Introduction to Rational Billiards II

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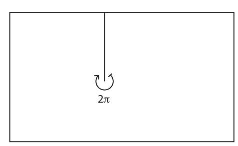
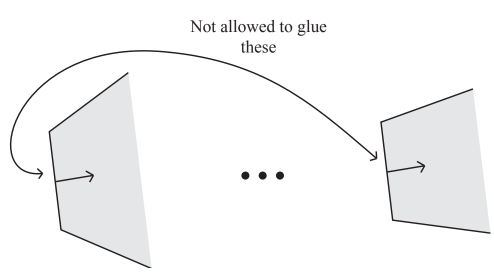
•Translation surfaces and their singularities

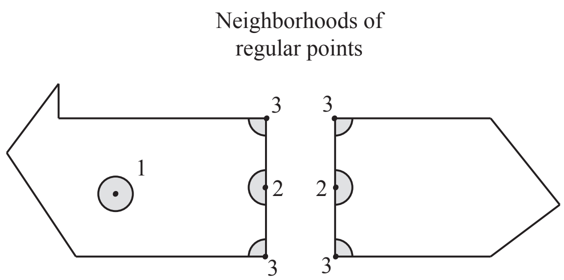
Last time we described the Zemlyakov-Katok construction for billiards on a triangular table, but this construction will work for billiards in any rational polygon. Such an object need not look like a polygon, for example the following rectangular table with a barrier, which can be thought of as a "heptagon" with one angle of 2π.



The objects obtained through this construction are examples of translation surfaces. A translation surface is a surface obtained from a finite collection of disjoint polygons in the plane by gluing parallel sides. We require the inward normals of glued edges to be in opposite directions, to ensure directions inherited from the plane are well defined at the edges.

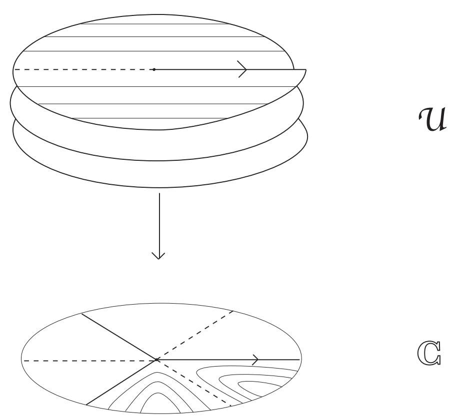


A translation surface comes equipped with a family of maps $U \subset S \to \mathbb{R}^2$ that cover the surface and are defined up to translation.



On any translation surface, there are finitely many points p_j where the cone angle at p_j is $2\pi n_j$ for some $n_j > 1$, and at such points the map is not a local homeomorphism. Away from the cone points, we have an atlas where the change of coordinate function are translations.

We can define a conformal atlas for the whole surface as follows. Suppose the cone angle at a point p is $2\pi n$. We can give polar coordinates $\{(r,\theta) \mid r \in [0, r_0],\ \theta \in [0, 2\pi n]\}$ for a neighborhood U of p, and define a chart on U by $(r,\theta) \mapsto (r^{\frac{1}{n}}, \frac{\theta}{n}) \in \mathbb{C}$. These charts are 1-1 and conformal.



A complex analyst will tell you that what you need to put a translation structure on a Riemann surface is an abelian differential.

If the cone angle at p is $c = 2\pi n$, then the curvature at p is $2\pi - c$. Exercise: Prove the Gauss-Bonnet formula in this context, i.e. show that if the cone points p_j of a translation surface S have curvatures κ_j for $1 \le j \le n$, then $\sum_{j=1}^{n} \kappa_j = 2\pi\chi(S)$. Hint: Triangulate.

Every translation invariant structure on $\mathbb{R}^2$ gives rise to a corresponding structure on translation surfaces. Some useful examples of such structures are

- translation invariant vector fields, which give rise to directional flows
- the standard metric on $\mathbb{R}^2$
- the area form on $\mathbb{R}^2$
- the notion of horizontal and vertical directions
- the forms dx and dy
- horizontal and vertical foliations

•Orbit Closures of directional flows

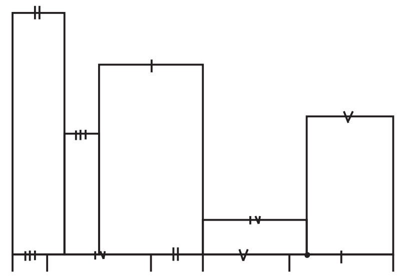
We would like to prove a characterization of orbit closures for directional flows due to Fox and Kershner (1936).

Consider the vertical flow ϕ_t on a translation surface S, and let I be a closed horizontal segment. The "first return map" to the segment I, is a map $f : I \to I$ sending $p \mapsto \phi_{t_0}(p)$, where t_0 is the first positive time t for which $\phi_t(p) \in I$. You may worry that this map is not well defined, and in fact there are points where it isn't.

