# Homotopy type of some homogeneous spaces of simple Lie groups of dimension 75

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## Contents



## 1 Introduction and History

Work of Jacobson, Morozov, Kostant and Mal'cev showed that for every nilpotent element in a Lie algebra $\mathfrak{g}$, there exists an $\mathfrak{sl}(2)$ triple with this element as its "positive nilpotent element". In fact it is shown that there is a one-to-one bijection between nilpotent orbits in a Lie algebra and Lie algebra homomorphisms $\phi : \mathfrak{sl}(2) \to \mathfrak{g}$ up to inner automorphism by $G$, where $\mathfrak{g} = \mathrm{Lie}(G)$. Now, by considering the induced maps $\Phi : \mathrm{SL}_2 \to G$ given by these Lie algebra homomorphisms $\phi$ and taking the quotient of $G$ by the image $\Phi(SL_2)$ of the induced map, we obtain a plethora of homogeneous spaces.

Given two simple Lie groups $G$ and $G'$ and two copies of $\Phi(SL_2)$, say $H \subset G$ and $H' \subset G'$ induced from nilpotent orbits of $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}' = \mathrm{Lie}(G')$ respectively, we may ask whether $X = {}^{G}\!/_{H}$ and $X' = {}^{G'}\!/_{H'}$ are homotopy equivalent, that is, if there exists a map $f : X \to X'$ and a map $g : X' \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity map on $X$ and $X'$ respectively. In the case that $G \neq G'$ this question can almost always be answered via a dimension argument alone, since the homogeneous spaces will be compact manifolds of differing dimension each with non-trivial cohomology in top degree. One interesting family of exceptions is when $G = B_n$ and $G' = C_n$. In fact, the focus of this report will be when $n = 6$, as here we have $\dim(B_6) = \dim(C_6) = \dim(E_6) = 78$, where $E_6$ is the rank 6 exceptional Lie group.

We begin our task of comparing homotopy types of these homogeneous spaces by computing the rational homotopy groups, this will only have a chance of being successful in the case $G \neq G'$, since the rational homotopy groups of the homogeneous space will not depend on the chosen non-zero nilpotent orbit. Next, we will compute some of the lower homotopy groups (with coefficients in $\mathbb{Z}$), and it is here that we make use of the so-called Dynkin index. These lower homotopy groups along with the Dynkin index will allow us to determine that the majority of homogeneous spaces are not homotopy equivalent to any other.

However, for a small number of spaces the question of whether they are homotopy equivalent to any other will still not be answered, and we will find ourselves solely in the case of comparing the homotopy types of homogeneous spaces where the simple Lie group is equal, that is, comparing homogeneous spaces where $G = G'$. To do this we will look to utilise the K-theory of each of the spaces, and use the fact that two spaces with differing K-theory cannot be homotopy equivalent. This allows us to show that in all but three cases, different homogeneous spaces of $G = B_6, C_6$ or $E_6$ are not homotopy equivalent.

Homogeneous spaces have been the subject of study by many mathematicians, dating back as far as 1929 when Cartan developed the theory of (relative) Lie algebra cohomology as a tool to study these spaces. In 1958, Bott and Samuelson studied a more specific family of spaces called Symmetric spaces in their paper, "Applications of the theory of Morse to Symmetric spaces", [2]. In 1974 Aloff and Wallach studied a particular family of spaces, namely the 7−manifolds $SU(3)/_{T^1}$, where $T^1$ is a closed connected 1 dimensional subgroup of $SU(3)$ in [1]. These spaces have since been named "Aloff-Wallach" spaces. Shortly after, although completely unrelated, in 1975, Minami published his paper "K groups of symmetric spaces I", [14], which successfully calculated the K groups of symmetric spaces via use of a spectral sequence, which we will also make use of.

# 2 The theory of nilpotent orbits and the motivating example

In this section we collect results on the theory of nilpotent orbits in (semi-)simple Lie algebras. This theory will be a key component in producing the homogeneous spaces we wish to compare. We will then give the motivating example of nilpotent orbits in type $A_n = \mathfrak{sl}_{n+1}$. This section follows [3], particularly chapters 3.5 and 8.4.

Throughout this section $G$ is a simple Lie group with simple Lie algebra $\mathfrak{g}$. We let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ denote the adjoint representation of $\mathfrak{g}$ and $\mathrm{Ad} : G \to GL(\mathfrak{g})$ denote the adjoint representation of $G$. We also let $\mathfrak{sl}(2, \mathbb{C})$ have standard basis $\{h, x, y\}$ satisfying Lie bracket relations $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$.

**Definition 2.1.** Let $X$ be a linear operator, then we say $X$ is nilpotent if $X^n = 0$ for some $n \in \mathbb{N}$.

**Remark 2.2.** It is well known by the preservation of the Jordan decomposition (see [10, Chapter 6.4]) that $X \in \mathfrak{g}$ is nilpotent if and only if $\phi(X)$ is a nilpotent operator as an element of $\mathfrak{gl}(V)$ for any Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. In particular when $\phi = \mathrm{ad}$, we have that $X \in \mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(X)^n = 0$ for some $n \in \mathbb{N}$.

As $\mathfrak{g}$ is the Lie algebra of $G$, there exists a natural action of $G_{\mathrm{ad}}$ on $\mathfrak{g}$ given by the adjoint representation of $G$. Thus for any element $X \in \mathfrak{g}$ we may define the orbit $\mathcal{O}_X$ of this element with respect to the action of $G_{ad}$, that is $\mathcal{O}_X = \left\{X' \in \mathfrak{g} : gXg^{-1} = X' \text{ for some } g \in G\right\}$. In particular, when $X$ is nilpotent we say that $\mathcal{O}_X$ is a nilpotent orbit.

**Definition 2.3.** We say that $\{H, X, Y\} \subset \mathfrak{g}$ is a standard triple in $\mathfrak{g}$ if the relations $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$ are satisfied.

Observe that a standard triple $\{H, X, Y\} \subset \mathfrak{g}$ is equivalent to a non-zero Lie algebra homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$, given by $\phi(x) = X, \phi(y) = Y$ and $\phi(h) = H$, letting $\mathrm{Hom}^{\times}(\mathfrak{sl}_2, \mathfrak{g})$ denote the set of non-zero homomorphisms from $\mathfrak{sl}_2$ to $\mathfrak{g}$ we see that we have a bijection

$$\left\{\text{standard triples in } \mathfrak{g}\right\} \longleftrightarrow \mathrm{Hom}^{\times}(\mathfrak{sl}_2, \mathfrak{g}).$$

One sees that both of these sets are invariant under the obvious action of $G_{\mathrm{ad}}$, and to this end we define the following set:

$$\begin{aligned} \mathcal{A}_{\text{triple}} &= \left\{G_{\mathrm{ad}}\text{-conjugacy classes of standard triples in } \mathfrak{g}\right\} \\ &= \left\{G_{\mathrm{ad}}\text{-conjugacy classes in } \mathrm{Hom}^{\times}(\mathfrak{sl}_2, \mathfrak{g})\right\}. \end{aligned}$$

We now define the following map, and this leads to the crucial theorem.

$$\Omega: \quad \mathcal{A}_{\text{triple}} \longrightarrow \big\{\text{non-zero nilpotent orbits in } \mathfrak{g}\big\}; \quad \Omega\big(\{H,X,Y\}\big) \mapsto \mathcal{O}_X.$$

**Theorem 2.4** (Theorem 3.2.10, [3])**.** *The map $\Omega$ is a one-to-one correspondence between the set $\mathcal{A}_{triple}$ and the set of non-zero nilpotent orbits of $\mathfrak{g}$. In particular we have a bijection of sets:*

$$\big\{\textit{non-zero nilpotent orbits in } \mathfrak{g}\big\} \longleftrightarrow \big\{G_{ad}\textit{-conjugacy classes in } \mathrm{Hom}^{\times}(\mathfrak{sl}_2, \mathfrak{g})\big\}$$

**Definition 2.5.** For each $\mathfrak{sl}_2$ triple $\{H,X,Y\} \subset \mathfrak{g}$, let $\mathcal{O}_H = \{H' \in \mathfrak{g} : gHg^{-1} = H' \text{ for some } g \in G\}$. Then $\mathcal{O}_H$ is called a distinguished semi-simple orbit.

**Corollary 2.6.** *There exists a one-to-one correspondence between the nilpotent orbits and the distinguished semi-simple orbits of $\mathfrak{g}$.*

By the above corollary, each nilpotent orbit $\mathcal{O}_X$ comes with an associated semi-simple orbit $\mathcal{O}_H$. After choosing a canonical element $\tilde{H} \in \mathcal{O}_H$, which we will discuss in the specific cases of interest later, we may label the Dynkin diagram of $\mathfrak{g}$ by assigning the value $\alpha(\tilde{H})$ to each simple root $\alpha$, called a weighted Dynkin diagram. Thus, we may view nilpotent orbits via their weighted Dynkin diagrams.

## 2.1 The motivating example - nilpotent orbits in Lie algebras of type $A_n$.

We now see the above theory in practice through the motivating example of nilpotent orbits in $A_n$. We begin with the following definition:

**Definition 2.7.** Given a positive integer $n$, let $\mathcal{P}(n)$ denote the set of partitions of $n$. For example, $\mathcal{P}(5) = \big\{[5],[4,1],[3,2],[3,1^2],[2^2,1],[2,1^3],[1^5]\big\}$.

For any positive integer $i$, we may construct the $i \times i$ matrix:

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

It's easy to see that $J_i$ is a nilpotent element of $\mathfrak{sl}_i$, now given any partition $[d_1, ..., d_k]$ of $n$, define

$$X_{[d_1,\dots,d_k]} := \begin{pmatrix} J_{d_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & J_{d_2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_{d_{k-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & J_{d_k} \end{pmatrix}.$$

$J_i$ is called the elementary Jordan block of size $i$, we see that $X_{[d_1,\dots,d_k]}$ is a nilpotent element of $\mathfrak{sl}_{d_1+\dots d_k} = \mathfrak{sl}_n$. We have $(\mathrm{SL}_n)_{\mathrm{ad}} = \mathrm{PSL}_n$. With this we define the nilpotent orbit $\mathcal{O}_{[d_1,\dots,d_k]}$ by

$$\mathcal{O}_{[d_1,\dots,d_k]} = \mathrm{PSL}_n \cdot X_{[d_1,\dots,d_k]}.$$

Given two different partitions $[d_1, ..., d_k]$ and $[c_1, ..., c_l]$, observe that there orbits are different by the uniqueness of the Jordan normal form. Moreover, we have that the $\mathrm{GL}_n$, $\mathrm{SL}_n$ and $\mathrm{PSL}_n$ conjugacy classes in $\mathfrak{sl}_n$ coincide (see [3, Chapter 1.2]), therefore if $X \in \mathfrak{sl}_n$ is a nilpotent element then there exists a partition $[d_1, ..., d_k]$ such that $X_{[d_1,\dots,d_k]} \in \mathcal{O}_X$, In this case, we write $\mathcal{O}_X = \mathcal{O}_{[d_1,\dots,d_k]}$. This gives us the very useful result:

**Proposition 2.8.** *[3, Proposition 3.1.7] There is a one-to-one correspondence between the nilpotent orbits of $\mathfrak{sl}_n$ and the set $\mathcal{P}(n)$ of partitions of $n$. The correspondence is given by:*

$$[d_1, ..., d_k] \longleftrightarrow PSL_n \cdot \begin{pmatrix} J_{d_1} & 0 & \dots & 0 & 0 \\ 0 & J_{d_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & J_{d_{k-1}} & 0 \\ 0 & 0 & \dots & 0 & J_{d_k} \end{pmatrix}$$

Having now classified the nilpotent orbits of $\mathfrak{sl}_n$ via partitions of $n$, we now wish to find explicit formula for the $\mathfrak{sl}_2$ triple $\{H_{[d_1,\dots,d_k]}, X_{[d_1,\dots,d_k]}, Y_{[d_1,\dots,d_k]}\}$ associated to the nilpotent element $X_{[d_1,\dots,d_k]}$. Given a fixed positive integer $r$, we define the map $\rho_r : \mathfrak{sl}_2 \to \mathfrak{sl}_{r+1}$ given by:

$$\rho_r(h) = \begin{pmatrix} r & 0 & 0 & \dots & 0 & 0 \\ 0 & r-2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -r+2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -r \end{pmatrix}$$

$$\rho_r(x) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\rho_r(y) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mu_1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \mu_r & 0 \end{pmatrix}$$

where $\mu_i = i(r+1-i)$ for $1 \leq i \leq r$. We have the following standard result.

