
\end{aligned}$$

We used $\lambda_+\lambda_- = 1$ and $|\lambda^+| > 1$ to simplify the equations.

Then

$$\begin{aligned}
\|w_i\| &= \left\| \mathrm{B}^{-1} \begin{pmatrix} \alpha_i^+ \\ \alpha_i^- \end{pmatrix} \right\| \le \|\mathrm{B}^{-1}\| \left\| \begin{pmatrix} \alpha_i^+ \\ \alpha_i^- \end{pmatrix} \right\| \le \|\mathrm{B}^{-1}\|\, 2\max\left\{|\alpha_i^+|, |\alpha_i^-|\right\} \\
&\le 2\|\mathrm{B}^{-1}\| \frac{\max\left\{\sup_k |\beta_k^+|, \sup_k |\beta_k^-|\right\}}{1-|\lambda_-|} \le 2\|\mathrm{B}^{-1}\|\,\|\mathrm{B}\| \frac{\sup_k \|u_k\|}{1-|\lambda_-|} \,.
\end{aligned}$$

Let

$$C_0 = \frac{2\|\mathrm{B}^{-1}\|\,\|\mathrm{B}\|}{1-|\lambda_-|}$$

Then

$$\sup_i \|w_i\| \le C_0 \sup_k \|u_k\| \,.$$

The sequence $w_i = \alpha_i^+ v_+ + \alpha_I^- v_-$ satisfies the equation (1).

In order to complete the proof we need to demonstrate the uniqueness. Suppose $\tilde{w}_i$ is another bounded solution of the equation (1) . Then $z_i = \tilde{w}_i - w_i$ satisfies the equation $z_i - \mathrm{A}z_{i-1} = 0$ for all i so $z_i = \mathrm{A}z_{i-1}$. Using the induction we conclude that $z_i = \mathrm{A}^i z_0$ for all $i \in \mathbb{Z}$. Then $\mathrm{B}z_i = \Lambda^i \mathrm{B}z_0 = \mathrm{diag}(\lambda_+^i, \lambda_-^i)\mathrm{B}z_0$ for all i. This sequence is bounded if and only if $z_0 = 0$. Hence $\tilde{w}_i = w_i$. The uniqueness is proved. $\square$

COROLLARY 49. *Let $\delta > 0$ and $(x_i)_{i\in\mathbb{Z}}$ be a δ-pseudo orbit of the linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x) = \mathrm{A}x$. Then there is a unique trajectory $(y_i)_{i\in\mathbb{Z}}$ of F such that the sequence $\|y_i - x_i\|$ is bounded. Moreover, $\|y_i - x_i\| < C_0\delta$ for all i.*

PROOF. Let $u_i = F(x_{i-1}) - x_i$. The definition of the δ-pseudo orbit implies that $\|u_i\| \le \delta$. The previous lemma implies that there is a unique bounded sequence w_i such that $w_i - Aw_{i-1} = u_i$. Let $y_i = x_i + w_i$. Then $\|y_i - x_i\| = \|w_i\| \le C_0\delta$. Since F is linear we get

$$y_i = x_i + w_i = F(x_{i-1}) - u_i + Aw_{i-1} + u_i = F(x_{i-1} + w_{i-1}) = F(y_{i-1}).$$

Consequently $(y_i)_{i\in\mathbb{Z}}$ is a trajectory of F. □

PROOF OF THE SHADOWING PROPERTY. Let $\varepsilon_0 = \min\{\frac{1}{4}, \frac{C_0}{4}\}$ where C_0 is the constant from the corollary above. For any $\varepsilon \in (0, \varepsilon_0)$ let $\delta = \varepsilon/C_0$. Obviously, $\delta \in (0, \frac{1}{4})$. Let $(x_i)_{i\in\mathbb{Z}} \subset \mathbb{T}^2$ be a δ-pseudo orbit of f.

