Introduction to Rational Billiards

Talk by John Smillie

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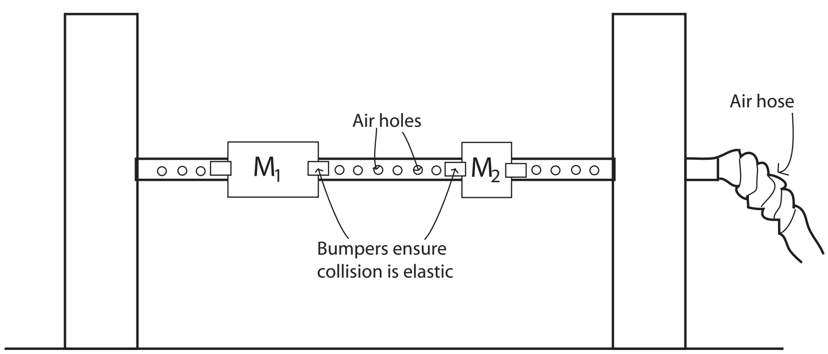
•References:

- *Masur, Howard; Tabachnikov, Serge*: Rational billiards and flat structures. *Handbook of dynamical systems, Vol. 1A*, 1015–1089, *North-Holland, Amsterdam*, 2002.
- *Vorobets, Ya. B.*: Plane structures and billiards in rational polygons: the Veech alternative. (Russian) *Uspekhi Mat. Nauk* **51** (1996), no. 5(311), 3–42; *translation in Russian Math. Surveys* **51** (1996), no. 5, 779–817
- *Zorich, Anton*: Flat surfaces. *Frontiers in number theory, physics, and geometry. I*, 437–583, *Springer, Berlin*, 2006.

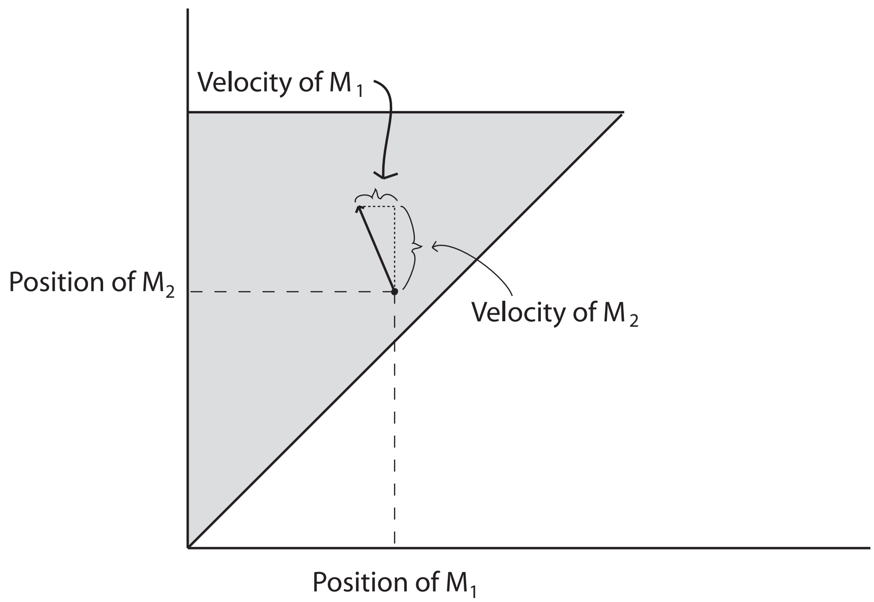
•Motivating questions

Consider the following physical system:



This setup consists of two sliders of different masses constrained to move along a straight line. The system is assumed to be frictionless, and the sliders, when set in motion, bounce off the walls and each other with no loss of kinetic energy. This can be thought of as a very simple toy model for a thermodynamics; instead of studying elastic collisions between millions of molecules with three degrees of freedom, we consider a system in which there are only two objects undergoing elastic collisions and moving in a single dimension.