**Lemma 2.9.** *[3, Lemma 3.2.6] $\rho_r$ is the unique irreducible representation of $\mathfrak{sl}_2$ of dimension $r+1$.*

Observe that for a given partition $[d_1, ..., d_k]$ of $n$, we have

$$X_{[d_1,\dots,d_k]} = \bigoplus_{i=1}^{k} \rho_{d_i - 1}(x)$$

and therefore we have

$$H_{[d_1,\dots,d_k]} = \bigoplus_{i=1}^{k} \rho_{d_i-1}(h), \quad Y_{[d_1,\dots,d_k]} = \bigoplus_{i=1}^{k} \rho_{d_i-1}(y).$$

**Example 2.10.** Let $[2, 2, 1] \in \mathcal{P}(5)$ be a partition of 5, then we have an $\mathfrak{sl}_2-$triple in $\mathfrak{sl}_5$ given by:

$$X_{[2,2,1]} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, H_{[2,2,1]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y_{[2,2,1]} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We conclude this section by discussing how to label the Dynkin diagram for nilpotent orbits of $\mathfrak{sl}_n$. From the proceeding discussion we have that

$$H_{[d_1,\dots,d_k]} = \mathrm{diag}(d_1 - 1, ..., -d_1 + 1, d_2 - 1, ..., -d_2 + 1, ..., d_k - 1, ... - d_k + 1).$$

We now define $\tilde{H}_{[d_1,...,d_k]}$ to be the diagonal matrix with the same entries as $H_{[d_1,...,d_k]}$, but with the entries in rearranged into decreasing order. For example,

$$H_{[2,2,1]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ gives } \tilde{H}_{[2,2,1]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

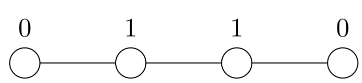
It is very well known that a basis of simple roots for $\mathfrak{sl}_{n+1}$ is given by $\{\alpha_i\}_{i=1}^n$, $\alpha_i = L_i - L_{i+1}$ where

$$L_i\left(\begin{pmatrix} h_1 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & h_n & 0 \\ 0 & 0 & 0 & 0 & h_{n+1} \end{pmatrix}\right) = h_i \text{ for all } i \in \{1, 2, ..., n+1\}.$$

Then for a given $\tilde{H} \in \mathfrak{sl}_{n+1}$ defined as above, we may label the Dynkin diagram of $\mathfrak{sl}_{n+1}$ as follows:


We will denote the weighted Dynkin diagram as above by $\triangle(\alpha_1(\tilde{H}), ..., \alpha_n(\tilde{H}))$, and may write $\mathcal{O}_{\triangle(\alpha_1(\tilde{H}),...,\alpha_n(\tilde{H}))}$ to denote the orbit with this weighted Dynkin diagram. We now see an example of a weighted Dynkin diagram for the partition $[2, 2, 1]$ of 5 i.e. the weighted Dynkin diagram associated to the orbit $\mathcal{O}_{[2,2,1]} = \mathcal{O}_{X_{[2,2,1]}}$

**Example 2.11.** For the partition $[2, 2, 1]$ of 5, we saw that $\tilde{H}_{[2,2,1]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$ and so the weighted Dynkin diagram is




and therefore we have $\mathcal{O}_{[2,2,1]} = \mathcal{O}_{\triangle(0,1,1,0)}$

## 2.2 Nilpotent orbits in Lie algebras $B_6$, $C_6$ and $E_6$ and the associated homogeneous spaces.

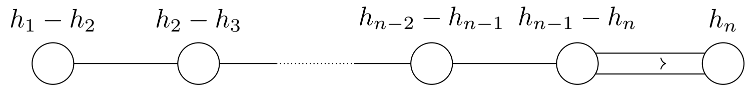
In this section we give a classification of the nilpotent orbits in types $B_6$, $C_6$ and $E_6$. This section closely follows [3, Chatper 5]. We first give the following two results.

**Theorem 2.12** (Theorem 5.1.2 [3])**.** *Nilpotent orbits in $B_n = \mathfrak{so}_{2n+1}$ are in one-to-one correspondence with the set of partitions of $2n+1$ in which even parts occur with even multiplicity.*

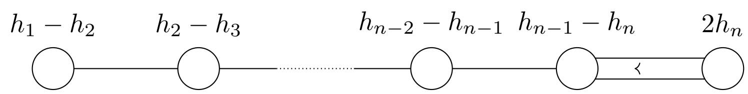
**Theorem 2.13** (Theorem 5.1.3 [3])**.** *Nilpotent orbits in $C_n = \mathfrak{sp}_{2n}$ are in one-to-one correspondence with the set of partitions of $2n$ in which odd parts occur with even multiplicity.*

**Example 2.14.** For $B_2 = \mathfrak{so}_5$ we require partitions of 5 such that even parts appear with even multiplicity. There are exactly four of these, namely $\{[5], [3, 1^2], [2^2, 1], [1^5]\}$.

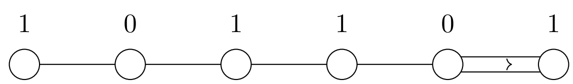
For $B_n$ we obtain the weighted Dynkin diagram as follows; given a partition $\boldsymbol{d} = [d_1, .., d_k]$ of $2n+1$, let $s_i = (d_i - 1, d_i - 3, ..., -d_1 + 3, -d_i - 1)$. Form the list $H$ by concatenating the $s_i$ for each $i \in \{1, ..., k\}$. Now let $\tilde{H}$ be the rearrangement of $H$ such that a 0 comes first, follows by the remaining non-negative terms in decreasing order, followed by their negatives, denote this list $\tilde{H} = (0, h_1, ..., h_n, -h_1, ..., h_n)$, then the weighted Dynkin diagram of the orbit $\mathcal{O}_{\boldsymbol{d}}$ of $B_n$ is




For $C_n$, the process is essentially the same; given a partition $\boldsymbol{d} = [d_1, .., d_k]$ of $2n$, let $s_i = (d_i - 1, d_i - 3, ..., -d_1 + 3, -d_i - 1)$. Form the list $H$ by concatenating the $s_i$ for each $i \in \{1, ..., k\}$. Now let $\tilde{H}$ be the rearrangement of $H$ such that non-negative terms come first, in decreasing order, followed by their negatives, denote this list $\tilde{H} = (h_1, ..., h_n, -h_1, ..., h_n)$, then the weighted Dynkin diagram of the orbit $\mathcal{O}_{\boldsymbol{d}}$ of $B_n$ is




**Example 2.15.** The partition $[5, 4, 4] \in \mathcal{P}(13)$ corresponds to a $B_6$ nilpotent orbit. We have $s_1 = (4, 2, 0, -2, -4)$, $s_2 = s_3 = (3, 1, -1, -3)$, and so $H = (4, 2, 0, -2, -4, 3, 1, -1, -3, 3, 1, -1, -3)$. Rearranging this as described above gives $\tilde{H} = (0, 4, 3, 3, 2, 1, 1, -4, -3, -3, -2, -1, -1)$, therefore the weighted Dynkin diagram is




and we have $\mathcal{O}_{[5,4,4]} = \mathcal{O}_{\triangle(1,0,1,1,0,1)}$.

Nilpotent orbits in $E_6$ do not have such a straightforward classification, it was nevertheless achieved through work of Dynkin, Bala and Carter [3, Chapter 8] via the study of so-called parabolic subalgebras. In particular, $E_6$ orbits do not have a labelling by partitions, but are instead labelled by the isomorphism class of the subalgebra. We merely give the table of nilpotent orbits of $E_6$, which can be found in Appendix A, or alternatively in [3, Chapter 8.4].

Now, we know that any nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ corresponds to a Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$, this induces a Lie group homomorphism $\Phi : SL_2 \to G$ which further produces a homogeneous space ${}^{G}\!/_{SL_2}$. We now begin our attempts to determine the homotopy type of such spaces where $G = B_6, C_6$ and $E_6$.

# 3 The rational homotopy groups

Rational homotopy theory was first discovered by Quillen and Sullivan, what makes the theory of rational homotopy groups so appealing is that the groups are often very easily computed. However, it has the disadvantage of losing some information about the homotopy groups, in particular, it removes all torsion elements.

Given a topological space $X$, we define the rational homotopy groups to be $\pi_*(X, \mathbb{Q}) = \pi(X) \otimes \mathbb{Q}$. One immediately observes that for any two topological spaces $X, Y$ we have

$$\pi_*(X, \mathbb{Q}) \not\cong \pi_*(Y, \mathbb{Q}) \implies \pi_*(X) \not\cong \pi_*(Y).$$

Note that the converse need not hold. With this in mind, we show that for any nilpotent orbits of $G = B_6$ or $C_6$ we have ${}^{G}\!/_{SL_2} \not\cong {}^{E_6}\!/_{SL_2}$ by showing that the rational homotopy groups are different. To do this we must first discuss the theory, which may be found in [5], particularly Chapter 12, and the article of Reeder titled "On the cohomology of compact Lie groups", see [17]. Given a smooth manifold $M$ we may consider the functor

$$A_{\mathrm{dR}} : \text{smooth manifolds} \to \text{commutative cochain algebras}$$

taking a smooth manifold to it's cochain complex of differential forms with differential, $d$, given by the exterior derivative. i.e. letting $\Omega^i(M)$ denote the space of $i-$forms on $M$ we have

$$A_{\mathrm{dR}}(M) = 0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \ldots$$

We now introduce a special type of commutative cochain algebra known as a Sullivan algebra.

**Definition 3.1.** A Sullivan algebra is a commutative cochain algebra of the form $(\bigwedge V, d)$ where

- $V = \{V^p\}_{p\geq 1}$, that is $V$ is positively graded, and where $\bigwedge V$ denotes the free graded commutative algebra on $V$.
- $V = \bigcup_{k=0}^{\infty} V(k)$ where $V(0) \subset V(1) \subset \ldots$ is an increasing sequence of graded subspaces such that
  - $d = 0$ in $V(0)$
  - $d\big(V(k)\big) \subset \bigwedge V(k-1)$ for all $k \geq 1$

**Definition 3.2.** Let $(A, d)$ be a commutative algebra, then

- A Sullivan model for $(A, d)$ is a quasi-isomorphism (that is, a map inducing an isomorphism on (co)homology groups) $m : (\bigwedge V, d) \to (A, d)$ from a Sullivan algebra $(\bigwedge V, d)$
- If $M$ is a smooth manifold then a Sullivan model for the commutative algebra $A_{dR}(M)$ is called a Sullivan model for $M$.
- The model is called minimal if $\mathrm{im}(d) \subset \bigwedge^+ V \cdot \bigwedge^+ V$

We now give two examples and a theorem which will play a key part in determining the rational homotopy groups of our spaces.

**Example 3.3** (Minimal Sullivan model for odd dimensional spheres)**.** Let $k$ be odd, we have

$$H^*(S^k, \mathbb{R}) = H^*\big(A_{\mathrm{dR}}(S^k)\big) = \begin{cases} \mathbb{R} & \text{if } * = 0, k \\ 0 & \text{else.} \end{cases}$$

Let $\phi \in H^k\big(A_{\mathrm{dR}}(S^k)\big)$ be a generator, and let $\omega \in \big(A_{\mathrm{dR}}(S^k)\big)$ be a cochain representative of $\phi$, that is $\phi = [\omega]$, where $[-]$ denotes the class of a cochain in cohomology. Then

$$m : \Big(\bigwedge\{e\}, 0\Big) \longrightarrow A_{\mathrm{dR}}(S^k)$$

where $\deg(e) = k$ and $m(e) = \omega$ is a Sullivan model for $S^k$. Note, $\mathrm{im}(d) = \mathrm{im}(0) = 0 = e \wedge e \in \bigwedge^+ V \cdot \bigwedge^+ V$ so the model is minimal. We remark that via the use of a slightly modified functor (of which details are omitted) we may obtain an almost completely analogous minimal Sullivan model for $S^k$ in which the homotopy groups are over $\mathbb{Q}$, as opposed to $\mathbb{R}$.

**Example 3.4.** Let $(\bigwedge V, d)$ be a minimal Sullivan model for a path connected space $X$ and $(\bigwedge W, d)$ be a minimal Sullivan model for a path connected space $Y$, then $(\bigwedge V, d) \otimes (\bigwedge W, d)$ is a minimal Sullivan model for $X \times Y$.

We now turn our attention to determining minimal Sullivan models for compact connected Lie groups. We begin with a definition, along with a couple of results from which the desired Sullivan models will follow.

**Definition 3.5.** A H-space is a topological space $X$, along with an element $e \in X$ and a map $\mu : X \times X \to X$ such that $\mu(e, e) = e$ and the maps $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity map $x \mapsto x$ through maps sending $e$ to $e$. In particular, a Lie group is a H-space.

**Lemma 3.6.** *[5, Chapter 12, Example 3] Let $X$ be a $H$-space, then $X$ has minimal Sullivan model of the form $(\bigwedge V, 0)$.*

**Theorem 3.7.** *[6, Theorem 3C.4][H. Hopf] Let $G$ be a compact connected Lie group. Then $G$ has the rational cohomology of a product of odd dimensional spheres, that is, $H^*(G,\mathbb{Q}) = H^*\left(\prod_i S^{2d_i-1}, \mathbb{Q}\right)$.*

As promised, we obtain the following description of minimal Sullivan models of compact connected Lie groups.

**Corollary 3.8.** *$G$ has minimal Sullivan model $\left(\bigwedge\{e_{2d_i-1}\}, 0\right)$ where $deg(e_{2d_i-1}) = 2d_i - 1$.*

*Proof.* We know by the lemma that $G$ has minimal Sullivan model $(\bigwedge V, 0)$ for some graded vector space $V$, so it remains to determine what $V$ is. To begin, it is a standard result that

$$H^k\left(\prod_i^n S^{2d_i-1}, \mathbb{Q}\right) = \left\{ I \subset \{2d_1 - 1, ..., 2d_n - 1\} : \sum_{i\in I} 2d_i - 1 = k \right\}.$$

Now, one observes that $\left(\bigwedge\{e_{2d_i-1}\}, 0\right)$ shares the same cohomology, thus $V$ must be $\bigwedge\{e_{2d_i-1}\}$. ☐

In fact, a stronger statement holds, namely that $G$ has the same rational homotopy type as a product of spheres, this is a well know result, and in our case is easiest to see as a consequence of the following result.

