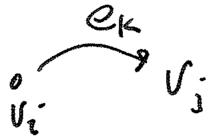
Graphs and dynamics. Topological Markov Chains.

~~Digraph~~

~~Recall that we have def~~

Graph terminology: Graph consists of a set of vertices V and edges $\mathcal{E}$ where $e_k \in \mathcal{E}$ goes from some vertex v_i to some vertex v_j.

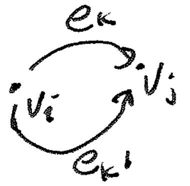


Write $t(e_k) = v_i$ and $h(e_k) = v_j$.

So formally a graph is determined by two sets $\mathcal{E}$ and V and two maps $t: \mathcal{E} \to V$ and $h: \mathcal{E} \to V$.

~~Def~~

We say that a graph has no multiple edges if we don't have:



If G has no multiple edges define the

$$\Sigma_G^V = \{ (\omega)_{i=-\infty}^{\infty} : \omega_i \in V \text{ ~~with~~ } \omega_i = t(\alpha_i),\ \omega_{i+1} = h(\alpha_i) \text{ for (unique) } \alpha_i \in \mathcal{E}\ \cancel{h(\omega_i) = t(\omega_{i+1})} \text{ for } i \in \mathbb{Z} \}.$$

σ is the ~~std~~ left shift.

(Can also consider the 1-sided version.)

(Whether or not G has multiple edges)

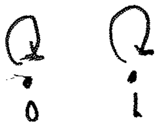
~~If~~ For any G we can define

$$\Sigma_G^e = \{ (\omega)_{i=-\infty}^{\infty} : \omega_i \in \mathcal{E} \text{ and } h(\omega_i) = t(\omega_{i+1}) \}$$

σ is the left shift.

Topological Markov Chains.

Examples:

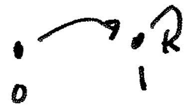
$\mathcal{A} =$ 

$\mathcal{G} =$ 

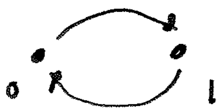
$\Sigma_{\mathcal{G}}^{\vee}$ consists of the single point ω
$\omega = (\dots 00.00 \dots)$.

$\mathcal{G} =$ 

$\Sigma_{\mathcal{G}}^{\vee}$ is empty. ~~There is no~~
If $\omega_i = 0$ then $\omega_{i+1} = 1$ and there is no way of filling in ω_{i+2}.

$\mathcal{G} =$ 

or 

$\mathcal{G} =$ 

$\Sigma_{\mathcal{G}}^{\vee}$ contains 2 points
$\omega = (\dots 010.101 \dots)$ and
$\omega' = (\dots 101.010 \dots)$.
$\sigma(\omega) = \omega'$ and $\sigma(\omega') = \omega$.

$\mathcal{G} =$ 

$\Sigma_{\mathcal{G}}^{\vee}$ contains ∞ many points

$\omega^{-} = (\dots 00.00 \dots)$

$\omega^{+} = (\dots 11.111 \dots)$

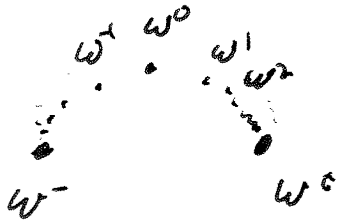
and $\omega^{j} = (\dots 000.\underbrace{00}_{j}111 \dots)$

for $n \in \mathbb{Z}$.

$\sigma(\omega^{j}) = \omega^{j+1}$

$\lim_{n \to \infty} \sigma^{n}(\omega^{j}) = \omega^{+}$

$\lim_{n \to -\infty} \sigma^{n}(\omega^{j}) = \omega^{-}$.



Prop. If $\mathcal{G}$ has no double edges then $\Sigma_{\mathcal{G}}^{v}$ and $\Sigma_{\mathcal{G}}^{e}$ are ~~som~~ topologically conjugate. (B)

$v_i \xrightarrow{e_j} v_k$ Say $v_i = t(e_j)$, $v_k = h(e_j)$

$(\dots v_{-1}. v_0\, v_1\, v_2 \dots)$ $(\dots e_{-1}. e_0\, e_1\, e_2 \dots)$

$\alpha : \Sigma_{\mathcal{G}}^{e} \to \Sigma_{\mathcal{G}}^{v}$ Define $\alpha(\dots e_{-1}. e_0\, e_1\, e_2 \dots) = (\dots t(e_{-1}). t(e_0)\, t(e_1) \dots)$

$\beta : \Sigma_{\mathcal{G}}^{v} \to \Sigma_{\mathcal{G}}^{e}$ Define $\beta(\dots v_{-1}. v_0\, v_1\, v_2 \dots)$

$= (\dots e_{-1}. e_0\, e_1 \dots)$

where e_j is the unique arrow ~~from v_{j-1} to v_j~~ v_j to v_{j+1} with $t(e_j) = v_j$ and $h(e_j) = v_{j+1}$.