Since $\delta < \frac{1}{4}$, there is a unique sequence $(x'_i)_{i\in\mathbb{Z}} \subset \mathbb{R}^2$, such that $\pi(x'_i) = x_i$ and $\|x'_i - F(x'_{i-1})\| \le \delta$ for all i. This sequence is a δ-pseudo orbit for F. Then the corollary above implies that there is a unique orbit (y'_i) which shadows (x'_i) and $\|y'_i - x'_i\| < \varepsilon$. The projection of this orbit on the torus shadows (x_i).

The shadow orbit is unique. Indeed, suppose that there is another trajectory $(\tilde{y}_i)$ of f which ε-shadows (x_i). Then the triangle inequality implies that $\mathrm{dist}(\tilde{y}_i, y_i) < 2\varepsilon$. Then there is a sequence $(\tilde{y}'_i) \subset \mathbb{R}^2$ such that $\tilde{y}_i = \pi(\tilde{y}'_i)$ and $\|\tilde{y}'_i - y'_i\| < 2\varepsilon$ for all i. We note that $\tilde{y}'_i = A\tilde{y}'_{i-1}$. (Indeed, $\tilde{y}_i = A\tilde{y}_{i-1} +$ integer vector, but $\|\tilde{y}_i - A\tilde{y}_{i-1}\| < 2\varepsilon < \frac{1}{2}$. So the integer vector is the zero vector.) Then $w_i = \tilde{y}'_i - y'_i$ satisfies $w_i - \mathrm{A}w_{i-1} = 0$ and $\|w_i\| \le 2\varepsilon$. Lemma 48 with $u_i = 0$ for all i implies $w_i = 0$ for all i. □

4. Structural stability

Let (X, d_X) and (Y, d_Y) be compact metric spaces. Let $\mathrm{Hom}(X, Y)$ be the space of all homeomorphisms $X \to Y$. For any $f, g \in \mathrm{Hom}(X, Y)$ let

$$\mathrm{d}(f, g) = \sup_{x\in X} d_Y(f(x), g(x)) + \sup_{y\in Y} d_X(f^{-1}(y), g^{-1}(y)).$$

Then d is a metric on $\mathrm{Hom}(X, Y)$.

THEOREM 50. *Let $A \in \mathrm{SL}(2, \mathbb{Z})$ be hyperbolic and $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a associated hyperbolic automorphism. Then there is $\delta > 0$ such that any homeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ with $d(f, g) < \delta$ is topologically semiconjugate to f, i.e., there is a continuous surjective map $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that the diagram*

$$\begin{tikzcd} \mathbb{T}^2 \arrow[r, "g"] \arrow[d, "h"'] & \mathbb{T}^2 \arrow[d, "h"] \\ \mathbb{T}^2 \arrow[r, "f"'] & \mathbb{T}^2 \end{tikzcd}$$

commutes.

PROOF. Let $\varepsilon, \delta > 0$ be the constants from the Shadowing Theorem. Take any $x_0 \in \mathbb{T}^2$ and let $x_i = g^i(x_0)$ for all $i \in \mathbb{Z}$. Consequently, $x_i = g(x_{i-1})$ for all i and

$$d(x_i, f(x_{i-1})) = d(g(x_{i-1}), f(x_{i-1})) \le d(f, g) < \delta.$$

So the sequence $(x_i)_{i\in\mathbb{Z}}$ is a δ-pseudo orbit for f. Then the Shadowing Theorem implies that there is a unique sequence $(y_i)_{i\in\mathbb{Z}}$ such that $y_i = f^i(y_0)$ and $d(x_i, y_i) < \varepsilon$ for all i. Let $h(x_0) = y_0$.

The map h has the following properties:

(i) $d(x, h(x)) < \varepsilon$ for all $x \in \mathbb{T}^2$.
Let $x_0 = x$, then $h(x) = y_0$ and the property follows directly from $d(x_0, y_0) < \varepsilon$.

(ii) $h(g(x)) = f(h(x))$ for all $x \in \mathbb{T}^2$.
Let $x_0 = x$ and use the definition of h to check that $h(x_i) = y_i$ for all $i \in \mathbb{Z}$. Then $h(g(x_0)) = h(x_1) = y_1 = f(y_0) = f(h(x_0))$.