**Proposition 3.9.** *[5, Chapter 17] We have a bijection*

$$\{\text{rational homotopy types}\} \xleftrightarrow{\cong} \{\text{isomorphism classes of minimal Sullivan algebras over } \mathbb{Q}\}.$$

*In particular, $G$ and $\prod_i S^{2d_i-1}$ are rationally homotopy equivalent.*

*Proof.* For the in particular part, observe that for $k, l$ odd we have $\left(\bigwedge e_k, 0\right) \otimes \left(\bigwedge e_l, 0\right) \cong \left(\bigwedge(e_k, e_l), 0\right)$, thus $G$ and $\prod_i S^{2d_i-1}$ have isomorphic minimal Sullivan models. ☐

This immediately gives us the rational homotopy groups of $G$.

$$\pi_*(G,\mathbb{Q}) = \pi_*\left(\prod_i S^{2d_i-1}, \mathbb{Q}\right) = \prod_i \pi_*\left(S^{2d_i-1}, \mathbb{Q}\right) = \begin{cases} \mathbb{Q} & \text{if } * = 2d_i - 1 \\ 0 & \text{else.} \end{cases}$$

We now wish to determine the $d_i$ for $G = B_6, C_6$ and $E_6$. We first recall some definitions from Lie theory. Let $G$ be a Lie group of rank $l$, with maximal torus $T$. Then $T$ is acted on by the Weyl group $W = {N(T)}\big/{Z(T)}$ via conjugation, where $N(T)$ and $Z(T)$ denote the normaliser and centraliser of $T$, respectively. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{t} = \mathrm{Lie}(T)$ be the Lie algebras of $G$ and $T$, and let $S = S(\mathfrak{t}^*)$ be the symmetric algebra on $\mathfrak{t}^*$, then there is a natural action of $W$ on $S$ induced from the action of $W$ on $T$.

**Example 3.10.** Let $G = SL_n(\mathbb{C})$, then $G$ has maximal torus $T = \left\{ \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & x_n \end{pmatrix} : x_1 x_2 ... x_n = 1 \right\}$ and $W \cong S_n$ acts by permuting the $x_i$. That is, for $\sigma \in S_n, \sigma \cdot x_i = x_{\sigma(i)}$. Next, we have Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{t} = \left\{ \begin{pmatrix} h_1 & 0 & \dots & 0 \\ 0 & h_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & h_n \end{pmatrix} : h_1 + h_2 + ... + h_n = 0 \right\}$. We have that $W$ acts on $\mathfrak{t}$ by permuting the $h_i$. Next, define linear functionals $\varepsilon_i \in \mathfrak{t}^*$ by $\varepsilon_i \cdot \mathrm{diag}(h_1, ..., h_n) = h_i$, note that we have the relation $\varepsilon_1 + \varepsilon_2 + ... + \varepsilon_n = 0$. Observe that for $\sigma \in W$, $\sigma$ acts on $\varepsilon_i$ by $\sigma \cdot \varepsilon_i = \varepsilon_{\sigma^{-1}(i)}$.

Write $S = S(\mathfrak{t}^*) = {\mathbb{C}[\varepsilon_1, ..., \varepsilon_n]}\big/{(\varepsilon_1 + ... + \varepsilon_n)}$, then we have $\sigma \cdot \sum_i c_i\, \varepsilon_1^{i_1} ... \varepsilon_n^{i_n} = \sum_i c_i\, \varepsilon_{\sigma^{-1}(1)}^{i_1} ... \varepsilon_{\sigma^{-1}(n)}^{i_n}$.

We now have the following result:

**Theorem 3.11.** *[17] Let $G$ be a compact connected Lie group, then the cohomology ring $H^*(G,\mathbb{Q})$ has generators in degrees $2d_i - 1$, where $d_i$ are the degrees of the fundamental invariant polynomials of $S(\mathfrak{t}^*)^W$, the space of polynomials in $S(\mathfrak{t}^*)$ which are invariant under the action of $W$.*

**Example 3.12.** Continuing from our previous example, where $S = {\mathbb{C}[\varepsilon_1, ..., \varepsilon_n]}/{(\varepsilon_1 + ... + \varepsilon_n)}$, we have fundamental invariant polynomials given by

$$F_k = \varepsilon_1^k + ... + \varepsilon_n^k \text{ for } 2 \geq k \geq n.$$

These can be readily checked to be $W-$invariant. To check that they form a complete set of fundamental invariant polynomials takes more work, but nonetheless is true, as can be seen in [9, Chapter 3.9]. Thus we have

$$\pi_*(SL_n(\mathbb{C}), \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 3, 5, 7, ..., 2n-1 \\ 0 & \text{else.} \end{cases}$$

The degrees of the fundamental polynomials for $G = B_6, C_6$ and $E_6$ are given as follows [9, Table 1, Chapter 3.7].

| Type | Degree of fundamental invariant polynomials |
|---|---|
| $B_6$ | $2, 4, 6, 8, 10, 12$ |
| $C_6$ | $2, 4, 6, 8, 10, 12$ |
| $E_6$ | $2, 5, 6, 8, 9, 12$ |

Thus we have $\pi_*(B_6, \mathbb{Q}) = \pi_*(C_6, \mathbb{Q}) \neq \pi_*(E_6, \mathbb{Q})$. Finally, observe from our example we have

$$\pi_*(\mathrm{SL}_2(\mathbb{C}), \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 3 \\ 0 & \text{else.} \end{cases}$$

and so we conclude this section with the following result.

**Proposition 3.13.** *For any nilpotent orbit of $G = B_6, C_6$ or $E_6$ with induced copy of $SL_2(\mathbb{C}) \subset G$ we have:*

$$\pi_*\left({B_6}/{SL_2(\mathbb{C})}, \mathbb{Q}\right) = \pi_*\left({C_6}/{SL_2(\mathbb{C})}, \mathbb{Q}\right) = \begin{cases} \mathbb{Q} & \text{if } * = 7, 11, 15, 19, 23 \\ 0 & \text{else.} \end{cases}$$

$$\pi_*\left({E_6}/{SL_2(\mathbb{C})}, \mathbb{Q}\right) = \begin{cases} \mathbb{Q} & \text{if } * = 9, 11, 15, 17, 23 \\ 0 & \text{else.} \end{cases}$$

*Therefore, for any induced copy of $H = SL_2(\mathbb{C})$ in $B_6$ or $C_6$, and any induced copy of $K = SL_2(\mathbb{C})$ in $E_6$, we have*

$$ {B_6}/{H} \not\simeq {E_6}/{K} \text{ or } {C_6}/{H} \not\simeq {E_6}/{K}$$

*where $\simeq$ denotes the homotopy equivalence of two spaces.*

*Proof.* The map $\Phi : SL_2(\mathbb{C}) \to G$ induces a non-zero map $\Phi_* : \pi_3(SL_2(\mathbb{C})) \to \pi_3(G)$, which gives an isomorphism between $\pi_3(SL_2(\mathbb{C}), \mathbb{Q})$ and $\pi_3(G, \mathbb{Q})$. The result now follows from the corresponding long exact sequence. ☐

The computation of the rational homotopy groups has enabled us to distinguish between homogeneous spaces of $E_6$ and homogeneous spaces of $B_6$ or $C_6$. However, it does not help us compare homogeneous spaces of $B_6$ and $C_6$, or indeed compare two homogeneous spaces arising from nilpotent orbits of the same $G$. Thus, in the next section we compute some lower homotopy groups (with coefficients in $\mathbb{Z}$) of our homogeneous spaces in hopes of finding another invariant.

# 4 Homotopy groups and the Dynkin index

In this section, we compute some of the lower homotopy groups for homogeneous spaces ${}^{G}\!/\!_{SL_2}$ where $G = B_6, C_6$ or $E_6$. We will find a purely algebraic way to compute the third homotopy group of our homogeneous spaces, which will turn out to depend on the nilpotent orbit.

## 4.1 First computations of the homotopy groups of homogeneous spaces

To begin, we recall some definitions and results from algebraic topology.

**Definition 4.1** (Exact sequences)**.** Given $A, B, C$ groups/rings/modules over a ring and $f : A \to B$ and $g : B \to C$ morphisms we may form the sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

We say this sequence is (short) exact if $\mathrm{im}(f) = \ker(g)$, $f$ is injective and $g$ is surjective. Furthermore, given objects $A_1, A_2, ..., A_n$ and morphisms $f_0 : 0 \to A_1, f_1, f_2, ..., f_{n-1}, f_n : A_n \to 0$ between them we say the sequence

$$0 \xrightarrow{f_0=0} A_1 \xrightarrow{f_1} A_2 \to \cdots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n=0} 0$$

is (long) exact if $\mathrm{im}(f_i) = \ker(f_{i+1})$ for $0 \leq i \leq n-1$.

**Example 4.2.** The following are examples of short exact sequences:

- $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\overline{1}} \mathbb{Z}_n \to 0$ for $n \geq 1$ is exact, since $\ker(\overline{1}) = \mathrm{im}(n) = n\mathbb{Z}$.
- Let $G$ be a Lie group and $H \subset G$ a closed Lie subgroup. Then:

$$0 \to H \to G \to {}^{G}\!/\!_{H} \to 0$$

  is exact, where the maps are the obvious ones.

We now state a result on obtaining a long exact sequence of homotopy groups from certain short exact sequences of spaces.

**Lemma 4.3.** *Let $p : E \to B$ be a (Serre) fibration with $B$ path connected and fiber $F$. Then we have a long exact sequence of homotopy groups*

$$\cdots \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(E) \to \pi_{i-1}(B) \to \cdots \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0$$

*where we note that the maps on $\pi_0$ are not group homomorphisms, but are exact in the sense that the relevant kernels and images are equal.*

**Remark 4.4.** We merely remark that for $G$ a Lie group and $H \subset G$ a Lie subgroup, $p : G \to {}^{G}\!/\!_{H}$ with fiber $H$ is a fibration. Thus for $H$ connected[1] the conditions on the above lemma hold, in particular for $H = \mathrm{SL}_2(\mathbb{C})$ the conditions hold.

Finally, before we calculate homotopy groups of our homogeneous spaces, we give a table of the lower homotopy groups of $B_6$,$C_6$ and $E_6$. The fact that $\pi_2(G) = 0$ and $\pi_3(G) = \mathbb{Z}$ are standard results. Also $\pi_*(\mathrm{SL}_2)$, $\pi_*(B_6)$ and $\pi_*(C_6)$ are fairly standard, using that $\pi_*(\mathrm{SL}_2) = \pi_*(S^3)$, $B_6 = \mathrm{SO}(13)$ and $C_6 = \mathrm{Sp}(2n)$. Finally, the groups $\pi_*(E_6)$ can be verified in [2, Chapter III.2, Theorem V].

We also note that $\mathrm{SL}_2(\mathbb{C})$ is simply connected, thus $\pi_1(\mathrm{SL}_2(\mathbb{C})) = 0$. We now give an example of calculating some homotopy groups of our homogeneous spaces ${}^{G}\!/\!_{SL_2}$ before giving a table of results.

[1] For Lie groups, connected and path connected are equivalent.

| | $\mathrm{SL}_2(\mathbb{C})$ | $B_6$ | $C_6$ | $E_6$ |
|---|---|---|---|---|
| $\pi_2$ | 0 | 0 | 0 | 0 |
| $\pi_3$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $\pi_4$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | 0 |
| $\pi_5$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | 0 |
| $\pi_6$ | $\mathbb{Z}_{12}$ | 0 | 0 | 0 |
| $\pi_7$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |

**Example 4.5.** Let $G = B_6$, then the projection $p : B_6 \to B_6/SL_2$ with fiber $SL_2(\mathbb{C})$ gives a long exact sequence

$$\cdots \to \pi_7(\mathrm{SL}_2(\mathbb{C})) \to \pi_7(B_6) \to \pi_7\left(B_6/SL_2\right) \to \pi_6(\mathrm{SL}_2(\mathbb{C})) \to \cdots \to \pi_2(B_6) \to \pi_2\left(B_6/SL_2\right) \to \ldots$$

In particular we have:

- $\overbrace{0}^{\pi_2(\mathrm{SL}_2(\mathbb{C}))} \longrightarrow \pi_2(B_6) \longrightarrow \pi_2\left(B_6/SL_2\right) \longrightarrow \overbrace{0}^{\pi_1(\mathrm{SL}_2(\mathbb{C}))} \implies \pi_2\left(B_6/SL_2\right) = \pi_2(B_6) = 0$
- $\cdots \longrightarrow \overbrace{\mathbb{Z}}^{\pi_3(\mathrm{SL}_2(\mathbb{C}))} \xrightarrow{d} \overbrace{\mathbb{Z}}^{\pi_3(B_6)} \longrightarrow \pi_3\left(B_6/SL_2\right) \longrightarrow \overbrace{0}^{\pi_2(\mathrm{SL}_2(\mathbb{C}))} \implies \pi_3\left(B_6/SL_2\right) = \mathbb{Z}/d\mathbb{Z}$
  where $d : \mathbb{Z} \to \mathbb{Z}$ will be discussed shortly.