$\alpha\beta(\dots v_{-1}. v_0\, v_1\, v_2 \dots)$

$= \alpha(\dots e_{-1}. e_0\, e_1 \dots)$

where $t(e_j) = v_j$ and $h(e_j) = v_{j+1}$

$\boxed{\text{$\alpha, \beta$ commute with the corresponding shift maps.}}$

~~$\alpha(\dots$~~

$= (\dots t(e_{-1}). \underset{h(e_{-1})}{t(e_0)} \dots) = (\dots v_{-1}. v_0 \dots)$

Could consider 1-sided or 2-sided.

Define sequence $(\dots e_{-1}. e_0\, e_1 \dots)$ satisfies

$h(e_j) = t(e_{j+1})$

$\beta\alpha(\ldots\ e_{-1}.e_0\,e_1\ldots)$

$=\beta(\ldots\ t(e_{-1}).\,t(e_0)\ t(e_1)\ldots)$

$\overset{\ast}{=}(\ \ldots\ e'_{-1}.e'_0 e'_1\ldots\ )$

where $t(e_j')=t(e_j)$ and $h(e_j')=t(e_{j+1})=h(e_j)$.

Since e_j is determined ~~once we know~~ by its head and tail we have $e_j'=e_j$.

fixed points of f

~~A~~ Relationship between growth of periodic points and growth of words of length n

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~~Count growth of periodic poin~~

Asymptotics

Understanding f_A ~~on~~ on a fine scale is equivalent to understanding behaviour on words of length n.

Can we take some of the insights from this coding process and apply them to more general dynamical systems?

$f(\omega) = \omega'$

Continuity: Given an n there is an m so that knowing digits $-m$ to m of ω determines digits $-n$ to n of ω'. (5)

For α, β ~~the~~ take $m = n+1$.

Two sequences ω, ω' are close if they agree in positions i $\omega_i = \omega_i'$ for $|i| \le n$.

Continuity of α means that given an n there is an m so that knowing ω_i for $|i| \le m$ determines $\alpha(\omega)_i$ for $|i| \le n$.

For α ~~let~~ we can take $m = n$.

For β we can take $m = n+1$.

Remark. $G^{(1)}$ does not have multiple edges between vertices.

~~Proof.~~

Proof. An edge in $G^{(1)}$ is a path of length 2 so $\cdot \xrightarrow{i} \cdot \xrightarrow{j} \cdot$ this edge goes from i to j. ~~so~~

~~As Any edge This is a path~~ A path from i to j is a path of length 2 whose first component is i and second component is j.

We have seen how we can get a symbolic description of orbits for maps of the circle.

Next we want to describe the corresponding phenomenon for diffeomorphisms.

Since diffeomorphisms are invertible it makes sense to talk about itineraries that extend for forwards and backwards time.

We introduce bi-infinite sequence spaces.

In this section we write Σ_2 for $\{(\omega_j)_{j=-\infty}^{\infty} : \omega_j \in \{0,1\}\}$.

We give Σ_2 a metric. For $\omega, \omega' \in \Sigma_2$ we let

$$d(\omega,\omega') = \max_{n\in\mathbb{Z}} \frac{\varepsilon(\omega_j,\omega_j')}{2^{|j|}}$$

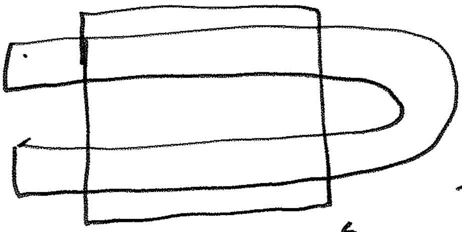
two sided

where $\varepsilon(j,k) = \begin{cases} 1 & \text{if } j \neq k \\ 0 & \text{if } j = k. \end{cases}$

We define a shift: $\sigma : \Sigma_2 \circlearrowleft$. $\sigma(\omega) = \omega'$ where $\omega'_n = \omega_{n+1}$. left shift. Unlike the one-sided case σ is invertible.

Prop. Σ_2 is a compact metric space. σ is a ~~continuous function~~ homeomorphism.

We want to model a diffeomorphism of the plane f that has the following effect on ~~there~~ a square $\quad f:\mathbb{R}^2 \to \mathbb{R}^2$



$f(S)$

S

Let us assume that a point which leaves S in forwards time ~~never~~ ie $x \in S$ $f(x) \notin S$ never returns to S ie $f^n(x) \notin S$ for $n \geq 1$.

Similarly we assume that if $x \in S$ and $f^{-1}(x) \notin S$ then $f^{-n}(x) \notin S$ for all $n \geq 1$.

We want to keep track of points whose orbits remain in the square for all time.

Say $\Lambda = \{p \in \mathbb{R}^2 : f^n(p) \in S \text{ for all } n\}$ so in fact it does not ~~ma~~ matter how ~~w~~ we define f outside of S.