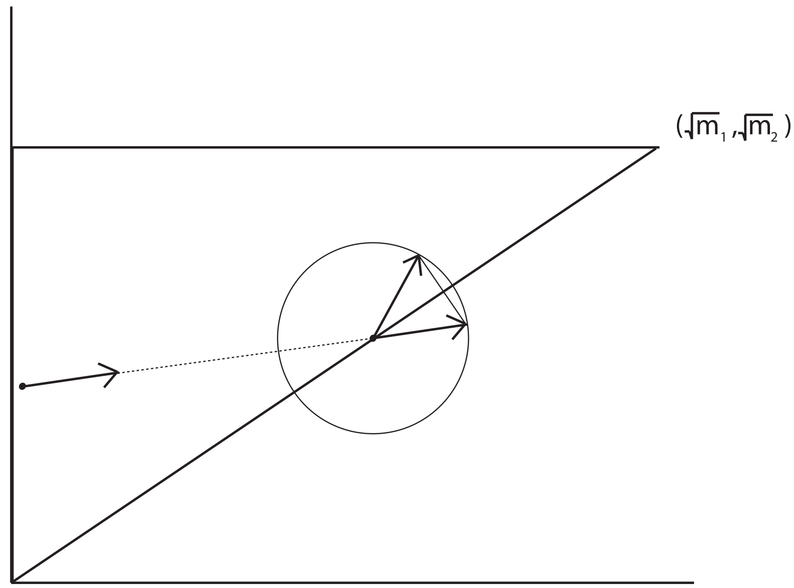
The phase space for this system can be identified with the tangent bundle on a triangle, as the following diagram illustrates.



For a point in the interior of the triangle, the system evolves according to the geodesic flow in the plane. When a vector hits the top of the triangle or the left-hand side of the triangle, this corresponds to one of the sliders hitting the edge of the apparatus. In this case, the slider colliding with the wall simply reverses directions while the other slider continues on with the same velocity. It follows that when a vector runs into on of these walls it is simply reflected in the side it hits.

When the vector runs into the diagonal line, this corresponds to the two sliders colliding. Consider the case when both masses are equal. Conservation of energy tells us that $m_1\dot{x_1}^2 + m_2\dot{x_2}^2$ stays the same before and after the collision, so in this case $\dot{x_1}^2 + \dot{x_2}^2$, the norm-squared of the vector stays the same. Conservation of momentum tells us that $m_1\dot{x_1} + m_2\dot{x_2} = (m_1, m_2) \cdot (\dot{x_1}, \dot{x_2}) = (1, 1) \cdot (\dot{x_1}, \dot{x_2})$ stays the same, so the projection of the vector onto the line of slope 1 is preserved. Putting this information together, we see that the vector must be reflected through the diagonal.

If $m_1 \neq m_2$, using the change of coordinates $y_1 = \sqrt{m_1}x_1,\ y_2 = \sqrt{m_2}x_2$, we get that energy is given by $\dot{y_1}^2 + \dot{y_2}^2$, and momentum is given by $(\dot{y_1}, \dot{y_2}) \cdot (\sqrt{m_1}, \sqrt{m_2})$, so the same analysis goes through, and we see that a vector hitting the diagonal is reflected in the diagonal. Note that we don't try to define a trajectory that hits the corners of phase space, so for a countable set of directions the our dynamical system is not defined for all time.



Using the above model, we are able to study the behavior of the original system by studying billiard trajectories in triangles. The questions we are interested in are those about the long-term average behavior of the system. As an example, we can consider the energy of slider one. If P is the triangle, this can be considered as a function from $P \times S^1 \to \mathbb{R}$. Let $\phi_t(p_0)$ denote the billiard flow starting with initial conditions given by a point p_0 in phase space. Motivated by the thermodynamic interpretation, we can ask is whether the time average

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\phi_T(p_0))dt$$

is equal to the average over phase space

$$\frac{1}{2\pi A} \int_{P \times S^1} f d\mu$$

where A is the area of the triangle and μ is Lebesgue measure on the $P \times S^1$.

If the space and time averages are to be related, it should be the case that the orbits are uniformly distributed. Questions about the distribution of orbits in phase space include:

- Are orbits uniformly distributes? How many orbits are uniformly distribute?
- Are the orbits dense in phase space? How many orbits are dense in phase space?
- Are there periodic orbits? How many orbits are periodic?
- Can we describe the sequence of collisions concretely? When is such a concrete description possible?
- Energy is necessarily an invariant of the system, are there other invariants?

•The case of rational angles

When a vector is reflected in one of the edges e_j of the triangle, it undergoes a transformation by the element r_j of $O(2)$ corresponding to reflection in that edge. Knowing that a trajectory with velocity vector v hits the edges $e_1, e_2, e_3, ..., e_n$ in that order tells us that the billiard trajectory has velocity vector $r_n r_{n-1} ... r_2 r_1 v$ after this sequence of collisions.