It is a routine exercise involving short exact sequences to verify the following table:

| | $B_6/\mathrm{SL}_2$ | $C_6/\mathrm{SL}_2$ | $E_6/\mathrm{SL}_2$ |
|---|---|---|---|
| $\pi_2$ | 0 | 0 | 0 |
| $\pi_3$ | $\mathbb{Z}/d\mathbb{Z}$ | $\mathbb{Z}/d\mathbb{Z}$ | $\mathbb{Z}/d\mathbb{Z}$ |
| $\pi_4$ | 0 | $\begin{cases} \mathbb{Z}_2 & \text{if } b_4 = 0 \\ 0 & \text{if } b_4 = 1 \end{cases}$ | 0 |
| $\pi_5$ | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ or } \mathbb{Z}_4 & \text{if } (b_4, b_5) = (1,1) \\ \mathbb{Z}_2 & \text{if } (b_4, b_5) = (1,0) \text{ or } (0,1) \\ 0 & \text{if } (b_4, b_5) = (0,0) \end{cases}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $\pi_6$ | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{cases} \mathbb{Z}_2 & \text{if } b_5 = 0 \\ 0 & \text{if } b_5 = 1 \end{cases}$ | $\mathbb{Z}/2\mathbb{Z}$ |

where the values $d$, $b_4$ and $b_5$ are given as follows:

- $d$ represents the map $d : \mathbb{Z} \to \mathbb{Z}$ from $\pi_3(\mathrm{SL}_2(\mathbb{C})) = \mathbb{Z}$ to $\pi_3(G) = \mathbb{Z}$ for each of $G = B_6$, $C_6$ and $E_6$.
- $b_4$ represents the map $b_4 : \mathbb{Z}_2 \to \mathbb{Z}_2$ from $\pi_4(\mathrm{SL}_2(\mathbb{C})) = \mathbb{Z}_2$ to $\pi_4(C_6) = \mathbb{Z}_2$.
- $b_5$ represents the map $b_5 : \mathbb{Z}_2 \to \mathbb{Z}_2$ from $\pi_5(\mathrm{SL}_2(\mathbb{C})) = \mathbb{Z}_2$ to $\pi_5(C_6) = \mathbb{Z}_2$.

Our first point of investigation will be the map $d : \mathbb{Z} \to \mathbb{Z}$, and as we will soon see this map does depend on which nilpotent orbit we choose. Therefore this will provide us with our first tool to compare homogeneous spaces internally, that is, compare homogeneous spaces of the same Lie group $G$. We will then investigate $b_4$ and $b_5$, as this will give us a way to distinguish between the spaces $B_6/SL_2(\mathbb{C})$ and $C_6/SL_2(\mathbb{C})$.

## 4.2 The Dynkin index

The value of $d : \mathbb{Z} \to \mathbb{Z}$ turns out to be given by the so-called Dynkin index of a $\mathfrak{sl}_2(\mathbb{C})$ subalgebra in a simple lie algebra $\mathfrak{g}$. We begin by discussing what the Dynkin index is how to compute it, before discussing how it is related to the homotopy groups of our homogeneous spaces. This sections closely follows [16], as well as [15].

Let $\mathfrak{g}$ be a simple finite dimensional algebra of rank $n$, and let $\mathfrak{h}$ be a Cartan subalgebra with set of roots $\triangle \subset \mathfrak{h}^*$. Choose a set of positive roots $\triangle^+$ and a set of simple roots $\Pi$. We normalise a non-degenerate $\mathfrak{g}-$invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ by requiring that $(\beta,\beta) = 2$ for any long root $\beta \in \triangle$ where, by abuse of notation, $(\cdot,\cdot)$ denotes the induced non-degenerate bilinear form on $\mathfrak{h}^*$. We denote the normalised bi-linear form on $\mathfrak{g}$ by $(\cdot,\cdot)_{\mathfrak{g}}$.

**Example 4.6.** Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Then $\mathfrak{h} = \{\mathrm{diag}(h_1,h_2,h_3) : h_1+h_2+h_3 = 0\}$ and $\Pi = \{\alpha_1,\alpha_2\}$ with $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$ where we define $\varepsilon_i \in \mathfrak{h}^*$ by

$$\varepsilon_i \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} = h_i.$$

A $\mathfrak{g}-$invariant non-degenerate bilinear form on $\mathfrak{g}$ gives an isomorphism from $\mathfrak{h}$ to $\mathfrak{h}^*$. As we know, all roots in $\triangle \subset \mathfrak{h}^*$ have the same length. Thus we require $(\alpha_1,\alpha_1) = 2$. This is equivalent, due to the isomorphism $\varepsilon_i \longleftrightarrow h_i$ to requiring that $\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 2$, thus we see that $(X,X)_{\mathfrak{g}} = \mathrm{tr}(X^2)$.

**Definition 4.7.** Let $\phi : \mathfrak{s} \to \mathfrak{g}$ be a homomorphism of simple Lie algebras. For any $x, y \in \mathfrak{s}$, the bilinear form $(x,y)to(\phi(x),\phi(x))_{\mathfrak{g}}$ is proportional to $(x,y)_{\mathfrak{s}}$, and the index of $\phi$ in $\mathfrak{g}$ is defined as

$$\mathrm{ind}(\mathfrak{s} \xrightarrow{\phi} \mathfrak{g}) = \frac{(\phi(x),\phi(y))_{\mathfrak{g}}}{(x,y)_{\mathfrak{s}}}.$$

In particular, when $\mathfrak{s} \subset \mathfrak{g}$ is a subalgebra, we say the index of $\mathfrak{s}$ in $\mathfrak{g}$ is

$$\mathrm{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g}) = \frac{(x,y)_{\mathfrak{g}}}{(x,y)_{\mathfrak{s}}}.$$

Finally, if $(,V)$ is a representation of $\mathfrak{g}$ then we define the index of the representation to be

$$\mathrm{ind}(\mathfrak{g},V) = \mathrm{ind}(\mathfrak{g} \xrightarrow{\rho} \mathfrak{sl}(V)) = \frac{\mathrm{tr}(\rho(x)^2)}{(x,x)_{\mathfrak{g}}}$$

The Dynkin index enjoys the following two nice properties:

- Multiplicativity: If $\mathfrak{s} \subset \mathfrak{h} \subset \mathfrak{g}$ are simple Lie algebras then we have

$$\mathrm{ind}(\mathfrak{s} \hookrightarrow \mathfrak{h}) \cdot \mathrm{ind}(\mathfrak{h} \hookrightarrow \mathfrak{h}) = \mathrm{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g}).$$

- Additivity: If $V_1$, $V_2$ are two representations of $\mathfrak{g}$ then we have

$$\mathrm{ind}(\mathfrak{g}, V_1 \oplus V_2) = \mathrm{ind}(\mathfrak{g},V_1) + \mathrm{ind}(\mathfrak{g},V_2)$$

Before stating how one computes the index for $\mathfrak{sl}_2$ subalgebras of $\mathfrak{g}$, we state the following very useful consequence of the multiplicative property. Let $\mathfrak{s} \subset \mathfrak{g}$ be a subalgebra, and let $V$ be a representation of $\mathfrak{g}$, then we have

$$\mathrm{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g}) = \frac{\mathrm{ind}(\mathfrak{s},V)}{\mathrm{ind}(\mathfrak{g},V)}.$$

Recall from Section 1 that $\mathfrak{sl}_2$ subalgebras in $\mathfrak{g}$, up to conjugation by $G$, are in one-to-one correspondence with nilpotent orbits in $\mathfrak{g}$. Moreover, for the classical Lie algebras these nilpotent orbits were classified by partitions. Following [16], for a partition $\boldsymbol{d} = [d_1,...,d_k]$, let $A_1(\boldsymbol{d})$ denote the corresponding $G_{\mathrm{ad}}-$orbit of the $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$. We have the following result:

**Theorem 4.8.** *[16, Theorem 2.1] Let $A_1(\boldsymbol{d}) \in \mathfrak{g}$ be an $\mathfrak{sl}_2$-subalgebra with associated partition $\boldsymbol{d} = [d_1, ..., d_k]$, then we have*

- $\operatorname{ind}\big(A_1(\boldsymbol{d}) \hookrightarrow \mathfrak{sp}\big) = \displaystyle\sum_{i=1}^{k} \binom{d_i + 1}{3}$
- $\operatorname{ind}\big(A_1(\boldsymbol{d}) \hookrightarrow \mathfrak{so}\big) = \displaystyle\frac{1}{2}\sum_{i=1}^{k} \binom{d_i + 1}{3}$

For the exceptional Lie algebras, and in particular for $\mathfrak{e}_6$, the Jordan normal form of each nilpotent element $X$ is determined in [12, Table 5]. Using this, we may assign a partition to each $\mathfrak{sl}_2$-subaglebra of $\mathfrak{e}_6$. Lastly, we have $\operatorname{ind}(\mathfrak{e}_6 \to \mathfrak{so}(27)) = 6$, and this along with the previous theorem allows us to compute the indices of the $\mathfrak{sl}_2$-subalgebras of $\mathfrak{e}_6$. The indices of all $\mathfrak{sl}_2$ subalgebras of $\mathfrak{b}_6$, $\mathfrak{c}_6$ and $\mathfrak{e}_6$ may be found in Appendix A.

The following remarkable theorem relates the Dynkin index of a subalgebra to the third homotopy group of the corresponding homogeneous space:

**Theorem 4.9.** *[15] Let $G$ and $H$ be connected simple compact Lie groups with $H \subset G$ and suppose $d = \operatorname{ind}(\mathfrak{h} \hookrightarrow \mathfrak{g})$ is the index of $\mathfrak{h}$ in $\mathfrak{g}$, where $\mathfrak{g} = \operatorname{Lie}(G)$ and $\mathfrak{h} = \operatorname{Lie}(H)$, then we have*

$$\pi_3\left({}^{G}\!/\!_{H}\right) = {}^{\mathbb{Z}}\!/\!_{d\mathbb{Z}}.$$

Before applying the above theorem to our cases of interest, we first investigate the maps $b_4$ and $b_5$ from the table of homotopy groups to conclude that for any $H = \mathrm{SL}_2 \subset B_6$ and $H' = \mathrm{SL}_2 \subset C_6$ we have ${}^{B_6}\!/\!_{H} \not\simeq {}^{C_6}\!/\!_{H'}$.

Let $f_{\mathrm{prin}} : \mathrm{SL}_2(\mathbb{C}) \to C_6$ be the induced Lie group homomorphism given by the inclusion $\mathfrak{sl}_2 \subset \mathfrak{b}_6$ with index 1, known as the principle map. Also define $m_n : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ by $m_n(A) = A^n$ for all $A \in \mathrm{SL}_2(\mathbb{C})$. Note that $m_n$ is not a group homomorphism, but it is a perfectly good map of topological spaces. We have the following result

**Proposition 4.10.** *Let $\Phi : SL_2(\mathbb{C}) \to C_6$ be the induced Lie group homomorphism given by an inclusion $\mathfrak{sl}_2 \subset \mathfrak{c}_6$ with index $n$, then we have homotopic maps (of topological spaces) $\Phi \simeq f_{prin} \circ m_n$, that is*

$$\pi_*(\Phi) = \pi_*(f_{prin} \circ m_n) = \pi_*(f_{prin}) \circ \pi_*(m_n) = n \cdot \pi_*(f_{prin}).$$

*Moreover, we have $\pi_k(f_{prin}) = 1$ for $k \leq 5$, so the maps on the $3^{rd}$, $4^{th}$ and $5^{th}$ homotopy groups are determined by the Dynkin index.*

*Proof.* We have two things to show, firstly that $\Phi \simeq f_{\mathrm{prin}} \circ m_n$, and secondly that $\pi_k(f_{\mathrm{prin}}) = 1$ for $k \leq 5$. For the first, it is well known that any compact Lie group deformation retracts onto its maximal compact subgroup $K$, in the case of $G = SL_2(\mathbb{C})$, we have $K = SU(2) \cong S^3$, the 3-sphere. We have by definition of the index, that $\pi_3(\Phi) = n = \pi_3(f_{\mathrm{prin}}) \cdot \pi_3(m_n)$, and by definition of $\pi_3$ that $\Phi \simeq f_{\mathrm{prin}} \circ m_n$.

For the second part we reproduce the proof supplied by Michael Albanese, [8]. We begin by noting that $SL_2(\mathbb{C}) \cong SU(2) \cong Sp(1)$, so our induced map $\pi_*(SL_2(\mathbb{C})) \xrightarrow{f_{\mathrm{prin}}} \pi_*(C_6)$ may be thought of as a map $\pi_*(Sp(1)) \to \pi_*(Sp(6))$. Also, note that since $\pi_4(C_n) = \pi_5(C_n) = \mathbb{Z}_2$, showing that $\pi_4(f_{\mathrm{prin}}) = \pi_5(f_{\mathrm{prin}}) = 1$ is equivalent to showing that the maps are isomorphisms. Now, $Sp(n)$ is the quaternionic unitary group, and so it acts transitively on $S^{4n-1} \subset \mathbb{H}^n$, with stabiliser $Sp(n-1)$. Fix $x = (0, 0, ..., 0, 1) \in \mathbb{H}^n$, let $i_n : Sp(n-1) \to Sp(n)$ be the inclusion given by $i(A) = \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}$, and $\rho : Sp(n) \to \mathbb{H}^n$ be given by $\rho(A) = Ax$. Then we have a fiber bundle $Sp(n-1) \to Sp(n) \xrightarrow{i} S^{4n-1}$ which induces a long exact sequence in homotopy groups:

$$\cdots \to \pi_{k+1}(S^{4n-1}) \to \pi_k(Sp(n-1)) \xrightarrow{(i_n)_*} \pi_k(Sp(n)) \xrightarrow{p_*} \pi_k(S^{4n-1}) \to \pi_{k-1}(Sp(n-1)) \to \dots$$

One notices that we have $\pi_k(S^{4n-1}) = \pi_{k+1}(S^{4n-1}) = 0$ for $k \leq 4n-3$ and so it immediately follows that $(i_n)_* : \pi_k(Sp(n-1)) \to \pi_k(Sp(n))$ is an isomorphism for $k \leq 4n-3$.