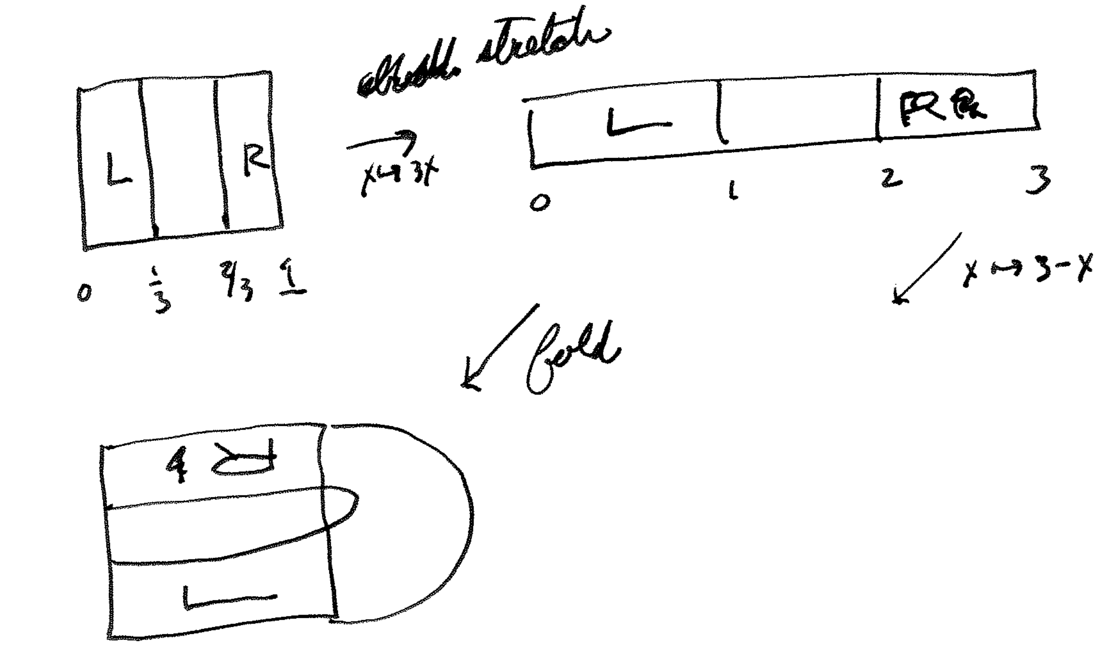
Say we have a explicit word

$$\ldots w_{-3}\, w_{-2}\, w_{-1}\; w_0\; w_1\; w_2\; w_3 \ldots$$

$$\ldots 0 \quad 0 \quad 1 \,.\, 0 \quad 0 \quad 1 \quad 0 \ldots$$

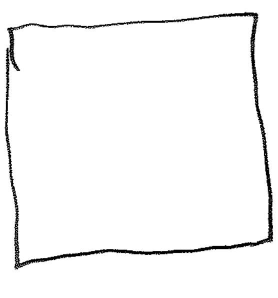
We want to be able to locate the 0-th position (hence all positions). We make the convention that we insert a decimal point between w_0 and w_{-1}. σ acts by shifting the sequence left (or shifting the decimal point right).

For concreteness we choose $\Delta = [0,1]\times[0,1]$.



$$f(x,y) = \begin{cases} (3x,\ \tfrac{1}{3}y) & (x,y)\in L \\ (3-3x,\ 1-\tfrac{1}{3}y) & (x,y)\in R \end{cases}$$

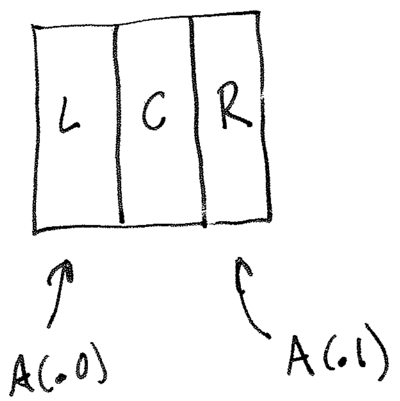
Thm. Let $\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(\Delta)$. Then $f(\Lambda)=\Lambda$ and $f|\Lambda$ is $f^{-1}(\Lambda)=\Lambda$ topologically conjugate to the shift map σ on Σ_2.



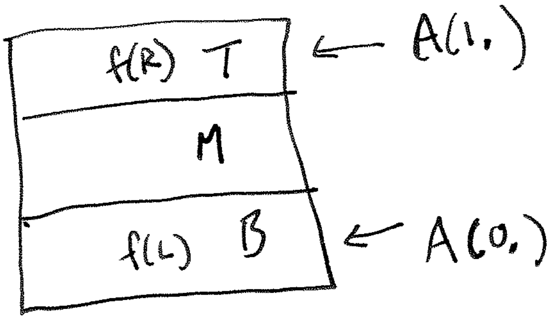
If $p \in \Delta$ we define an itinerary by

$$\omega_j(p) = 0 \quad \text{if } f^j(p) \text{ is defined and } f^j(p) \in L$$
$$= 1 \quad \text{"} \qquad \text{"} \qquad f^j(p) \in R$$
$$= * \quad \text{if } f^j(p) \text{ is not defined.}$$

Let ~~A(ω)~~ $w = \omega_{-j} \dots \omega_{-1} . \omega_0 \dots \omega_{k-1}$ be a finite word ~~lets~~ consisting of 0's and 1's.



Let $A(\omega) = \{ p : f^\ell(p) = \omega_\ell$ for $-j \le \ell \le k-1$



f^{-1} is defined at these points.

