On the representation theory of $GL_3(\mathbb{F}_p)$

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Contents

Introduction

Given $GL_3(\mathbb{F}_p)$ we consider a very special subgroup called the torus, which is defined as

$$T := \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} : x, y, z \in \mathbb{F}_p^{\times} \right\}.$$

We can restrict any representation of $GL_3(\mathbb{F}_p)$ to a representation of T, from this we find that any and every representation can be broken up into a collection of weight-spaces such that T acts as $x^a y^b z^c$ for some (a, b, c) where a,b,c depend on which weight-space we are acting on. In the case of $GL_3(\mathbb{F}_p)$ we call the 3-tuple (a, b, c) a *weight*. If we restrict ourselves to the set of weights in which a, b, c are weakly decreasing (called the dominant region) then each (a, b, c) has a corresponding representation, which we denote $W(a, b, c)$ throughout. For weights in the dominant region such that $a - b, b - c < p$ (we call this the p-restricted region) we have a irreducible sub-representation of $W(a, b, c)$, which we denote $F(a, b, c)$. In representation theory an important question to ask is, if possible, how any representation decomposes into irreducible ones so throughout this paper we will look into the decomposition of $W(a, b, c)$, as well as the decompositions of $F(\lambda) \otimes F(\mu)$ where λ, μ denote weights in the p-restricted region, that is, $F(\lambda)$ and $F(\mu)$ are irreducible representations themselves. To do this we will use some known theorems and results such as the “Strong Linkage Principle”, “The Littlewood Richardson rule”, “Steinberg’s Theorem” and others.

Section 1 will serve as an introduction and explanation of these theorems and results that will come in useful during our study. Section 2 is mainly calculation, we look into the complete decomposition of tensor products of irreducible representations into irreducible representations themselves, and investigate some properties of these decompositions and the terms that arise. A part of the issue with generalizing our work from Section 2 to general primes ‘p’ is the issue of knowing how the W representations decompose outside the p-restricted region, so Section 3 looks into these decompositions for $p = 5$, then gives a few Lemmas & Propositions to end. This then sees us return in Section 4 to the problem we had in Section 2, where we start to look into the complete decomposition of tensor products of irreducible representations for general p, and due to lack of time the section ends with a general question, rather than an answer. Finally, Section 5 provides a link to some Python code I created in order to compute explicit decompositions once I realised that they can take some time to do by hand.

Acknowledgements

I would like to thank the London Mathematical Society and Durham University for providing financial support throughout my 8 week placement. I would also like to give a massive thank you to my supervisor, Jack Shotton, firstly for volunteering to supervise me, secondly for teaching me a lot of the theory I know to date on this wonderful subject, and finally for proof-reading and helping me correct parts of this paper.

1 Theorems, Tools and Terminology.

1.1 Short Exact Sequences and the Grothendieck group

Definition 1.1. (Short exact sequence) Given a group G and U, V, W representations. A short exact sequence is a pair of G-intertwiners $f : U \to V$ and $g : V \to W$ with f injective, g surjective and $\mathrm{Im}(f) = \ker(g)$.

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

Definition 1.2. (Grothendieck group) Given a finite group G, define the Grothendieck group of the representations of G to be the quotient of the free abelian group generated by all representations $[V]$ by the relation $[V] - [U] - [W]$ for every SES as defined above. Note that the Grothendieck group is itself free abelian if all representations are irreducible.

Remark 1.3. If we have $V = U \oplus W$ for some representations U, V, W then f and g are as follows:

$$\begin{array}{ll} f : U \to V & \qquad g : V \to W \\ u \to (u, 0) & \qquad (v, w) \to w \end{array}$$

In this case we say the SES *splits*. As it is still a SES (albeit a special one) we can write, in the Grothendieck group, $[V] = [U] + [W]$. This is good because if $V = U \oplus W$ as representations we certainly want it to be the case that $[V] = [U] + [W]$ in the abelian group formed.

Example 1.4. To give a straightforward example of a short exact sequence which doesn't split we will consider a SES of groups rather than representations, this means we require f and g to be group homomorphisms rather than G-intertwiners. Our group SES is

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0$$

where f and g are as follows:

$$\begin{array}{cc} f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} & \qquad g : V \to W \\ \overline{0} \to \overline{0} & \qquad \overline{0}, \overline{2} \to \overline{0} \\ \overline{1} \to \overline{2} & \qquad \overline{1}, \overline{3} \to \overline{1} \end{array}$$

It can be checked that f and g are indeed group homomorphisms and satisfy $\mathrm{Im}(f) = \ker(g)$, so in the Grothendieck group $[\mathbb{Z}/4\mathbb{Z}] = [\mathbb{Z}/2\mathbb{Z}]+[\mathbb{Z}/2\mathbb{Z}]$, however it's not the case that $\mathbb{Z}/4\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, to see this consider the order of the elements.