(iii) h is continuous.
Indeed, since g is continuous, then g^n is continuous for every n. So for any $\eta > 0$ and any $N \in \mathbb{N}$ there is $\delta_1 > 0$ such that

$$d(x_0, \tilde{x}_0) < \delta_1 \qquad \Longrightarrow \qquad d(x_n, \tilde{x}_n) < \eta \qquad \forall |n| \le N$$

Then

$$\begin{aligned} d(f^n(h(x_0)), f^n(h(\tilde{x}_0))) &= d(h(g^n(x_0)), h(g(\tilde{x}_0))) \\ &= d(h(x_n), h(\tilde{x}_n)) \\ &\le d(h(x_n), x_n) + d(x_n, \tilde{x}_n) + d(h(\tilde{x}_n), \tilde{x}_n) \\ &\le \eta + 2\varepsilon\,. \end{aligned}$$

Since f is obtained from a hyperbolic linear map, there is $C > 0$ with the following property: if the inequality $d(f^n(y_0), f^n(\tilde{y}_0)) < \tilde{\varepsilon}$ with $\tilde{\varepsilon} \in (0, \frac{1}{4})$ holds for all $|n| \le N$, then

$$d(y_0, \tilde{y}_0) \le C \frac{\tilde{\varepsilon}}{|\lambda_+|^N}\,.$$

It follows that

$$d(h(x_0), h(\tilde{x}_0)) \le C \frac{\eta + 2\varepsilon}{|\lambda_+|^N}\,.$$

Now fix $\eta = \varepsilon$.

Take any $\epsilon_1 > 0$ and choose N sufficiently large to ensure that $\frac{3C\varepsilon}{|\lambda_+|^N} < \epsilon_1$. Then choose δ_1 using these η and N. It follows that $d(x_0, \tilde{x}_0) < \delta_1$ implies $d(h(x_0), h(\tilde{x}_0)) < \epsilon_1$. Hence h is continuous.

(iv) h is surjective.
If $h : \mathbb{T}^2 \to \mathbb{T}^2$ is continuous and $d(x, h(x)) < \frac{1}{4}$ for all $x \in \mathbb{T}^2$, then h is surjective (see your notes from the lectures).

□

Structural stability. Now let $\mathrm{Diff}(\mathbb{T}^2)$ be the space of all diffeomorphisms $\mathbb{T}^2 \to \mathbb{T}^2$. The following equation defines a metric on this space:

$$d_{C^1}(f,g) = \sup_{x\in\mathbb{T}^2} d(f(x),g(x)) + \sup_{x\in\mathbb{T}^2} \|D_x f - D_x g\| \ ,$$

where $f,g \in \mathrm{Diff}(\mathbb{T}^2)$ and $D_x f$ defines the derivative of f at the point x.

We say that a diffeomorphism f is structurally stable if there is $\delta > 0$ such that any diffeomorphism g with $d(f,g) < \delta$ is topologically conjugate to f.

THEOREM 51. *Linear hyperbolic automorphisms of the torus are structurally stable.*

We will not discuss the proof of this theorem.

CHAPTER 4

Complex Dynamical Systems

In the last part of these lectures we briefly discuss dynamical systems defined by iterations of holomorphic (=analytic) maps. Let us start with some definitions.

Let $\mathbb{C}$ be a complex plane ($z \in \mathbb{C}$, $z = x + iy$, $x, y \in \mathbb{R}$, $i^2 = -1$) and $D \subseteq \mathbb{C}$ be an open non-empty subset. A map $f : D \to \mathbb{C}$ is called *holomorphic* (or *analytic*) if it is differentiable at every point of its domain of definition, i.e., it has a derivative defined by

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

at every point $z \in D$. Holomorphic maps have many remarkable properties. In particular, a holomorphic map f has derivatives of all orders and its Taylor expansion converges. Moreover, holomorphic maps are *conformal*, i.e., they preserve angles.