The first thing that could go wrong is that the trajectory could hit a cone point, in which case the flow is not defined. The number of vertical trajectories that enter a cone point of cone angle $2\pi n$ is n, however, so there aren't that many bad trajectories to worry about. We also count as bad points trajectories whose first return hits the endpoint of the interval since these lead to discontinuities for the first return map, but there are only two of these (unless our endpoints are cone points) so we exclude these from the domain of f as well. What remains is a partition of the interval into finitely many subintervals I_j, and each I_j consists of points that stay together under the vertical flow until their first return to I, and return time is constant along such intervals (this would not be true if we hadn't decided to throw out points that flow to the endpoints of I).

For the remaining points in the domain, the only thing to worry about is that the flow doesn't come back at all. Each point p_j lies in some interval I_j, which, as described above, is a set of fellow-travelers. Consider the area swept out by I_j by the vertical flow. If I_j does not return to I under the vertical flow, then I_j sweeps out infinite area (which is impossible as S has finite area) or the region swept out by I_j begins to overlap with itself. If $p \in \phi_{t_1}(I_j) \cap \phi_{t_2}(I_j)$ for $0 < t_1 < t_2$, then $\phi_{-t_1}(p) \in I, \rightarrow\leftarrow$, so overlapping is not possible. Thus the first return map is defined on each of the intervals I_j.

From this discussion it is evident that we can reconstruct a piece of the original translation surface by gluing together the strips given by flowing in I_j's.



The first return map is an example of an "interval exchange map." The intervals are exchanged according to the rules shown in the above picture, and the relative lengths of the intervals along with the instructions for shuffling determine the dynamics.

It is also clear that if one point in an interval i_j is periodic with period 1 with respect to the first return map, then every point in I_j is periodic with period 1. If a point is periodic with period n, a similar analysis is possible. Let B be the set of bad points of the interval. We can look at the set $f^{-1}(B) \cup f^{-2}(B) \cup ... \cup f^{-n}(B)$, which will divide the set into intervals I_k which are fellow-travelers at least until the n-th time the I_k intersects I. Thus if one point in such an interval has period n with respect to the first return map, all points in this interval have period n. Thus we have shown

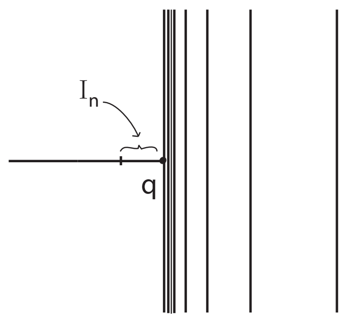
Theroem: Every closed geodesic is contained in an open cylinder of closed geodesics of the same length and the boundary of the cylinder is a union is a union of saddle connections.

This analysis shows that each periodic point is contained in an interval of periodic points with the same period. The boundary of this interval has to hit a singular point in forward time (if it hits an end point of I, simply extend I as necessary). Applying this analysis to ϕ_{-t}, we see that an endpoint of such an interval must also hit a singular point in backwards time. A trajectory hitting singular points in both forward and backward time is called a "saddle connection." We have shown that cylinders of periodic points have boundaries which are unions of saddle connections.

For points that are not periodic, we have the following theorem:

Theorem: (Fox-Kershner) Given a bi-infinite (non-periodic) trajectory $\phi_t(p)$ of the vertical flow in a translation surface S, the closure of $\{\phi_t(p) \mid t \in \mathbb{R}\}$ is a subsurface of S whose boundary is a union of saddle connections.

Proof: If the the closure A of $\{\phi_t(p) \mid t \in \mathbb{R}\}$ has no boundary points, then by connectivity the theorem holds, so assume we have a boundary point q. Construct a horizontal segment I through q that intersects A only in the point q. This is possible because q is a boundary point and A is a closed set. Arguing by contradiction, we assume that the orbit of q does not hit a singular point. As above, we can construct subintervals I_j of I such that interior points of each I_j are fellow travelers until the first return of I_j to I. One of these intervals, I_n, will have q as one of its endpoints.



As above, I_n must return to I, so q must return to I by continuity as it doesn't hit a saddle connection. But q must also return to a point of the orbit closure, as the orbit closure is invariant under the flow (if $\phi_{t_i} \to q$ then $\phi_{t_i+s} \to \phi_s(q)$). By assumption, q is the only point of A in I, so $f(q) = q$, and hence q is periodic. But we already have shown that periodic points lie in the interior of cylinder of periodic points, so it is impossible for the set $\{\phi_t(p)\}$ to approach q $\rightarrow\leftarrow$. Thus q must flow into a singular point, and running this argument for backward time, we see q must be on a saddle connection.

It isn't hard to show that there aren't very many saddle connections. One way to see this is to look at the holonomy map from the set of saddle connections to $\mathbb{R}^2$ sending $\sigma \mapsto (\int_\sigma dx, \int_\sigma dy)$. It is easy to see from our construction of translation surfaces that this map is discrete and has bounded multiplicity. As an impressive sounding conclusion, we have

Corollary: The directional flow is minimal (i.e. every orbit is dense) in all but countably many directions.