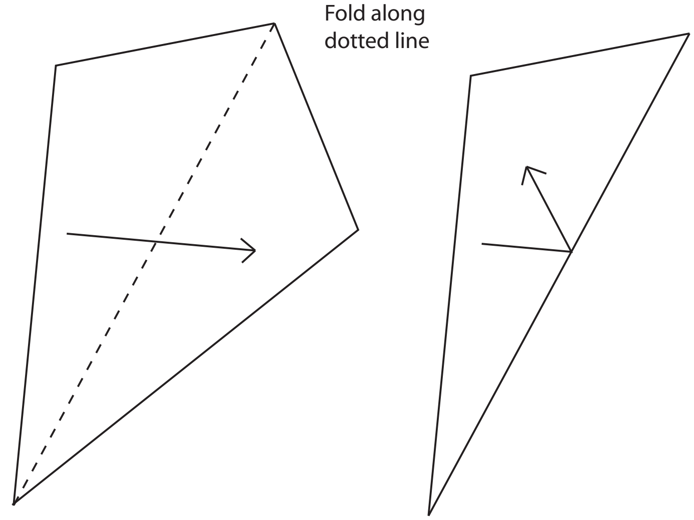
Let $\Gamma \subset O(2)$ be generated by the r_j's. Then Γ is either finite or it is a dense subgroup of $O(2)$. If Γ is finite, then a billiard trajectory is stuck in a small subset of phase space. We can rephrase the questions listed above to ask how orbits are distributed within this subset. It is easy to see that Γ is finite if and only if the angles of the billiard table are rational multiples of π.

There are many reasons to care about rational billiards:

- The irrational case is much more difficult and few results are known.
- Physicists are interested in rational billiards as they relate to physical situations that lie on the boundary between classical and quantum mechanics.
- Rational billiards lead to interesting structures on surfaces.
- The idea of renormalization, which is a useful tool in dynamics, comes up in a nice way in the study of rational billiards.

•The Zemlyakov-Katok Construction

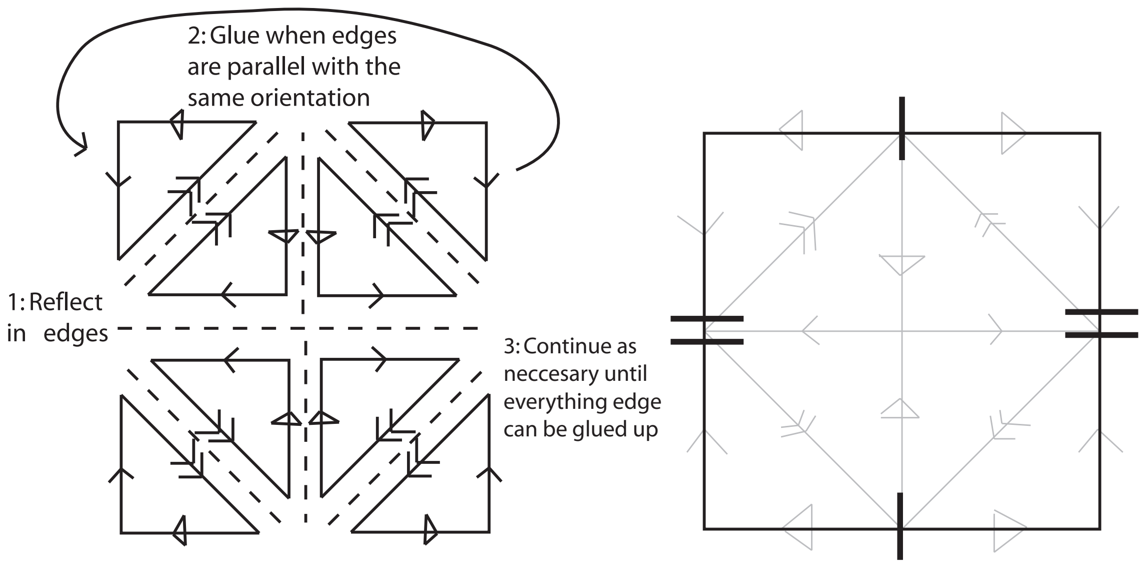
The Zemlyakov-Katok construction takes a polygonal billiard table and creates a surface in which billiard trajectories follow straight lines. When a billiard trajectory hits a side of the table, instead of reflecting the velocity vector in the edge, we can choose to reflect the table in the edge and let the billiard trajectory carry on straight. If we were to fold along this edge, the resulting trajectory would give the billiard trajectory.