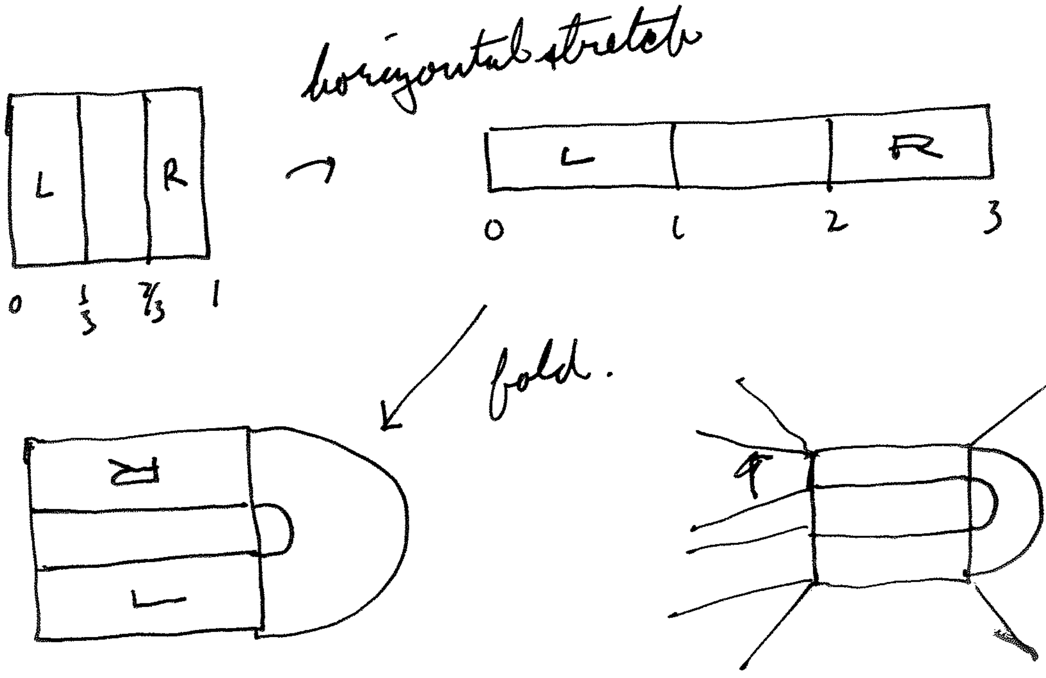
Let $p \in \Delta$

For $j > 0$ we say $f^j(p)$ is not defined if for some $k \le j$ $f^k(p) \in C$.

For $j < 0$ we say $f^j(p)$ is not defined if for some $j \le k$, $f^j(p) \in M$.

Our "model ~~no~~ diffeomorphism" $f: \mathbb{R}^2 \circlearrowleft$ is constructed in order to clearly show the important dynamical behavior. In fact it is quite artificial but we will worry about that later.

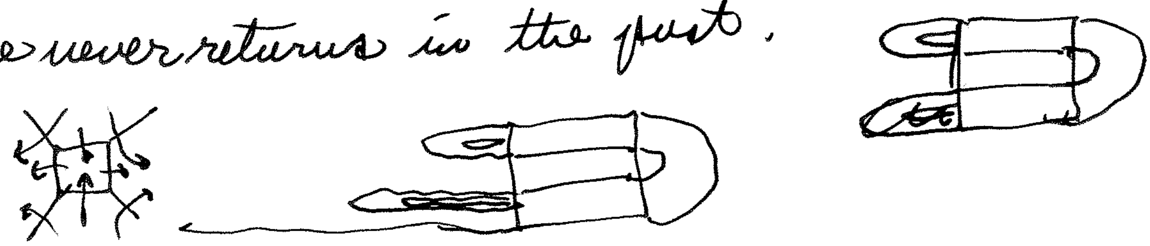
Let $\Delta = [0,1] \times [0,1]$. We will focus on ~~the~~ $f|\Delta$ and $f^{-1}|\Delta$.



$$f(x,y) = \begin{cases} (3x, \tfrac{1}{3}y) & (x,y) \in L \\ (3-3x, 1-\tfrac{1}{3}y) & (x,y) \in R \end{cases}$$

We do not specify f outside of Δ but we assume that any point in Δ which leaves Δ at some ~~time~~ future time never returns ~~in to~~ to Δ in the future.

We assume that any point leaving Δ at some past time never returns in the past.



Thm. Let $\Omega = \bigcap_{n=-\infty}^{\infty} f^n(\Delta)$. Then Ω is an invariant set under $f|\Omega$ is topologically conjugate to the shift on Σ on two symbols.

The proof will follow the outline of the construction of the semi-conjugacy for an expanding map of the circle.
(Version in the notes improves on my lecture.)