Now let $i = i_n \circ ... \circ i_2$, then $i : Sp(1) \to Sp(n)$ is the standard inclusion map $A \to \begin{pmatrix} A & 0 \\ 0 & I_{2n-2} \end{pmatrix}$ and $i_* = (i_n)_* \circ ... \circ (i_2)_*$. Observe that for $m \geq 2$ we have $4, 5 \leq 4m - 3$ and so

$$(i_m)_* : \pi_4(Sp(m-1)) \to \pi_4(Sp(m)) \text{ and } (i_m)_* : \pi_5(Sp(m-1)) \to \pi_5(Sp(m))$$

are isomorphisms, and since a composition of isomorphisms is also an isomorphism we have that

$$i_* : \pi_4(Sp(1)) \to \pi_4(Sp(n)) \text{ and } i_* : \pi_5(Sp(1)) \to \pi_5(Sp(n))$$

are isomorphisms, as required. □

**Corollary 4.11.** *For all $\mathfrak{sl}_2$-subalgebras of $\mathfrak{b}_6$ and $\mathfrak{c}_6$ with associated homogeneous spaces ${}^{B_6}\!/_{SL_2(\mathbb{C})}$ and ${}^{C_6}\!/_{SL_2(\mathbb{C})}$ we have ${}^{B_6}\!/_{SL_2(\mathbb{C})} \not\simeq {}^{C_6}\!/_{SL_2(\mathbb{C})}$.*

*Proof.* From the proposition we have $b_4 \equiv b_5 \pmod 2$ since $b_4$ and $b_5$ are determined by the parity of the Dynkin index. In either case we have $\pi_5\left({}^{C_6}\!/_{\mathrm{SL}_2(\mathbb{C})}\right) \neq \pi_5\left({}^{B_6}\!/_{\mathrm{SL}_2(\mathbb{C})}\right)$ and so ${}^{B_6}\!/_{\mathrm{SL}_2(\mathbb{C})} \not\simeq {}^{C_6}\!/_{\mathrm{SL}_2(\mathbb{C})}$. □

With this corollary we have now determined that no two homogeneous spaces with differing $G$, where $G = B_6, C_6$ or $E_6$ are homotopy equivalent. We now turn our attention to internal comparisons of homogeneous spaces, and conclude this section with an immediate but sizable result following from the theorem relating the Dynkin index and $\pi_3$.

**Proposition 4.12.** *Labelling homogeneous spaces by their associated weighted Dynkin diagram of the $\mathfrak{sl}_2$ subalgebra we have:*

- *The only possibilities for homogeneous spaces of $B_6$ to be homotopy equivalent are:*
  - $\triangle(0,2,0,2,0,0)$ *and* $\triangle(1,0,1,1,0,1)$
  - $\triangle(2,0,2,0,0,0)$ *and* $\triangle(2,1,0,0,0,1)$ *and* $\triangle(0,1,1,0,1,0)$
  - $\triangle(2,1,0,1,0,0)$ *and* $\triangle(0,2,0,0,0,1)$
  - $\triangle(2,2,0,0,0,0)$ *and* $\triangle(0,2,0,1,0,0)$
  - $\triangle(0,2,0,0,0,0)$ *and* $\triangle(1,0,0,0,1,0)$
  - $\triangle(1,0,1,0,0,0)$ *and* $\triangle(0,0,0,0,0,1)$
  - $\triangle(2,0,0,0,0,0)$ *and* $\triangle(0,0,0,1,0,0)$
- *The only possibilities for homogeneous spaces of $C_6$ to be homotopy equivalent are:*
  - $\triangle(2,0,1,0,0,0)$ *and* $\triangle(0,1,0,0,1,0)$
  - $\triangle(2,1,0,0,0,0)$ *and* $\triangle(0,1,0,1,0,0)$
- *No two homogeneous spaces of $E_6$ are homotopy equivalent.*

*Proof.* Compare Dynkin indices from the tables in Appendix A, the lists in the proposition are the only nilpotent orbits with the same Dynkin index. □

The Dynkin index has proven to be an excellent tool in distinguishing between our homogeneous spaces, however we still have nine cases (including a triple comparison) to consider. Higher homotopy groups do not appear to have a useful invariant in the same way $\pi_3$ did, and so we turn our attention away from homotopy groups in hopes of finding a new invariant, and thus begins our discussion and computation of the K-theory of our indistinguishable orbits.

# 5 The K-theory of a homogeneous space

We now introduce our final tool for distinguishing between homogeneous spaces, the K-theory of the space. First we review the definition of a vector bundle. A nice reference for an introduction to vector bundles and K-theory is [11].

## 5.1 Defintions and terminology

**Definition 5.1.** Let $X$ be a topological space. A vector bundle on $X$ is a space $E$ with a surjective map $\rho : E \to X$ such that $\rho^{-1}(\{x\})$ is a vector space for each $x \in X$ and $\rho$ satisfies a "locally trivial" condition, that is, for each $x \in X$ there exists an open neighbourhood $U_x$ containing $x$ such that $\rho^{-1}(U_x) \cong V \times U_x$ for some vector space $V$.

**Example 5.2** (The trivial bundle)**.** Let $X$ be a space and $V$ a vector space, we may form the bundle $E = V \times X$ with projection $\rho : V \times X \to X$ the projection onto the second factor. This is clearly a vector bundle, called the trivial bundle. If $\dim(V) = n$ then we denote this bundle by $\varepsilon_n$.

**Example 5.3.** Let $X = S^1 \in \mathbb{R}^2$, $E = TS^1$ and $\rho : E \to S^1$ such that $\rho(T_xS^1) = x$, that is, $E$ is the tangent bundle of $S^1$. We show that $E$ is a vector bundle. By definition of $E$, for each $x \in S^1$, $\rho^{-1}(\{x\})$ is a vector sapce, namely the tangent space at $x$. We must show that $\rho$ is locally trivial. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^2$, and let $U_x = \{y \in S^1 : \langle x, y \rangle > 0\}$, roughly speaking, $U$ consists of the points on $S^1$ which are less than a quarter of the circumference away from $x$ in both directions. Now define $\phi : \rho^{-1}(U) \to \mathbb{R} \times U$ by projecting the tangent plane for each $y \in U$ to the tangent plane $T_xS^1$ of $x$. This gives the required isomorphism, since $T_yS^1$ and $T_xS^1$ are not orthogonal for $\langle x, y \rangle \neq 0$.

**Remark 5.4.** One has direct sums and tensor products of vector bundles, which behave as one would expect.

We now introduce two separate equivalence relations on the set of vector bundles of a space $X$ as follows, let $E_1$ and $E_2$ be two vector bundles of a space $X$, we say $E_1$ is stably isomorphic to $E_2$, written $E_1 \approx_S E_2$, if $E_1 \oplus \varepsilon_n \approx E_2 \oplus \varepsilon_n$ for some $n \in \mathbb{N}$, similarly, we say that $E_1 \sim E_2$ if $E_1 \oplus \varepsilon_m \approx E_2 \oplus \varepsilon_n$ for some $m, n \in \mathbb{N}$, where $\approx$ denotes an isomorphism of vector bundles and we recall that $\varepsilon_n$ is the trivial bundle of dimension $n$.

**Proposition 5.5.** *If $X$ is a compact Hausdorff space, then the set of $\sim$ equivalence classes of vector bundles on $X$ forms an abelian group with addition given by the direct sum, $\oplus$. This group is called the $0^{th}$ reduced K-group of $X$, and denoted $\tilde{K}^0(X)$.*

Given a compact space $X$ we may also form the abelian group $K^0(X)$, called the $0^{\text{th}}$ K-group of $X$, consisting of formal differences $E_1 - E_2$ of vector bundles $E_1$ and $E_2$ on $X$, with the equivalence relation $E_1 - E_1' = E_2 - E_2' \iff E_1 \oplus E_2' \approx_S E_2 \oplus E_1'$. This group has the addition property

$$(E_1 - E_1') + (E_2 - E_2') = (E_1 \oplus E_2) - (E_2 \oplus E - 2')$$

with the zero element given by the equivalence class of $E - E$ for any vector bundle $E$. Furthermore, $K^0(X)$ can be given a multiplication, realising it as a ring, this multiplication is given by

$$(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 - E_1 \otimes E_2' - E_1' \otimes E_2 + E_1' \otimes E_2'$$

with multiplicative identity $\varepsilon_1$. Observe that every element of $K^0(X)$ can be expressed as $E - \varepsilon_n$ for some $n$, as given $E' - E''$, it is a fact that we can find a bundle $F$ such that $E'' \oplus F = \varepsilon_n$. Set $E = E' \oplus F$, then we have $E' - E'' = E - \varepsilon_n$ in $K^0(X)$.

We now define a map $K^0(X) \to \tilde{K}^0(X)$ by sending $E - \varepsilon_n$ to the $\sim$-equivalence class of $E$. One checks that this is well defined, surjective and has kernel $\{\varepsilon_m - \varepsilon_n\} \cong \mathbb{Z}$. In fact, if one chooses a base point $x_0 \in X$ and considers the restriction of vector bundles $K^0(X) \to K^0(x_0) \cong \mathbb{Z}$, one obtains an isomorphism $K^0(X) \simeq \tilde{K}^0(X) \oplus \mathbb{Z}$.

We know have the following result, which gives us a key corollary in distinguishing homogeneous spaces up to homotopy.

**Proposition 5.6** (Theorem 7.2 [11]). *Let $X$ be a compact space, and $f_0, f_1 : X \to Y$ be two homotopic maps. If $E$ is a vector bundle over $Y$, the the pullback bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic.*

**Corollary 5.7.** *Let $X$ and $Y$ be homotopy equivalent compact spaces. Then $K^0(X) = K^0(Y)$ and $\tilde{K}^0(X) = \tilde{K}^0(Y)$ .*

Recall that given a space $X$, we may define the suspension of $X$, denoted $SX$, given as follows:

$$SX = {X \times [0,1]} \big/ {\big(X \times \{0\}, X \times \{1\}\big)}$$

We may then define, as in [11], $K^{-1}(X) = \tilde{K}^0(SX)$. It is often useful to define the $1^{st}$ K-group in the unreduced case too, and in fact the correct defintion here is $K^{-1}(X) = \tilde{K}^{-1}(X)$. Furthermore, these K-groups are $\mathbb{Z}_2$ graded, so $K^{2k+i}(X) = K^i(X)$ for all $k \in \mathbb{Z}$, and similarly for the reduced K groups. This gives the following result.

**Corollary 5.8.** *Let $X$ and $Y$ be homotopy equivalent compact spaces, then $K^1(X) = K^1(Y)$.*

We now briefly discuss some definitions and results regarding equivalent K-theory, before turning our attention to methods of computation of K-groups. For this part we follow [18, Section 1]. Let $G$ be a group, we say a topological space $X$ is a $G$-space if $X$ admits a continuous $G$-action $G \times X \to X$. To begin, we define a $G$-vector bundle.

**Definition 5.9.** Let $G$ be a group, and $X$ a $G$-space then a $G$-vector bundle on $X$ is a $G$-space $E$ together with a map $\rho : E \to X$ such that

- $\rho$ is a $G-$map, that is, $g \cdot \rho(\xi) = \rho(g \cdot \xi)$ for all $g \in G$, $\xi \in E$.
- $\rho : E \to X$ is a vector bundle, in the usual sense.
- for any $g \in G$ and $x \in X$ the group action $g : E_x \to E_{g \cdot x}$ is a map of vector spaces, where $E_y =^{-1} (y)$ for all $y \in Y$.

Now comes a very significant example which will be at the heart of our computations of K-groups of homogeneous spaces.

**Example 5.10.** [Homogeneous vector bundles, [18]] Let $G$ be a group, $H$ a closed subgroup (closed in the topological sense) and let $X = {G}\big/{H}$, the coset space. Let $\rho : E \to X$ be a $G$-vector bundle, and let $E_0 =^{-1} (eH)$, where $eH \in X$ denotes the identity coset. Then one sees that $E_0$ is a $H$-module, which turns out to determine $E$ completely.

The action of $G$ on $E$ gives a map $\alpha : G \times_H E_0 \to E$, where $G \times_H E_0$ is the space of orbits of $G \times E_0$ under the $H$ action given by $(h, g, v) \mapsto (gh^{-1}, hv)$. If we define a $G$-action on $G \times_H E_0$ by $(g, g', v) \mapsto (gg', v)$ then one checks that $\alpha$ is a $G$-map. In fact, it is a homeomorphism, and we can explicitly construct it's inverse. For details see [18].

Conversely, if $H$ is locally compact, and $E_0$ is any H-module, then $G \times_H E_0$ is a $G$-vector bundle on $X$, thus $H$-modules are in correspondence with $G$-vector bundles over $X = {G}\big/{H}$.

Given a group $G$ and a $G$-space $X$, we define the $G$-invariant K-theory of $K^0_G(X)$ (or just $K_G(X)$, since we will have no need for high $G$-invariant K-groups) analogously to how we defined $K^0(X)$ of a space $X$, except we use $G$-vector bundles in place of vector bundles.

In particular we have the following result, which follows immediately from example 5.10.

**Lemma 5.11.** *Let $G$ be a group, $H \subset G$ a closed subgroup, then*

$$K_G\left({G}\big/{H}\right) = R(H)$$

*where $R(H) = \mathbb{Z}\big[\overline{V} :\ V$ is a finite dimensional representation of $H\big]$ and where $\overline{V}$ denotes the equivalence class of $V$ under the relations $V \sim W \iff V \cong W$ as $H$-representations and $\overline{V} + \overline{W} \sim \overline{V \oplus W}$. In particular we have*

- $K_G(G) = R\big(\{e\}\big) \cong \mathbb{Z}$ *with an isomorphism given by* $[V] \longmapsto \dim(V)$.
- $K_G\big(\{e\}\big) = R(G)$.