The **Riemann sphere** is the set $\hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \}$ equipped with a metric defined with the help of a stereographic projection. The stereographic projection establishes a bijection between a sphere without its north pole and the plane $\mathbb{C}$. Then the north pole is mapped to ∞. In this way we get a bijection between the sphere and $\hat{\mathbb{C}}$. The sphere is a metric space (the distance between two points is equal to the length of the shortest arc which connects the points). Then the distance between two points in $\hat{\mathbb{C}}$ is defined as the distance between their stereographic images on the sphere.

The definition implies that the stereographic projection is an isometry. Hence, the Riemann sphere $\hat{\mathbb{C}}$ is compact.

It is possible to describe this metric in a more direct way. Let $\gamma : [0, 1] \to \mathbb{C}$ be a smooth curve. We can define its length by

$$\text{length}(\gamma) = \int_\gamma \frac{2|dz|}{1 + |z|^2} = \int_0^1 \frac{2|\dot{\gamma}(t)|}{1 + |\gamma(t)|^2} dt\,.$$

Then the distance between two points $z_0, z_1 \in \mathbb{C}$ is defined by

$$d(z_0, z_1) = \inf_{\gamma\,:\,\gamma(0)=z_0,\,\gamma(1)=z_1} \text{length}(\gamma)\,.$$

In other words, the distance is equal to the length of the shortest curve which connects the points. This metric coincides with the one obtained via the stereographic projection (if the radius of the sphere is chosen appropriately).

The definition of a holomorphic function can be extended to the Riemann sphere. Let $D \subseteq \hat{\mathbb{C}}$ be open and $f : D \to \hat{\mathbb{C}}$. We say that f is holomorphic in D if f has a derivative at every point of D, i.e. there is $f' : D \to \hat{\mathbb{C}}$.

The definition of the derivative requires some explanations as $\hat{\mathbb{C}}$ includes ∞. We will not discuss the formal definition but provide a tool which can be used to test if a function is holomorphic or not by mapping ∞ to 0 with the help of the function $z \mapsto z^{-1}$.

Take a point $z_0 \in D$.

If $z_0 \neq \infty$ and $f(z_0) = \infty$, then f is holomorphic in a neighbourhood of z_0, iff $1/f(z)$ is holomorphic in a neighbourhood of z_0.

If $z_0 = \infty$ and $f(\infty) \neq \infty$, then f is holomorphic in a neighbourhood of ∞, iff $f(z^{-1})$ is holomorphic in a neighbourhood of 0.

If $z_0 = \infty$ and $f(\infty) = \infty$, then f is holomorphic in a neighbourhood of ∞, iff $1/f(z^{-1})$ is holomorphic in a neighbourhood of 0.

Remark. A holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ can be extended onto the Riemann sphere by setting $f(\infty) = \lim_{z\to\infty} f(z)$ provided the limit exists.

Examples of holomorphic functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$:

(1) $f(z) = z^d$, $d \in \mathbb{N}$;
(2) $f(z) = z + 1$;
(3) $f(z) = \frac{1}{z}$. This function maps a neighbourhood of ∞ to a neighbourhood of 0 on the Riemann sphere.

Example. $f(z) = \mathrm{e}^z$ is not holomorphic on the Riemann sphere (the limit $z \to \infty$ does not exists).

1. Rational maps

Let $P(z), Q(z)$ be polynomials:

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad Q(z) = b_0 + b_1 z + \cdots + b_m z^m$$

where $a_k, b_k \in \mathbb{C}$. A map f is called *rational* if it can be written as a quotient of two polynomials:

$$f(z) = \frac{P(z)}{Q(z)}\,.$$

We assume that the polynomials are coprime (i.e., no common zeroes). We note that Q has exactly m roots (counting with multiplicity). So assume $Q(z_k) = 0$ for $k = 1, \ldots, m$ and $P(z_k) \neq 0$.

The differentiation rule for a quotient implies that f is differentiable at every point z such that $Q(z) \neq 0$. So

$$f : \mathbb{C} \setminus \{z_1, \ldots, z_m\} \to \mathbb{C}$$

is analytic. The degree of f is defined by

$$\deg(f) = \max\{\deg(P), \deg(Q)\}\,.$$

We say that $z \in \mathbb{C}$ is a critical point of f if $f'(z) = 0$. If z is a critical point of f then $w = f(z)$ is called a critical value of f. If w is not a critical value, then w is called a *regular value*. If z is not a critical point, then z is called *regular*. In a neighbourhood of a regular point the map f is a local diffeomorphism.