Let $\rho_j = r_j + c_j$ be the element in the group of Euclidean isometries given by reflecting in the edge e_j of a polygon P. Let Γ_{Euc} be the group generated by the ρ_j's. Consider the product $\Gamma_{\text{Euc}} \times P$. We have a natural map $\Gamma_{\text{Euc}} \times P \to \mathbb{R}^2$ which takes $P_\gamma = (\gamma, P) \mapsto \gamma(P)$. Let $\widetilde{S}$ be $\Gamma_{\text{Euc}} \times P/\sim$, where $\sim$ identifies the edge e_j in P_γ with e_j in $P_{\gamma\rho_j}$. We then have a local homeomorphism from $\widetilde{S}$ to $\mathbb{R}^2$ which turns billiard orbits into straight lines.

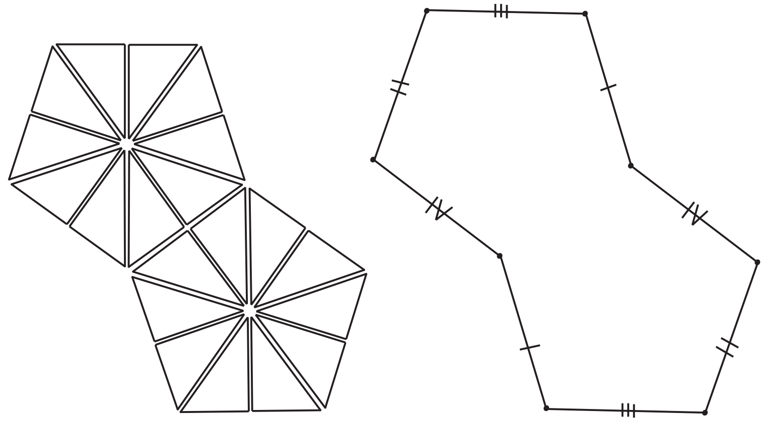
The surface so constructed is too large, as after many unfoldings, we must come back to a surface that is a translate of our original surface. A translation will take lines in a given direction to lines in the same direction, and hence will send billiard orbits to billiard orbits, so quotienting out by these translations doesn't interfere with the billiard trajectories. Let Γ' be the kernel of the derivative map $\Gamma_{\text{Euc}} \to \Gamma$, and let $S = \widetilde{S}/\Gamma'$. This surface will consist of finitely many copies of the original polygon, so S gives a finite area flat surface on which geodesic flow corresponds to the billiard flow.

There is an easy recipe for constructing these surfaces. Label and orient the sides of the billiard table. Unfold along an edge, and if there are any parallel edges with the same labels glue them together. Any time you see an unglued edge, you can reflect in that edge and perform all allowable gluings. This process will terminate in a finite number of steps. The following figure shows the construction of the surface S corresponding to an equilateral right triangle.

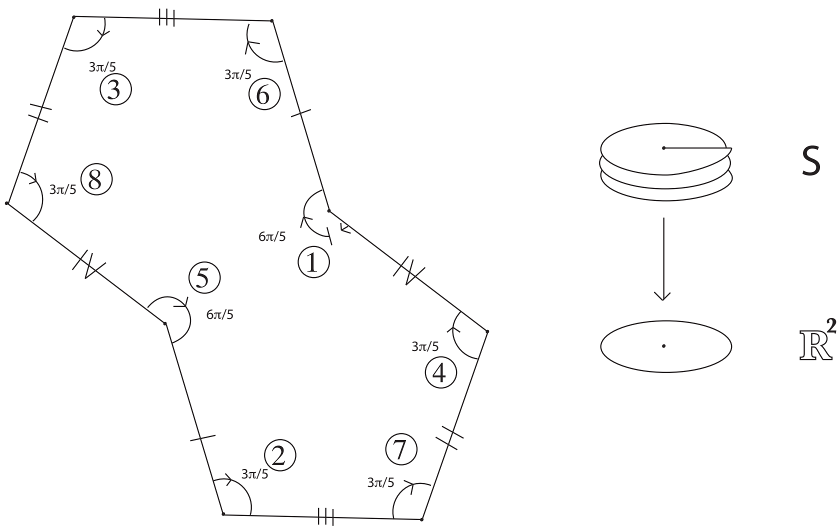


In this case the corresponding surface is a torus.

The following figure shows a more complicated example coming from billiards in a right triangle with an angle of $\frac{\pi}{5}$. Chasing through the identifications, it is easy to see that there is only one vertex after gluing, so by computing the Euler characteristic we can see that the surface has genus 2.



The above surface exhibits a feature that all such surfaces of genus greater than one share, namely it has points of concentrated negative curvature, or cone points.



As can be see from the picture, the angle around this vertex is 6π. A neighborhood of such a point looks like a branched cover of a disc in the plane.