## 5.2 The Kunneth spectral sequence in equivariant K-theory

We have now defined the K-groups and reduced K-groups for a space $X$ and seen that if two spaces are homotopy equivalent, then they have the same K-groups. This suggests a way to distinguish between our homogeneous spaces, however to do this we need a way to compute these groups. This is achieved via the so-called Kunneth spectral sequence in equivariant K-theory, first discovered by Hodgkin, [7].

We begin by stating the main result, which can be found in [14]. We note that "collapses" here means that all differentials $d_r$ vanish for $r \geq 2$.

**Theorem 5.12.** *Let $G$ be a compact connected Lie group such that $\pi_1(G)$ is torsion free and $H$ a closed subgroup of $G$, then the spectral sequence*

$$\begin{aligned} E_2^{*,0} &= Tor^*_{R(G)}\left(K^0_G(G), K^0_G\left({}^{G}\!/\!_{H}\right)\right) \\ &= Tor^*_{R(G)}\left(\tilde{\mathbb{Z}}, R(H)\right) \implies K^*\left({}^{G}\!/\!_{H}\right) \end{aligned}$$

*collapses, where $\tilde{\mathbb{Z}}$ is the $R(G)$ module given by action $R(G) \times \tilde{\mathbb{Z}} \to \tilde{\mathbb{Z}}$; $[V] \cdot 1 = \dim([V])$. We write $\tilde{\mathbb{Z}}$ so as not to confuse it with the trivial module.*

By grading considerations we see that

$$K^0\left({}^{G}\!/\!_{H}\right) = \bigoplus_{k=0} E_2^{2k,0}$$

$$K^1\left({}^{G}\!/\!_{H}\right) = \bigoplus_{k=0} E_2^{2k+1,0}.$$

Thus, we wish to compute the Tor groups above, we begin with the definition of a Koszul complex, details of which can be found in [4]

**Definition 5.13.** Let $R$ be a commutative ring and $M$ a free $R-$module of finite rank $r$. Then given a $R-$linear map $s : E \to R$ we define the Koszul complex of $s$ as:

$$K(s) : 0 \to \bigwedge^r E \xrightarrow{D_r} \bigwedge^{r-1} E \xrightarrow{D_{r-1}} \dots \xrightarrow{D_2} \bigwedge^1 E \xrightarrow{D_1} R \to 0$$

where $D_k(e_1 \wedge e_2 \wedge ... \wedge e_k) = \sum_{i=1}^{k} (-1)^{i+1} s(e_i) e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_k$.

Note that we have $\bigwedge^r E \cong R$, $\bigwedge^1 E = E$ and $d_1 = s$.

The following result is a specific case of [Theorem 16.5, [13]].

**Proposition 5.14.** *Let $R$ be a polynomial ring over a ring $k$ with indeterminates $x_1, ..., x_n$. Let $s : R^n \to R$ be given by $s = (x_1, ..., x_n)$, then:*

$$K(x_1, ..., x_n) := K(s) = 0 \to \bigwedge^n R^n \xrightarrow{D_n} \bigwedge^{n-1} R^n \xrightarrow{D_{n-1}} \dots \xrightarrow{D_2} R^n \xrightarrow{s} R \to 0$$

*is a free resolution of ${}^{R}\!/\!_{(x_1, ..., x_n)}$.*

It is well known that for compact simply-connected Lie groups $G$, the representation ring $R(G)$ is a polynomial ring in rank$(G)$ indeterminates. Thus, for $G = B_6, C_6$ or $E_6$ we have $R(G) = \mathbb{Z}[x_1, ..., x_6]$, where $x_i$ are the fundamental characters and moreover we have $R(SL2) = \mathbb{Z}[t]$. Observe that given a subgroup $H \subset G$, $R(H)$ is a $R(G)-$module via restriction, that is

$$\begin{aligned} R(G) \times R(H) &\to R(H) \\ (\chi, \psi) &\longmapsto \chi|_H \cdot \psi \end{aligned}$$

where $\cdot$ denotes multiplication of characters in $R(H)$. From now on, $G$ is $B6$, $C6$ or $E6$, and $H = SL2$. Let $d_i = \dim(x_i)$, the $i^{th}$ fundamental representation of $G$, then we have

$$\tilde{\mathbb{Z}} = \mathbb{Z}[x_1, ..., x_6] \big/ (x_1 - d_1, ..., x_6 - d_6)$$

as $R(G) = \mathbb{Z}[x_1, ..., x_6]-$modules. Thus we have a free-resolution of $\tilde{\mathbb{Z}}$ given by

$$0 \to \bigwedge^6 \mathbb{Z}[x_1, ..., x_6]^6 \xrightarrow{D_6} \bigwedge^5 \mathbb{Z}[x_1, ..., x_6]^6 \xrightarrow{D_5} \dots \xrightarrow{D_2} \mathbb{Z}[x_1, ..., x_6]^6 \xrightarrow{D_1} \mathbb{Z}[x_1, ..., x_6] \to 0$$

where $D_1 = (x_1 - d_1, ..., x_6 - d_6)$ and $\mathbb{Z}[x_1, ..., x_6]^k := \bigoplus^k \mathbb{Z}[x_1, ..., x_6]$.

We wish to find a canonical way to write the differentials $D_i$, and for this we require a canonical basis of $\bigwedge^k \mathbb{Z}[x_1, ..., x_6]^6$ as a $\mathbb{Z}[x_1, ..., x_6]$-module. Firstly, $\mathbb{Z}[x_1, ..., x_6]^6$ has basis $\{e_1, ..., e_6\}$, and in order to use this to define a canonical basis of the exterior powers we define a lexicographic ordering on the exterior powers.

**Definition 5.15.** Let $e_{i_1} \wedge .... \wedge e_{i_k}$ and $e_{j_1} \wedge .... \wedge e_{j_k}$ be two elements of $\bigwedge^k \mathbb{Z}[x_1, ..., x_6]^6$, we say that $e_{i_1} \wedge .... \wedge e_{i_k}$ comes before $e_{j_1} \wedge .... \wedge e_{j_k}$ if $e_i = j_i$ for $i < m$ for some $1 \leq m \leq 6$, and $e_m < j_m$.

With this ordering we may give a canonical ordered basis to each exterior power, and this along with the fact that $\bigwedge^k \mathbb{Z}[x_1, ..., x_6]^6 = \mathbb{Z}[x_1, ..., x_6]^{\binom{6}{k}}$ gives us the following form of the free resolution:

$$0 \to \mathbb{Z}[x_1, ..., x_6] \xrightarrow{D_6} \mathbb{Z}[x_1, ..., x_6]^6 \xrightarrow{D_5} \dots \xrightarrow{D_3} \mathbb{Z}[x_1, ..., x_6]^{15} \xrightarrow{D_2} \mathbb{Z}[x_1, ..., x_6]^6 \xrightarrow{D_1} \mathbb{Z}[x_1, ..., x_6] \to 0$$

where, $D_i$ is viewed as a $\binom{6}{i-1} \times \binom{6}{i}$ matrix, for example

$$D_1 = (x_1 - d_1, x_2 - d_2, ..., x_6 - d_6)$$

$$D_5 = \begin{pmatrix}
x_5 - d_5 & x_6 - d_6 & 0 & 0 & 0 & 0 \\
-(x_4 - d_4) & 0 & x_6 - d_6 & 0 & 0 & 0 \\
0 & -(x_4 - d_4) & -(x_5 - d_5) & 0 & 0 & 0 \\
x_3 - d_3 & 0 & 0 & x_6 - d_6 & 0 & 0 \\
0 & x_3 - d_3 & 0 & -(x_5 - d_5) & 0 & 0 \\
0 & 0 & x_3 - d_3 & x_4 - d_4 & 0 & 0 \\
-(x_2 - d_2) & 0 & 0 & 0 & x_6 - d_6 & 0 \\
0 & -(x_2 - d_2) & 0 & 0 & -(x_5 - d_5) & 0 \\
0 & 0 & -(x_2 - d_2) & 0 & x_4 - d_4 & 0 \\
0 & 0 & 0 & -(x_2 - d_2) & -(x_3 - d_3) & 0 \\
x_1 - d_1 & 0 & 0 & 0 & 0 & x_6 - d_6 \\
0 & x_1 - d_1 & 0 & 0 & 0 & -(x_5 - d_5) \\
0 & 0 & x_1 - d_1 & 0 & 0 & x_4 - d_4 \\
0 & 0 & 0 & x_1 - d_1 & 0 & -(x_3 - d_3) \\
0 & 0 & 0 & 0 & x_1 - d_1 & x_2 - d_2
\end{pmatrix}$$

**Remark 5.16.** One may view the full complex with explicit matrices given for all differentials on SageMath with the following code. I was unable to name my indeterminates "$x_i - D_i$" without returning an error, so one should read "$x_i - D_i$" instead of $xi$ if they run the code below.

```
#SageMath version 9.2
R.<x1,x2,x3,x4,x5,x6> = ZZ[]
K = KoszulComplex(R, [x1,x2,x3,x4,x5,x6])
ascii_art(K)
```

To compute $\mathrm{Tor}^*_{R(G)}\left(\tilde{\mathbb{Z}}, R(H)\right) = \mathrm{Tor}^*_{R(G)}\left(\tilde{\mathbb{Z}}, \mathbb{Z}[t]\right)$, we apply the functor $-\otimes\mathbb{Z}[t]$ to our free resolution of $\tilde{\mathbb{Z}}$. We observe that $(-\otimes\mathbb{Z}[t])(D_i) = D_i|_{\mathbb{Z}[t]}$, that is, the functor applied to $D_i$ can be explicitly viewed as replacing all $x_i$ inside the matrix by $x_i|_{\mathbb{Z}[t]}$, the restriction of $x_i$ to $\mathbb{Z}[t]$. By an abuse of notation, we also write $D_i$ for the restriction $D_i|_{\mathbb{Z}[t]}$, where there will be no confusion as to whether we mean $D_i$ or it's restriction. After applying the functor we have

$$0 \to \mathbb{Z}[t] \xrightarrow{D_6} \mathbb{Z}[t]^6 \xrightarrow{d_5} \dots \xrightarrow{D_3} \mathbb{Z}[t]^{15} \xrightarrow{D_2} \mathbb{Z}[t]^6 \xrightarrow{D_1} \mathbb{Z}[t] \to 0$$

with homology groups $E_2^{0,0} = {\mathbb{Z}[t]}\big/{\mathrm{im}(D_1)}$ and $E_2^{i,0} = {\ker(D_i)}\big/{\mathrm{im}(D_{i+1})}$ for $1 \leq i \leq 6$.

## 5.3 An example of calculating Tor groups and formation of a complex over $\mathbb{Z}$.

We now calculate the Tor groups for a specific orbit to see how the calculation goes. Then compute various modifications of the Tor groups for orbits in Proposition 4.12 to try distinguish them.

**Example 5.17.** For our explicit example, we will compute the complex for the $\Phi : SL_2 \to B_6$ with associated Dynkin diagram $\triangle(2,0,0,0,0,0)$. [2] We begin by finding the restriction of the fundamental characters $x_i$ of $B_6$ to $SL2$, using SageMath or otherwise, one obtains that the first fundamental character $x_1$ is given by

$$x_1 = h_1^{-1} + h_1^{-1}h_2 + h_2^{-1}h_3 + h_3^{-1}h_4 + h_4^{-1}h_5 + h_5^{-1}h_6^2 + 1 + h_5h_6^{-2} + h_4h_5^{-1} + h_3h_4^{-1} + h_2h_3^{-1} + h_1h_2^{-1} + h_1$$

The weighted Dynkin diagram $\triangle(2,0,0,0,0,0)$ tells us how the torus $T^1 \subset SL_2$ acts on the roots of $B_6$, to determine the action on the weights we apply $C^{-1}$, where $C$ denotes the Cartan matrix of $B_6$, we have

$$C^{-1}\cdot\begin{pmatrix}2\\0\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & -1 & 2 & -2\\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix}2\\0\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}2\\2\\2\\2\\2\\1\end{pmatrix}.$$

This now gives us a map

$$\begin{aligned} T^1 \subset SL_2 &\longrightarrow T^6 \subset B_6\\ \gamma &\longmapsto (\gamma^2,\gamma^2,\gamma^2,\gamma^2,\gamma^2,\gamma) \end{aligned}$$

which in turn gives us $x_1|_{SL_2} = \gamma^{-2} + 11 + \gamma^2$. Now we wish to write $x_1|_{SL_2}$ in terms of the fundamental character $t$ of $R(SL_2) = \mathbb{Z}[t]$, that is, the character of the fundamental 2 dimensional representation of $SL_2$. We begin by observing that $t = \gamma + \gamma^{-1}$, and so $x_1|_{SL_2} = t^2 + 9$. Next, evaluation of $x_i(h_1, ..., h_6)$ at $(1,1,1,1,1,1)$ will give us the dimension of the character (note that setting $\gamma = 1$ or $t = 2$ in $x_i|_{SL_2}$ also gives the dimension), we obtain $d_1 = 13$. Finally, we get $x_1 - d_1 = t^2 - 4$. Below is a table giving all $x_i|_{SL_2}$, $D_i$ and factored $x_i|_{SL_2} - D_i$ for $1 \leq i \leq 6$, note that $t-2$ is always a factor of $x_i|_{SL_2} - D_i$, since evaluation of $x_i|_{SL_2}$ at $t = 2$ is precisely $D_i$.

| $x_i\|_{SL_2}$ | $D_i$ | $x_i\|_{SL_2} - D_i$ |
|---|---|---|
| $t^2 + 9$ | 13 | $(t-2)(t+2)$ |
| $11t^2 + 34$ | 78 | $11(t-2)(t+2)$ |
| $55t^2 + 66$ | 286 | $55(t-2)(t+2)$ |
| $165t^2 + 55$ | 715 | $165(t-2)(t+2)$ |
| $330t^2 - 33$ | 1287 | $330(t-2)(t+2)$ |
| $32t + 32$ | 64 | $32(t-2)$ |

[2]This orbit was chosen as it is the first on the list of comparisons, it is no easier or harder to compute than any other orbit.