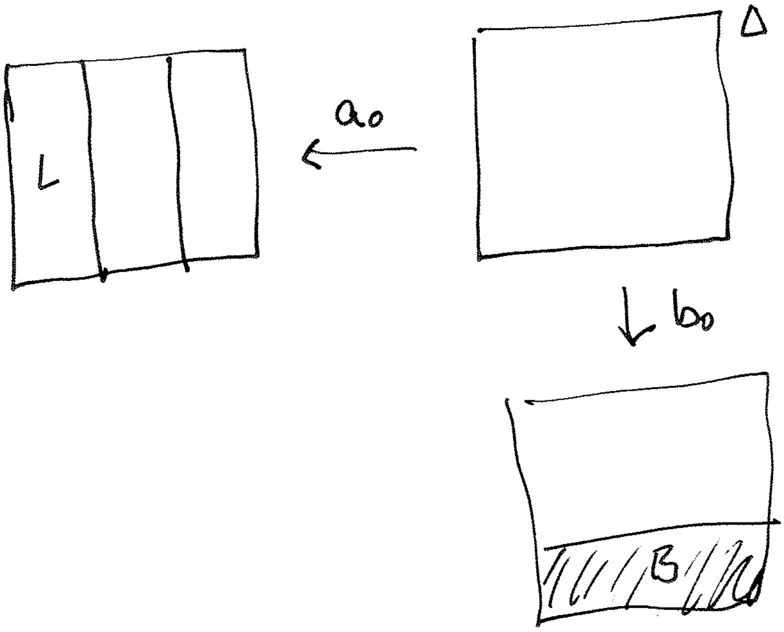
There is a connection between f and the ~~continued~~ ternary expansion of the coordinates

~~Let $\alpha_j(\Delta) = \frac{x}{3} + j$ for $j = 0, 1, 2$.~~

$$a_j(x,y) = \left(\frac{x+j}{3},\ y\right) \quad \text{for } j = 0, 1, 2.$$

$$b_j(x,y) = \left(x,\ \frac{y+j}{3}\right) \quad \text{for } j = 0, 1, 2.$$

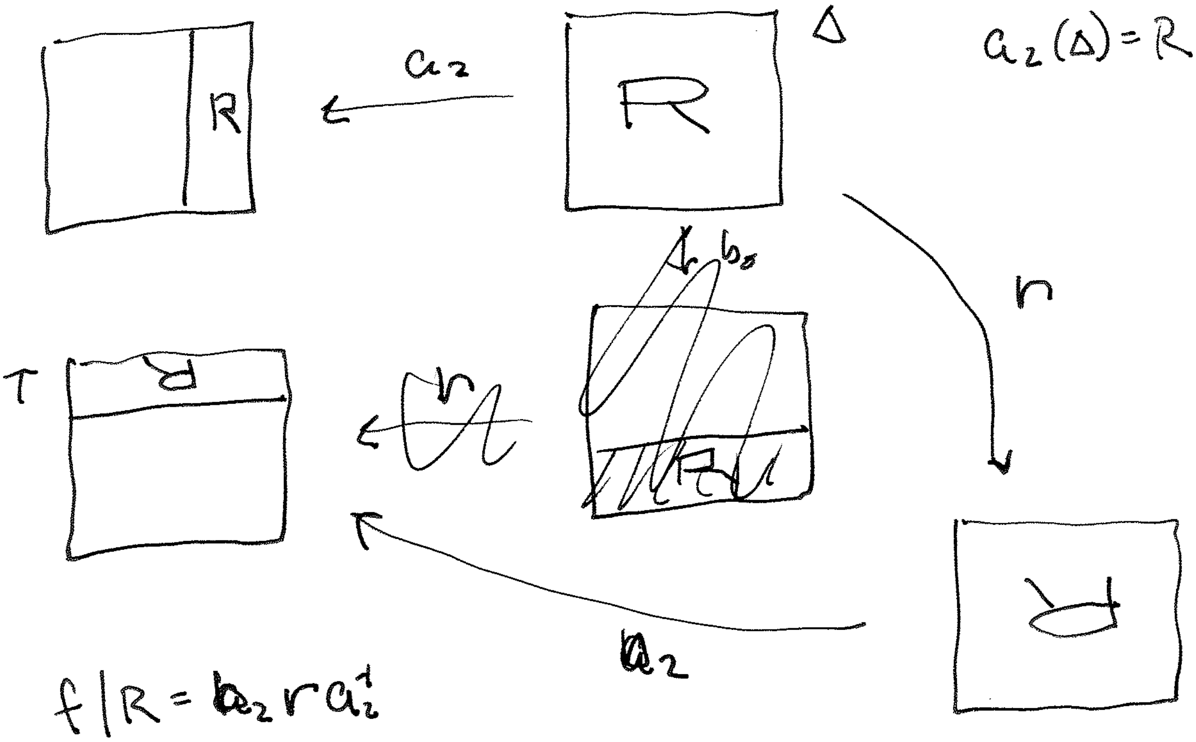
Let $r(x,y) = (1-x,\ 1-y)$.



$a_0(\Delta) = L$

$f|L = a_0\, b_0\, a_0^{-1}$ (composition is implicit)

$a_2(\Delta) = R$



$f|R = a_2\, r\, a_2^{-1}$

~~$f^{-1}|T\ldots$~~

$f^{-1}|B = a_0 \beta_0^{-1}$

$f^{-1}|T = a_2 r \beta_2^{-1}$.