Exercise. How many critical points and critical values can a rational map have?

Remark: A rational map can be extended to a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $f(z_k) = \infty$ and $f(\infty) = \lim_{z\to\infty} f(z)$. The extended map is continuous. It can be checked that f is not only continuous but also holomorphic on $\hat{\mathbb{C}}$. On the other hand, any holomorphic map on the Riemann sphere is rational.

Exercise. Let $w \in \mathbb{C}$ be a regular value of a rational map f and $f(\infty) \neq w$. Show that the number of preimages of w,

$$\#\{\, z \in \mathbb{C} : f(z) = w \,\} = \deg(f)\,.$$

2. Möbius transformations

A rational map of degree one is called a *Möbius transformation.* It can be written in the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. The last inequality is equivalent to the requirement that f is not constant.

The Möbius transformations is an invertible map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (each point has exactly one preimage). It can be checked directly that the inverse map is also a Möbius transformation. Moreover, the set of all Möbius transformations form a group under composition. It is called the Möbius group.

It is an automorphism group of the Riemann sphere. In particular, every Möbius transformation is a homeomorphism $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Exercise. Show that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}\ .$$

defines a homomorphism from $\mathrm{GL}(2,\mathbb{C})$ to the Möbius group.

We note that if f, h are two Möbius maps, then $g = h \circ f \circ h^{-1}$ is also a Möbius map. Since h is a homeomorphism of the Riemann sphere, the maps f and g are topologically conjugate.

2.1. Dynamics of a Möbius map.

Suppose that $f(z) = \frac{az+b}{cz+d}$ is a Möbius map and f is not the identity. A fixed point of the map satisfies the equation $f(z) = z$:

$$\frac{az+b}{cz+d} = z\,.$$

A solution of this equation is either given by $z = \infty$ (if $c = 0$) or satisfies the equation

$$cz^2 + (d-a)z - b = 0\,.$$

This is a quadratic equation. Its discriminant $D = (d-a)^2 + 4bc$ can be rewritten in the form

$$D = \mathrm{Tr}(\mathrm{A})^2 - 4\det \mathrm{A}\,.$$

The number of fixed points of f depends on the value of D:

(1) If $D = 0$ the Möbius transformation f has exactly one fixed point $z_1 \in \hat{\mathbb{C}}$. Then there is a Möbius transformation h such that $h(z_1) = \infty$ and $g = h \circ f \circ h^{-1}$ takes the form

$$g(z) = z + 1\,.$$

(2) If $D \neq 0$ the Möbius transformation f has exactly two fixed points $z_1, z_2 \in \hat{\mathbb{C}}$. Then there is a Möbius transformation h such that $h(z_1) = \infty$, $h(z_2) = 0$ and $g = h \circ f \circ h^{-1}$ takes the form

$$g(z) = \lambda z\,,$$

where $\lambda \in \mathbb{C}$, and $\lambda \notin \{0, 1\}$ as f is neither constant nor identity.

We see that a Möbius map is conjugated either to a translation or to multiplication by λ. The maps are classified according to the value of λ.

3. Periodic points

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map. Let $p \in \mathbb{C}$ be a periodic point of f of prime period n: $p = f^n(p)$. The number $\lambda = (f^n)'(p)$ is called a *multiplier* of the periodic point p.