Thus, we have complex

$$0 \to \mathbb{Z}[t] \xrightarrow{D_6} \mathbb{Z}[t]^6 \xrightarrow{d_5} \dots \xrightarrow{D_3} \mathbb{Z}[t]^{15} \xrightarrow{D_2} \mathbb{Z}[t]^6 \xrightarrow{D_1} \mathbb{Z}[t] \to 0$$

where the $D_i$ are given by substituting expressions for $x_i|_{SL_2} - D_i$ from the table into the matrices, for example

$$d_1 = \big((t-2)(t+2), 11(t-2)(t+2), 55(t-2)(t+2), 165(t-2)(t+2), 330(t-2)(t+2), 32(t-2)\big).$$

In general, since $\mathbb{Z}[t]$ is not an Euclidean domain, or even a P.I.D, calculating the homology groups is not a straightforward task. However, calculating $E_2^{0,0}$ is usually manageable, albeit not so useful on it's own. In this example one sees that

$$E_2^{0,0} = \mathrm{Tor}^0 = {\mathbb{Z}[t]}\big/{\mathrm{im}(D_1)} = {\mathbb{Z}[t]}\big/{\big(32(t-2),(t-2)(t+2)\big)}.$$

As alluded to in the example, calculation of the Tor groups over $\mathbb{Z}[t]$ would be a messy endeavor, and so instead we look to modify the complex so as to make the homology groups easier to calculate, whilst still ensuring that the resulting groups are a homotopy invariant. Our first modification is tensoring the complex by $- \otimes_{\mathbb{Z}[t]} {\mathbb{Z}[t]}\big/{(t-a)}$, otherwise known as evaluation of the complex at $t = a$ for some $a \in \mathbb{Z}$. We observe that if all Tor groups are isomorphic for two different orbits then their evaluations at any $a \in \mathbb{Z}$ will be isomorphic. Letting $K^*|_{t=a} := K^* \otimes {\mathbb{Z}[t]}\big/{(t-a)}$ denote the resulting $K$-groups after evaluation at $t = a$, we have

**Lemma 5.18.** *Let $H, H' \subset G$ be two copies of $SL_2$ with associated Dynkin Diagrams $\triangle$ and $\triangle'$. Then $G/H \cong G/H' \implies K^*\left(G/H\right) \cong K^*\left(G/H'\right) \implies K^*\left(G/H\right)|_{t=a} \cong K^*\left(G/H'\right)|_{t=a}$ for all $a \in \mathbb{Z}$.*

*In particular, if there exists an $a \in \mathbb{Z}$ such that $K^*\left(G/H\right)|_{t=a} \ncong K^*\left(G/H'\right)|_{t=a}$, then $G/H \ncong G/H'$*

**Remark 5.19.** We know that the K-theories are isomorphic as rings, however for the lemma to hold we would require that the $\mathbb{Z}[t]-$module isomorphism sends $t \to t$, this seems very likely since $t$ represents the $2-$dimensional irreducible representation of $SL_2$, so it does hold some concrete significance.

**Remark 5.20.** Using lemma 5.21 below, which in fact holds for all Dedekind domains, in particular for $\mathbb{F}_p[t]$, for some prime $p$, one could rephrase the lemma by instead applying $- \otimes_{\mathbb{Z}} \mathbb{F}_p[t]$, thus only requiring the isomorphism between K-theories to be one of rings, not of modules. Through good choices of $p$, one can arrive at similar conclusions to the ones found in the remainder of this section.

Given a complex $0 \to \mathbb{Z}[t] \xrightarrow{D_6} \mathbb{Z}[t]^6 \xrightarrow{D_5} \dots \mathbb{Z}[t]^6 \xrightarrow{D_1} \mathbb{Z}[t] \to 0$ we write $C(t = a)$ for the complex $0 \to \mathbb{Z} \xrightarrow{D_6} \mathbb{Z}^6 \xrightarrow{D_5} \dots \mathbb{Z}^6 \xrightarrow{D1} \mathbb{Z} \to 0$ given by evaluating at $t = a$. Once again, by abuse of notation, we write $D_i$ for $D_i(t = a)$. This is a useful construction due to the ease of computing homology groups as given by the following lemma.

**Lemma 5.21.** *Let $K(a_1, ..., a_n)$, $a_i \in \mathbb{Z}$ denote the Koszul complex of free $\mathbb{Z}$-modules*

$$0 \to \mathbb{Z} \xrightarrow{D_n} \mathbb{Z}^n \xrightarrow{D_{n-1}} \mathbb{Z}^{\binom{n}{n-2}} \xrightarrow{D_{n-2}} \dots \to \mathbb{Z}^{\binom{n}{2}} \xrightarrow{D_2} \mathbb{Z}^n \xrightarrow{D_1} \mathbb{Z} \to 0$$

*with differentials $D_i$ given by setting $x_i = a_i$ in the differentials of $K(x_1, ..., x_n)$. Setting $D_0 = D_{n+1=0}$ we have*

$$ker(D_i)\big/ im(D_{i+1}) = \bigoplus_{j=1}^{\binom{n-1}{i}} C_{gcd(a_1,..,a_n)},$$

*the cyclic group of order $gcd(a_1, .., a_n)$, for all $0 \le i \le n$. In particular, when $n = 6$, $K^*|_{t=a}$ is completely determined by $gcd\big(x_1|_{SL_2} - d_1, ..., x_6|_{SL_2} - d_6\big)|_{t=a}$.*

*Proof.* Let $P = \{p \text{ prime } : p \text{ divides one of the } a_i\}$. Given $p \in P$, let $\mathbb{Z}_{(p)} = \left\{\frac{a}{b} \in \mathbb{Q} : p \text{ does not divide } b\right\}$ denote the localisation of $\mathbb{Z}$ at $p$. Let $K(a_1, ..., a_n; \mathbb{Z}_{(p)})$ be the Koszul complex over $\mathbb{Z}_{(p)}$. It is clear that for any units $u_i \in \mathbb{Z}_{(p)}$, we have that $K(a_1, ..., a_n; \mathbb{Z}_{(p)})$ and $K(u_1a_1, ..., u_na_n; \mathbb{Z}_{(p)})$ are quasi-isomorphic via the obvious chain map, in particular, they share the same homology. Thus, by choosing the $u_i$ such that $u_ia_i = p^{k_i}$ for some $k_i$, and without loss of generality assuming that $k_1 \leq ... \leq k_n$, we obtain that $H_\bullet\big(K(a_1, ..., a_n; \mathbb{Z}_{(p)})\big) = H_\bullet\big(K(p^{k_1}, ..., p^{k_n}; \mathbb{Z}_{(p)})\big)$.

By inspection of the differentials $D_i$, and using that for indeterminates $x_i$, $K(x_1, ..., x_n)$is exact to guide us, we observe that $\ker(D_i)$ is spanned by the columns of $D_{i+1}$ containing a "$p^{k_1}$" term, and there are precisely $\binom{n-1}{i}$ of these, namely the number of ways to choose $i$ values from $\{p^{k_2}, ..., p^{k_n}\}$. Moreover, if $v \in \mathbb{Z}_{(p)}^N$ is such a column, we note that $\frac{1}{p^{k_1}} \cdot v \in \mathbb{Z}_{(p)}^N$ is in the kernel of $D_i$.

Combining both of these observations gives us that

$$\ker(D_i)/\mathrm{im}(D_{i+1}) = \bigoplus_{i=1}^{\binom{n-1}{i}} \mathbb{Z}_{(p)}/p^{k_1}\mathbb{Z}_{(p)} = \bigoplus_{i=1}^{\binom{n-1}{i}} \mathbb{Z}_{(p)}/\gcd\big(p^{k_1}, ..., p^{k_n}\big)\mathbb{Z}_{(p)}.$$

Finally, a direct sum of these local rings gives us the required result. ☐

**Remark 5.22.** In fact, this holds for any Dedekind domain, the proof is very similar.

**Example 5.23.** Returning to our example of calculating the complex associated to the orbit with Dynkin diagram $\triangle(2, 0, 0, 0, 0, 0)$ of $B_6$, recall that we had

$$\begin{aligned}
x_1|_{SL_2} - d_1 &= (t-2)(t+2)\\
x_2|_{SL_2} - d_2 &= 11(t-2)(t+2)\\
x_3|_{SL_2} - d_3 &= 55(t-2)(t+2)\\
x_4|_{SL_2} - d_4 &= 165(t-2)(t+2)\\
x_5|_{SL_2} - d_5 &= 330(t-2)(t+2)\\
x_6|_{SL_2} - d_6 &= 32(t-2).
\end{aligned}$$

We have

- For $t = 2$ the complex is $C(t = 2) = 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^6 \xrightarrow{0} \ldots \mathbb{Z}^6 \xrightarrow{0} \mathbb{Z} \to 0$ and so $E_2^{i,0}|_{t=a} = \mathbb{Z}^{\binom{6}{i}}$
- For $t = 3$ we have

$$\begin{aligned}
x_1|_{SL_2}(t=a) - d_1 &= 3\\
x_2|_{SL_2}(t=a) - d_2 &= 33\\
x_3|_{SL_2}(t=a) - d_3 &= 165\\
x_4|_{SL_2}(t=a) - d_4 &= 495\\
x_5|_{SL_2}(t=a) - d_5 &= 990\\
x_6|_{SL_2}(t=a) - d_6 &= 32.
\end{aligned}$$

  The complex is $C(t = 3) = 0 \to \mathbb{Z} \xrightarrow{D_6} \mathbb{Z}^6 \xrightarrow{D_5} \ldots \mathbb{Z}^6 \xrightarrow{D_1} \mathbb{Z} \to 0$, we have $\gcd(3, 33, 165, 495, 990, 32) = 1$ and so all Tor groups are trivial.

One may compute the homology groups for any $K(a_1, ..., a_6)$ using Lemma 5.21 or more concretely using SageMath with

```
#SageMath version 9.2
K = KoszulComplex(ZZ, [a_1, ..., a_6])
K.homology()
```

With Lemma 5.18 in mind, for each pair of orbits from Proposition 4.12 we look to find $a \in \mathbb{Z}$ such that $K^*\left({G}/{H}\right)|_{t=a}$ differ, thus showing that the two homogeneous spaces are not homotopy equivalent. One can verify the following table.

| $B_6$ comparisons | | | |
|---|---|---|---|
| Orbits being compared | $t = a$ | $\gcd(\{x_i\|_{SL_2} - d_i\})$ for LHS orbit | $\gcd(\{x_i\|_{SL_2} - d_i\})$ for RHS orbit |
| $\triangle(2,0,0,0,0,0)$ and $\triangle(0,0,0,1,0,0)$ | $a = 4$ | 4 | 8 |
| $\triangle(1,0,1,0,0,0)$ and $\triangle(0,0,0,0,0,1)$ | $a = 5$ | 27 | 9 |
| $\triangle(0,2,0,0,0,0)$ and $\triangle(1,0,0,0,1,0)$ | $a = 3$ | 5 | 1 |
| $\triangle(2,2,0,0,0,0)$ and $\triangle(0,2,0,1,0,0)$ | $a = 4$ | 4 | 8 |
| $\triangle(2,1,0,1,0,0)$ and $\triangle(0,2,0,0,0,1)$ | - | - | - |
| $\triangle(2,0,2,0,0,0)$ and $\triangle(2,1,0,0,0,1)$ | $a = 3$ | 5 | 1 |
| $\triangle(2,0,2,0,0,0)$ and $\triangle(0,1,1,0,1,0)$ | $a = 3$ | 5 | 1 |
| $\triangle(2,1,0,0,0,1)$ and $\triangle(0,1,1,0,1,0)$ | $a = 5$ | 27 | 9 |
| $\triangle(0,2,0,2,0,0)$ and $\triangle(1,0,1,1,0,1)$ | $a = 4$ | 4 | 8 |

Thus, for all pairs of homogeneous spaces of $B_6$ listed in Proposition 4.12, the only pair which cannot be shown to not be homotopy equivalent via evaluation at $t = a$ for some $a \in \mathbb{Z}$ is the pair $\triangle(2,1,0,1,0,0)$ and $\triangle(0,2,0,0,0,1)$. For $C_6$, from Proposition 4.12 we see that there are two pairs of spaces to compare, and unfortunately these cannot be distinguished via evaluation at $t = a$ for any $t = a$. We have a refinement of Proposition 4.12:

**Proposition 5.24.** *Labelling homogeneous spaces by their associated weighted Dynkin diagram of the $\mathfrak{sl}_2$ subalgebra we have:*

- *The only possibility for two homogeneous spaces of $B_6$ to be homotopy equivalent is:*
  - $\triangle(2,1,0,1,0,0)$ *and* $\triangle(0,2,0,0,0,1)$
- *The only possibilities for two homogeneous spaces of $C_6$ to be homotopy equivalent are:*
  - $\triangle(2,0,1,0,0,0)$ *and* $\triangle(0,1,0,0,1,0)$
  - $\triangle(2,1,0,0,0,0)$ *and* $\triangle(0,1,0,1,0,0)$
- *No two homogeneous spaces of $E_6$ are homotopy equivalent.*

Unfortunately, as remarked, Lemma 5.21 extends to complexes over $\mathbb{F}[t]$, for fields $\mathbb{F}$. This, along with the fact that computing homology of the original $\mathbb{Z}[t]$-complex is much too messy, means that distinguishing the remaining 3 pairs of homogeneous spaces has not been possible using any methods discussed thus far. Therefore, despite making significant progress in distinguishing the homogeneous spaces of $B_6$, $C_6$ and $E_6$, a complete result has not been achieved.