If $(X,Y) = (.t_0t_1t_2\ldots, Y)$ where $t_0t_1\ldots$ is the ternary expansion then

$a_j(X,Y) = (.j\,t_0t_1t_2\ldots, Y)$

$\beta_j(X, .s_0s_1s_2\ldots) = (X, .j\,s_0s_1s_2\ldots)$.

$a_ia_ja_k(X,Y) = (.ijk\,t_0t_1t_2\ldots)$

Relations between a_i, b_j and r.

a_i, b_j commute
a_i, a_j do not commute
b_i, b_j "

~~$ra_j(X,Y) = r(X/3+j, Y) = (1-$~~

Claim $ra_j = a_{(2-j)}r$.

$ra_j(X,Y) = r(\frac{X+j}{3}, Y) = (1-\frac{X+j}{3}, 1-Y)$

$= (\frac{3-j-X}{3}, 1-Y)$

$a_{(2-j)}r(X,Y) = a_{(2-j)}(1-X, 1-Y) = (\frac{1-X+2-j}{3}, 1-Y)$.

Similarly $rb_j = b_{(2-j)}r$

Consider a rectangle determined by specifying ternary expansions. We can compute the itinerary of points in this rectangle.

~~Box~~ ~~B~~ ~~R~~ $= \{(X,Y): X = 020 ***\ldots \quad Y = 20*** \}$

~~B~~ ~~R~~ $= a_0 a_2 a_0\, b_2 b_0(\Delta)$.

Apply f to ~~Rec~~ Box. ~~Rec~~ ~~Box~~ Box $\subset$ ~~R~~ L since it is in the image a_0. For each point in ~~Box~~ $\omega_0 = 0$

$f|L$ ~~f(Box)~~ $= b_0 a_0^{-1}$

$$b_0 a_0^{-1}(\text{Box}) = b_0 a_0^{-1} a_0 a_2 a_0 b_2 b_0(\Delta)$$
$$= b_0 a_2 a_0 b_2 b_0(\Delta)$$
$$= a_2 a_0 b_0 b_2 b_0(\Delta)$$

$f(\text{Box}) = \{(X,Y): X = 20**, Y = 020** \}$. $\quad \omega_1 = 2$.

Now: $f^2(\text{Box}) = f(a_2 a_0 b_0 b_2 b_0(\Delta))$ $\quad f(\text{Box}) \subset$ ~~L~~ R

$$= \underline{b_2 r a_2^{-1}} a_2 a_0 b_0 b_2 b_0(\Delta)$$
$$= b_2 r\, a_0 b_0 b_2 b_0(\Delta)$$
$$= b_2 a_2 r\, b_0 b_2 b_0(\Delta)$$
$$= b_2 a_2 b_2 r\, b_2 b_0(\Delta)$$
$$= b_2 a_2 b_2 b_0 r\, b_0(\Delta)$$
$$= b_2 a_2 b_2 b_0 b_2 r(\Delta)$$
$$= b_2 a_2 b_2 b_0 b_2(\Delta)$$
$$= a_2 b_2 b_2 b_0 b_2(\Delta)$$

$f^2(\text{Box}) \subset L$

$\omega_2 = 2$.

For any word $w_j \dots w_{k-1}$ with $j \le 0$, $k \ge 0$

Claim. $A^{j,k}(w)$ is a rectangle in Δ with horizontal and vertical sides. The height is $\frac{1}{3^{|j|}}$ and the width is $\frac{1}{3^{|k|}}$.

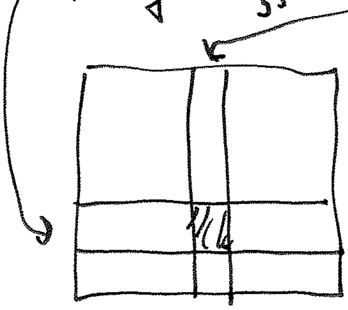
$.0$ $j=0$ $k=1$

height$(A(.0))=1$
width$(A(.0))=\frac{1}{3}$

Proof. By induction. It is true for words of length 1: $0.$, $1.$, $.0$, $.1$.

Assume it is true for words of length $< n$. Consider a word of length n: $w_j \dots w_1 . w_0 \dots w_{k-1}$ with $j \ge 1$ and $k \ge 1$. By induction hypothesis

$A(w_j \dots w_1 .)$ is a rectangle in Δ of width 1 and height $\frac{1}{3^j}$



$A(.w_0 \dots w_{k-1})$ is a rectangle in Δ of height 1 and width $\frac{1}{3^k}$.

Their intersection has width $\frac{1}{3^k}$ and height $\frac{1}{3^j}$. Thus the claim holds.

Now consider a word $.w_0 \dots w_{n-1}$ of length n. $j=1$ $k=n-1$

shift left

$A(w_0 . w_1 \dots w_{n-1})$ ~~satisfies~~ is a rectangle of width $\frac{1}{3^{n-1}}$ and height $\frac{1}{3}$. It is contained in L or R (depending on w_0) $f|L^{\text{or } R}$ is a linear map which multiplies height ~~width~~ by $\frac{1}{3}$ and ~~height~~ width by 3 so $f(A(w_0 . w_1 \dots w_{n-1}))$ $= A(.w_0 \dots w_{n-1})$ has height 1 and width $\frac{1}{3^n}$ as was to be shown.