Periodic points are classified according to the value of their multiplier:

$\lvert\lambda\rvert > 1$	p is repelling;
$0 < \lvert\lambda\rvert < 1$	p is attracting;
$\lambda = 0$	p is super-attracting;
$\lvert\lambda\rvert = 1$	p is neutral

If p is attracting (or super-attracting) we can define its *basin of attraction*:

$$\mathcal{B}(p) = \left\{ w \in \mathbb{C} : f^m(w) \text{ converges to the set } \{\, p, f(p), \dots, f^n(p) \,\} \right\}.$$

Proposition 52. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map and p an attractive periodic point of f. Then $\mathcal{B}(p)$ is a non-empty open set.*

Proof. Let $|\lambda| < c < 1$. Since $|(f^n)'(p)| = \lambda$ and the derivative is continuous, there is $r > 0$ such that $|(f^n)'(w)| < c$ for all $w \in B_r(p)$ (an open ball centered at p). Then

$$|f^n(w) - p| = |f^n(w) - f^n(p)| = \left| \int_p^w (f^n)'(z)dz \right| \leq c|w - p| < cr < r\,.$$

Consequently, $f^n(w) \subset B_r(p)$. Moreover, using induction we see that $|f^{mn}(w) - p| \leq rc^m \to 0$ as $m \to \infty$. Thus $B_r(p) \subset \mathcal{B}(p)$.

Finally, if $w \in \mathcal{B}(p)$ then there is $m \in \mathbb{N}$ such that $f^m(w) \in B_r(p)$. Then $\mathcal{B}(p) = \bigcup_{m\in\mathbb{N}} f^{-m}(B_r(p))$ is open as the preimage of any open set is open. □

4. Fatou set and Julia set

Definition. Let $U \subset \hat{\mathbb{C}}$ be open. A family of functions $f_n : U \to \hat{\mathbb{C}}$ is called *normal* (or *equicontinuous*) if $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $z, w \in U$ with $d(z, w) < \delta$ implies $d(f^n(z), f^n(w)) < \varepsilon$ for all $n \in \mathbb{N}$.

Let $f(z) = P(z)/Q(z)$ be a rational map.

Definition. The *Fatou set* $\mathcal{F} \subset \hat{\mathbb{C}}$ *of* f is the set of all $z \in \hat{\mathbb{C}}$ for which there is a neighbourhood V of z, such that the family $f_n := f^n$, $n \in \mathbb{N}$, is normal.

Definition. The *Julia set* $\mathcal{J}(f)$ is the complement of the Fatou set $\mathcal{F}(f)$.

PROPOSITION 53. *If f is a rational function then $\mathcal{J}(f)$ is closed. Moreover, if $\deg(f) \geq 2$, then $\mathcal{J}(f) \neq \emptyset$.*

PROOF. The definition implies that $\mathcal{F}(f)$ is open, so $\mathcal{J}(f)$ is closed.

Suppose $\mathcal{F}(f) = \hat{\mathbb{C}}$. Then $f^n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is normal. Arzelá-Ascoli theorem implies there is a uniformly convergent subsequence f^{n_k}. Montel's theorem implies the limit of f^{n_k} is an analytic function $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Hence g is a rational function and has a finite degree. On the other hand, $\deg(f^{n_k}) = \deg(f)^{n_k}$ is strictly increasing and unbounded, which leads to a contradiction. So $\mathcal{J}(f) \neq \emptyset$. □

Example: $f(z) = \lambda z$, $|\lambda| < 1$. Then $\mathcal{J}(f) = \{\infty\}$.

Recall that a subset E is called *completely invariant* if $f^{-1}(E) = E$. A set E is completely invariant iff its complement is completely invariant.

We list some elementary properties of the Fatou and Julia sets:

- $\mathcal{J}(f)$ is completely invariant.
- For any $m \in \mathbb{N}$, $\mathcal{J}(f) = \mathcal{J}(f^m)$.
- Every attracting periodic point belongs to $\mathcal{F}(f)$.
- Every repelling periodic point belongs to $\mathcal{J}(f)$.
- If p is an attracting periodic orbit then its basin of attraction $\mathcal{B}(p) \subset \mathcal{F}(f)$.

Example: $f(z) = z^d$, $d \geq 2$. $\mathcal{J}(f) = \{\, z \in \mathbb{C} : |z| = 1 \,\}$.

Exercise: Find Fatou and Julia sets for a Möbius transformation. How many different cases are there?

Exercise: Let $f(z) = z^2 + z$. Show that $0 \notin \mathcal{F}(f)$.