## A Tables for $B_6$, $C_6$ and $E_6$ orbits

| $B_6$ orbits | | |
|---|---|---|
| $\pi_i(B_6) \otimes \mathbb{Q} = \begin{Bmatrix} \mathbb{Q} & i = 3, 7, 11, .., 23 \\ 0 & \text{otherwise} \end{Bmatrix} \implies \pi_i \left( {B_6}/{SL_2} \right) \otimes \mathbb{Q} = \begin{Bmatrix} \mathbb{Q} & i = 7, 11, .., 23 \\ 0 & \text{otherwise} \end{Bmatrix}$ | | |
| Partition | Dynkin diagram<br>1 2 3 4 5 6 | Dynkin index |
| $[13]$ | $\triangle(2,2,2,2,2,2)$ | 182 |
| $[11,1,1]$ | $\triangle(2,2,2,2,2,0)$ | 110 |
| $[9,3,1]$ | $\triangle(2,2,2,0,2,0)$ | 62 |
| $[9,2,2]$ | $\triangle(2,2,2,1,0,1)$ | 61 |
| $[9,1,1,1,1]$ | $\triangle(2,2,2,2,0,0)$ | 60 |
| $[7,5,1]$ | $\triangle(2,0,2,0,2,0)$ | 38 |
| $[7,3,3]$ | $\triangle(2,2,0,0,2,0)$ | 32 |
| $[7,3,1,1,1]$ | $\triangle(2,2,0,2,0,0)$ | 30 |
| $[7,2,2,1,1]$ | $\triangle(2,2,1,0,1,0)$ | 29 |
| $[7,1,1,1,1,1,1]$ | $\triangle(2,2,2,0,0,0)$ | 28 |
| $[6,6,1]$ | $\triangle(0,2,0,2,0,1)$ | 35 |
| $[5,5,3]$ | $\triangle(0,2,0,0,2,0)$ | 22 |
| $[5,5,1,1,1]$ | $\triangle(0,2,0,2,0,0)$ | 20 |
| $[5,4,4]$ | $\triangle(1,0,1,1,0,1)$ | 20 |
| $[5,3,3,1,1]$ | $\triangle(2,0,0,2,0,0)$ | 14 |
| $[5,3,2,2,1]$ | $\triangle(2,0,1,0,1,0)$ | 13 |
| $[5,3,1,1,1,1,1]$ | $\triangle(2,0,2,0,0,0)$ | 12 |
| $[5,2,2,2,2]$ | $\triangle(2,1,0,0,0,1)$ | 12 |
| $[5,2,2,1,1,1,1]$ | $\triangle(2,1,0,1,0,0)$ | 11 |
| $[5,1,1,1,1,1,1,1,1]$ | $\triangle(2,2,0,0,0,0)$ | 10 |

| | | |
|---|---|---|
| $[4,4,3,1,1]$ | $\triangle(0,1,1,0,1,0)$ | 12 |
| $[4,4,2,2,1]$ | $\triangle(0,2,0,0,0,1)$ | 11 |
| $[4,4,1,1,1,1,1]$ | $\triangle(0,2,0,1,0,0)$ | 10 |
| $[3,3,3,3,1]$ | $\triangle(0,0,0,2,0,0)$ | 8 |
| $[3,3,3,2,2]$ | $\triangle(0,0,1,0,1,0)$ | 7 |
| $[3,3,3,1,1,1,1]$ | $\triangle(0,0,2,0,0,0)$ | 6 |
| $[3,3,2,2,1,1,1]$ | $\triangle(0,1,0,1,0,0)$ | 5 |
| $[3,3,1,1,1,1,1,1,1]$ | $\triangle(0,2,0,0,0,0)$ | 4 |
| $[3,2,2,2,2,1,1]$ | $\triangle(1,0,0,0,1,0)$ | 4 |
| $[3,2,2,1,1,1,1,1,1]$ | $\triangle(1,0,1,0,0,0)$ | 3 |
| $[3,1,1,1,1,1,1,1,1,1,1]$ | $\triangle(2,0,0,0,0,0)$ | 2 |
| $[2,2,2,2,2,2,1]$ | $\triangle(0,0,0,0,0,1)$ | 3 |
| $[2,2,2,2,1,1,1,1,1]$ | $\triangle(0,0,0,1,0,0)$ | 2 |
| $[2,2,1,1,1,1,1,1,1,1,1]$ | $\triangle(0,1,0,0,0,0)$ | 1 |
| $[1,1,1,1,1,1,1,1,1,1,1,1,1]$ | $\triangle(0,0,0,0,0,0)$ | 0 |

| $C_6$ orbits | | |
|---|---|---|
| $\pi_i(C_6)\otimes\mathbb{Q}=\begin{Bmatrix}\mathbb{Q} & i=3,7,11,..,23\\ 0 & \text{otherwise}\end{Bmatrix} \implies \pi_i\left({C_6}\big/{SL_2}\right)\otimes\mathbb{Q}=\begin{Bmatrix}\mathbb{Q} & i=7,11,..,23\\ 0 & \text{otherwise}\end{Bmatrix}$ | | |
| Partition | Dynkin diagram<br>1 2 3 4 5 6 | Dynkin index |
| $[12]$ | $\triangle(2,2,2,2,2,2)$ | 286 |
| $[10,2]$ | $\triangle(2,2,2,2,0,2)$ | 166 |
| $[10,1,1]$ | $\triangle(2,2,2,2,1,0)$ | 165 |
| $[8,4]$ | $\triangle(2,2,0,2,0,2)$ | 94 |
| $[8,2,2]$ | $\triangle(2,2,2,0,0,2)$ | 86 |
| $[8,2,1,1]$ | $\triangle(2,2,2,0,1,0)$ | 85 |
| $[8,1,1,1,1]$ | $\triangle(2,2,2,1,0,0)$ | 84 |
| $[6,6]$ | $\triangle(0,2,0,2,0,2)$ | 70 |
| $[6,4,2]$ | $\triangle(2,0,2,0,0,2)$ | 46 |
| $[6,4,1,1]$ | $\triangle(2,0,2,0,1,0)$ | 45 |
| $[6,3,3]$ | $\triangle(2,1,0,1,1,0)$ | 43 |
| $[6,2,2,2]$ | $\triangle(2,2,0,0,0,2)$ | 38 |
| $[6,2,2,1,1]$ | $\triangle(2,2,0,0,1,0)$ | 37 |
| $[6,2,1,1,1,1]$ | $\triangle(2,2,0,1,0,0)$ | 36 |
| $[6,1,1,1,1,1,1]$ | $\triangle(2,2,1,0,0,0)$ | 35 |
| $[5,5,2]$ | $\triangle(0,2,0,1,1,0)$ | 41 |
| $[5,5,1,1]$ | $\triangle(0,2,0,2,0,0)$ | 40 |
| $[4,4,4]$ | $\triangle(0,0,2,0,0,2)$ | 30 |
| $[4,4,2,2]$ | $\triangle(0,2,0,0,0,2)$ | 22 |
| $[4,4,2,1,1]$ | $\triangle(0,2,0,0,1,0)$ | 21 |
| $[4,4,1,1,1,1]$ | $\triangle(0,2,0,1,0,0)$ | 20 |

| | | |
|---|---|---|
| $[4,3,3,2]$ | $\triangle(1,0,1,0,1,0)$ | 19 |
| $[4,3,3,1,1]$ | $\triangle(1,0,1,1,0,0)$ | 18 |
| $[4,2,2,2,2]$ | $\triangle(2,0,0,0,0,2)$ | 14 - |
| $[4,2,2,2,1,1]$ | $\triangle(2,0,0,0,1,0)$ | 13 |
| $[4,2,2,1,1,1,1]$ | $\triangle(2,0,0,1,0,0)$ | 12 |
| $[4,2,1,1,1,1,1,1]$ | $\triangle(2,0,1,0,0,0)$ | 11 |
| $[4,1,1,1,1,1,1,1,1]$ | $\triangle(2,1,0,0,0,0)$ | 10 |
| $[3,3,3,3]$ | $\triangle(0,0,0,2,0,0)$ | 16 |
| $[3,3,2,2,2]$ | $\triangle(0,1,0,0,1,0)$ | 11 |
| $[3,3,2,2,1,1]$ | $\triangle(0,1,0,1,0,0)$ | 10 |
| $[3,3,2,1,1,1,1]$ | $\triangle(0,1,1,0,0,0)$ | 9 |
| $[3,3,1,1,1,1,1,1]$ | $\triangle(0,2,0,0,0,0)$ | 8 |
| $[2,2,2,2,2,2]$ | $\triangle(0,0,0,0,0,2)$ | 6 |
| $[2,2,2,2,2,1,1]$ | $\triangle(0,0,0,0,1,0)$ | 5 |
| $[2,2,2,2,1,1,1,1]$ | $\triangle(0,0,0,1,0,0)$ | 4 |
| $[2,2,2,1,1,1,1,1,1]$ | $\triangle(0,0,1,0,0,0)$ | 3 |
| $[2,2,1,1,1,1,1,1,1,1]$ | $\triangle(0,1,0,0,0,0)$ | 2 |
| $[2,1,1,1,1,1,1,1,1,1,1]$ | $\triangle(1,0,0,0,0,0)$ | 1 |
| $[1,1,1,1,1,1,1,1,1,1,1,1]$ | $\triangle(0,0,0,0,0,0)$ | 0 |

| $E_6$ orbits | | |
|---|---|---|
| $\pi_i(E_6) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 3, 9, 11, 15, 17, 23 \\ 0 & \text{otherwise} \end{cases} \implies \pi_i\left(E_6 / SL_2\right) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 9, 11, 15, 17, 23 \\ 0 & \text{otherwise} \end{cases}$ | | |
| Orbit name | Dynkin diagram<br>2<br>1 3 4 5 6 | Dynkin index |
| $E6$ | $\triangle(2, 2, 2, 2, 2, 2)$ | 156 |
| $E6(a1)$ | $\triangle(2, 2, 2, 0, 2, 2)$ | 84 |
| $D5$ | $\triangle(2, 2, 0, 2, 0, 2)$ | 60 |
| $E6(a3)$ | $\triangle(2, 0, 0, 2, 0, 2)$ | 36 |
| $D5(a1)$ | $\triangle(1, 2, 1, 0, 1, 1)$ | 30 |
| $A5$ | $\triangle(2, 1, 1, 0, 1, 2)$ | 35 |
| $A4 + A1$ | $\triangle(1, 1, 1, 0, 1, 1)$ | 21 |
| $D4$ | $\triangle(0, 2, 0, 2, 0, 0)$ | 28 |
| $A4$ | $\triangle(2, 2, 0, 0, 0, 2)$ | 20 |
| $D4(a1)$ | $\triangle(0, 0, 0, 2, 0, 0)$ | 12 |
| $A3 + A1$ | $\triangle(0, 1, 1, 0, 1, 0)$ | 11 |
| $2A2 + A1$ | $\triangle(1, 0, 0, 1, 0, 1)$ | 9 |
| $A3$ | $\triangle(1, 2, 0, 0, 0, 1)$ | 10 |
| $A2 + 2A1$ | $\triangle(0, 0, 1, 0, 1, 0)$ | 6 |
| $2A2$ | $\triangle(2, 0, 0, 0, 0, 2)$ | 8 |
| $A2 + A1$ | $\triangle(1, 1, 0, 0, 0, 1)$ | 5 |
| $A2$ | $\triangle(0, 2, 0, 0, 0, 0)$ | 4 |
| $3A1$ | $\triangle(0, 0, 0, 1, 0, 0)$ | 3 |
| $2A1$ | $\triangle(1, 0, 0, 0, 0, 1)$ | 2 |
| $A1$ | $\triangle(0, 1, 0, 0, 0, 0)$ | 1 |
| 0 | $\triangle(0, 0, 0, 0, 0, 0)$ | 0 |

# B Code

Unless stated otherwise, all code should be executed in SageMath (version 9.2).

## B.1 Main code - finding the $SL_2$ restricted polynomials for any orbit

This may be found at https://github.com/DylanJohnston. Note that this code works for all compact simple Lie groups $G$, and any orbit/weighted Dynkin diagram, not just $B_6$, $C_6$ and $E_6$.

## B.2 Koszul complex

To find the Koszul complex $K(x_1, x_2, ...x_6)$ use the below code, to find the complex in a different amount of variables, add or remove variables on both the first and second line.

```
#version 9.2
R.<x1,x2,x3,x4,x5,x6> = ZZ[]
K = KoszulComplex(R, [x1,x2,x3,x4,x5,x6])
ascii_art(K)
```

## B.3 Calculating Homology from complex over the integers.

First define a complex over the integers, for example.

```
K = KoszulComplex(R, [2,4,6])
ascii_art(K)
```

Output:

```
                                  [-4 -6  0]           [ 6]
                                  [ 2  0 -6]           [-4]
                 [2 4 6]          [ 0  2  4]           [ 2]
 0 <-- C_0 <---------- C_1 <-------------- C_2 <------ C_3 <-- 0
```

Now to calculate homology:

```
K.homology()
```

Output:

```
{0: C2, 1: C2 x C2, 2: C2, 3: 0}
```

**Remark B.1.** One notices that homology groups only consist of copies of $C_{\text{gcd}(2,4,6)} = C_2$.

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