Thm. There is ~~a~~ a conjugacy $h:\Sigma_2 \to \Lambda$ so that ~~$hf\sigma h$~~ σ ④
~~$fh = \sigma h$~~ $\sigma h = hf$

$$\begin{array}{ccc} \Lambda & \xrightarrow{f} & \Lambda \\ \downarrow h & & \downarrow h \\ \Sigma_2 & \to & \Sigma_2 \end{array}$$

Proof. Construction of conjugacy

Let $\omega \in \Sigma_2$. The sequence of sets

$A(\omega_{-n} \cdot \omega_{-1} \cdot \omega_0 \ldots \omega_{n-1})$ are nested and have

decreasing diameter.

Let $h(\omega) = \bigcap_{n=1}^{\infty} A(\omega_{-n} \ldots \omega_{n-1})$.

Let $\varepsilon > 0$. Choose m s.t. $\frac{1}{\sqrt{2}} 3^{-m} < \varepsilon$. Let $S = 2^{-}$

h is continuous, ~~fix~~ ~~ω~~. ~~Following Let $\varepsilon>0$. Choose~~ ~~n so that $3^{n}<\varepsilon$.~~ Given $m \geq 0$ and $\omega' \in \Sigma$ s.t.

$d(\omega, \omega') < 2^{-m}$. Then $\omega'_n = \omega_n$ for all $|n| < m$

So $h(\omega)$ and $h(\omega') \in A(\omega_{-n} \ldots \omega_{n-1})$. In particular

$\|h(\omega) - h(\omega')\| \leq \frac{1}{\sqrt{2}} \cdot 3^{-n}$. $\quad \square \ \frac{1}{3^n}$, $\frac{1}{3^n}$

h is injective: If $\omega \neq \omega'$ then there is some ~~n~~ k ~~all~~ with $\omega_k \neq \omega'_k$. Thus for $n \geq |k|+1$ $A(\omega_{-n} \ldots \omega_n)$ and $A(\omega'_{-n} \ldots \omega'_n)$ are disjoint. But $h(\omega)$ is in the first set and $h(\omega')$ is in the second.

h is surjective. Any point p in $\bigcap_{n=-\infty}^{\infty}(\Delta)$ is in $\bigcap_{n=-\infty}^{\infty} f^n(L \cup R)$. So it has ~~a coding~~ an itinerary ω. We have $h(\omega) = p$.

h is a homeomorphism: Since Σ and Λ are compact and h is a continuous bijection, h^{-1} is ~~a homeomorphism~~ continuous.

h is a topological conjugacy. ~~[illegible]~~
~~If The image of p is~~ The inverse image of p is determined by its itinerary. But the itinerary of $f(p)$ is the shift applied to the itinerary of p.

Construction of conjugacy:

Let $\omega \in \Sigma_2$. The sequence of sets

$A(\omega_{-n} \ldots \omega_{-1} . \omega_0 \ldots \omega_{n-1})$ are nested and have decreasing diameter.

Let $h(\omega) = \bigcap_{n=1}^{\infty} A(\omega_{-n} \ldots \omega_{n-1})$.

$d = d_\lambda$

h is continuous, Fix $\lambda > 1$. ~~Choosing let $\varepsilon > 0$. Choose n so that $\lambda^n < \varepsilon$.~~ Given $\nu > 0$ and $\omega' \in \Sigma$ s.t. $d(\omega, \omega') < \lambda^{-\nu}$. Then $\omega'_n = \omega_n$ for all $|n| < \nu$.

So $h(\omega)$ and $h(\omega') \in A(\omega_{-\nu} \ldots \omega_{\nu-1})$. In particular

$\|h(\omega) - h(\omega')\| \le \sqrt{2} \cdot 3^{-\nu}$.

h is injective: If $\omega \ne \omega'$ then there is some k with $\omega_k \ne \omega'_k$. Thus for $n \ge |k|+1$ $A(\omega_{-n} \ldots \omega_n)$ and $A(\omega'_{-n} \ldots \omega'_n)$ are disjoint. But $h(\omega)$ is in the first set and $h(\omega')$ is in the second, so $h(\omega) \ne h(\omega')$.

h is surjective. Any point p in $\bigcap_{n=-\infty}^{\infty} f^n(\Delta)$ is in

$p \in \Delta$, $f(p) \in \Delta$ $\Rightarrow p \in L \cup R$

$\bigcap_{n=-\infty}^{\infty} f^n(L \cup R)$. So it has ~~a coding~~ an itinerary ω. We have $h(\omega) = p$.

Looking at its itinerary we see that $p \in A(\omega_{-n} \ldots \omega_{n-1})$ for all n so $h(\omega) = p$.

h is a homeomorphism: Since Σ and Λ are compact and h is a continuous bijection, h^{-1} is ~~a homeomorphism~~ continuous.

h is a topological conjugacy. ~~[illegible]~~

~~If The image of p~~ The inverse image of p is determined by its itinerary. But the itinerary of $f(p)$ is the shift applied to the itinerary of p.

coding as p.

Can define the local unstable manifold of p to be the set of points q such that

$d(f^n(p), f^n(q)) \leq \varepsilon$ for $n \leq 0$ and

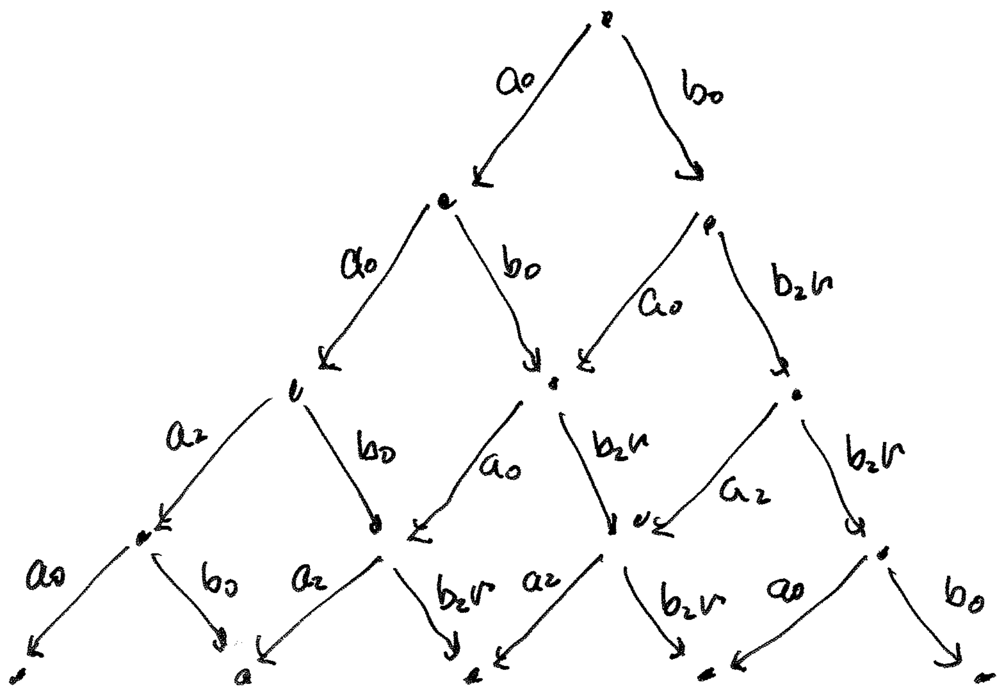
$\lim_{n \to \infty} d(f^{-n}(p), f^{-n}(q)) = 0$.

Note that the horseshoe is constructed so that local stable and unstable manifolds are easy to identify.

Another important property is reflected in the fact that if p and q in Λ are sufficiently close then $W^s_\varepsilon(p) \cap W^u_\varepsilon(q)$ at a unique point. This is related to the fact that ~~all~~ the sets $A(w)$ are rectangles. We will talk about this property again when we talk about shadowing and local product structure.

Example. Consider the word 0110.

$\rho \in A(0110)$ exactly when the following string of ops makes sense for ρ.



Fill in the edges. $A(.0110) = a_0 a_2 a_0 a_0 (\Delta)$

$A(0.110) = b_0 a_2 a_0 a_0 (\Delta) = a_2 b_0 a_0 a_0 (\Delta) = \dots$

$A(0110.) = b_0 b_2 r\, b_2 r\, b_0 (\Delta)$.

$a_0(\Delta) = \{(.0**, .***)\}$

$a_0 a_0(\Delta) = \{(.00**, .**)\}$

$b_2 a_0 a_0(\Delta) = \{(.00**, .2**)\}$

a_j inserts j and shifts sequence to the right.

wild cards

②

$A(.0110) = a_0 a_2 a_0 a_0(\Delta) = \{(.0020{*}{*}\ldots, .{*}{*})$ in ternary

$A(0.110) = b_0 a_2 a_0 a_0(\Delta) = \{(.002{*}{*}, .0{*}{*})\}$

$A(01.10) = \cancel{a_0} b_2 r\, a_0 b_0 a_0(\Delta)$

$= b_2 a_2 b_2 a_2 \cancel{r} r(\Delta)$

$= b_2 a_2 b_2 a_2(\Delta) \in Q$

$= b_2 b_2 a_2 a_2(\Delta) = \{(.22{*}{*}, .22{*}{*})\}$

~~A(0110~~

$A(0110.) = b_0 b_2 r\, b_2 r\, b_0(\Delta)$

$= b_0 b_2 b_0 r r\, b_0(\Delta)$

$= b_0 b_2 b_0 b_0 r r(\Delta)$

$= b_0 b_2 b_0 b_0(\Delta) = \{(.{*}{*}{*}, .0020{*}{*})\}.$

⟵⟶ See next page.

Prop. $\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(\Delta)$ consists of points for which the ternary expansion ~~can be written~~ of each coordinate can be written with only 0's and 2's.

Correspondence between words

$A(\omega_{-j} \ldots \omega_{-1} . \omega_0 \ldots \omega_{k-1})$ and rectangles $(.\varepsilon_1 \ldots \varepsilon_{j+k}, .\delta_1 \ldots \delta_{j+k})$

In fact this correspondence is a bijection since it is injective and # of elements is 2^{j+k} in both sets.