1.2 Mod p Representations of $SL_2(\mathbb{F}_p)$

We will first briefly discuss the representations of $SL_2(\mathbb{F}_p)$. This section will introduce some concepts and terminology which we can carry with us when we explore the representations of $GL_3(\mathbb{F}_p)$. To start let us recall what the group $SL_2(\mathbb{F}_p)$ is:

$$SL_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \overline{1} \ \&\ a, b, c, d \in \mathbb{F}_p \right\}$$

We see by letting $SL_2(\mathbb{F}_p)$ act on $\mathbb{F}_p^2$ in the usual manner and using orbit-stabiliser that $|SL_2(\mathbb{F}_p)| = p(p^2 - 1)$.

Definition 1.5. (The representation V_k) Let $k \in \mathbb{N}$, then define the representation V_k as

$$\begin{aligned} V_k &:= \{F^k(X, Y) \in \mathbb{F}_p(X, Y)\} \\ &= \{\text{“Homogeneous polynomials in variables } X \text{ and } Y \text{ of degree } k\text{”}\} \\ &\cong \mathrm{Sym}^k(\mathbb{F}_p^2). \end{aligned}$$

This is a representation of $SL_2(\mathbb{F}_p)$ over $\mathbb{F}_p$ where $SL_2(\mathbb{F}_p)$ acts on V_k via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X^\alpha Y^{k-\alpha} = \big(aX + cY)^\alpha (bX + dY)^{k-\alpha}\big)$$

Proposition 1.6. *[1, Theorem 3.2 with $r = 1$] For $k < p$ we have that V_k is irreducible.*

Remark 1.7. We usually shorten the term “irreducible representation” to “irrep”.

Now, consider the following subgroup, known as the torus of $SL_2(\mathbb{F}_p)$:

$$T := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{F}_p^\times \right\}$$

Then if we restrict the representation V_k of $SL_2(\mathbb{F}_p)$ to the subgroup T we have that

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot (X^\alpha Y^{k-\alpha}) = \lambda^{2\alpha - k}(X^\alpha Y^{k-\alpha})$$

The power of λ, ie “$2\alpha - k$”, is known as a weight, this can be thought of as a generalisation of the concept of an eigenvalue since it doesn't actually depend on the value of λ. Observe that in V_k, $\alpha \in \{0, 1, 2, ..., k\}$ so the weights take values $\big(-k, -(k-2), ...0, ..., k-2, k\big)$. These weights are integers differing by 2 and can be plotted on a straight line (This idea of plotting the weights becomes more useful when we move into $GL_3(\mathbb{F}_p)$).

Definition 1.8. (The representation W_k.) For $k \in \{0, 1, 2, ..., p-2\}$ define

$$W_k := \{f : \mathbb{F}_p^2 \to \mathbb{F}_p : f(\lambda x, \lambda y) = \lambda^k f(x, y) \ \forall \lambda \in \mathbb{F}_p^\times \ \&\ f(0, 0) = 0\}.$$

Remark 1.9. We have $\dim(W_k) = p + 1$ for all k. To see that the dimension is independent of k, simply list the homogeneous functions $f(x, y) = x^\alpha y^\beta$ with $\alpha + \beta \equiv k \bmod (p-1)$, $\alpha, \beta > 0$. Then by observing that after evaluation we have $x^{p-1} \equiv 1 \pmod p$ for all $x \in \mathbb{F}_p^\times$ and similarly for y we find ourselves with $p - 1$ linearly independent functions. Then by noting that $g(x, y) = x^k$ and $h(x, y) = y^k$ are a further 2 linearly independent functions after evaluation due to their evaluations at $(x, 0)$ and $(0, y)$ respectively we get $p + 1$ linearly independent functions in the space of homogeneous functions over $\mathbb{F}_p$. Any function in W_k can be expressed as a linear combination of these $p + 1$ linearly independent functions so we have found a basis, so we conclude the dimension is indeed $p + 1$ for all k.

For any $a \geq 0$ we have an evaluation map (where $[a] \equiv a \bmod (p - 1)$):

$$ev : V_a \to W_{[a]}$$

$$F(X, Y) \to ((x, y) \to F(x, y))$$

Lemma 1.10.

i) $\ker(ev) = \{F \in V_a : F$ *is divisible by* $\Phi(X, Y) := X^p Y - XY^p\}$

ii) If $a \geq p$ *then* ev *is surjective.*

iii) If $a \leq p$ *then* ev *is injective.*

Proof.

i) $\Phi(X, Y) = X^p Y - XY^p = XY(X - Y)(X - 2Y)...(X - (p - 1)Y)$, to see this observe X, Y are clearly factors. Then fix Y $\neq \overline{0}$ and notice that for $X = nY$, $n \in \{1, 2, ..., p - 1\}$ we have $\Phi(nY, Y) = 0$, so all $(X - nY)$ are factors. By a degree argument this is all the factors of $\Phi(X, Y)$. Now, the kernel consists of the a^{th} degree polynomials in x, y which evaluate at 0 for all $x, y \in \mathbb{F}_p$, ie. have $\Phi(X, Y)$ as a factor ie. are divisible by $\Phi(X, Y)$

ii) Let $P(X, Y) \in V_a$, then $P(X, Y) = \sum_{i=0}^{a} \alpha_i X^i Y^{a-i}$. Any function $f \in W_{[a]}$ is defined by where it sends $p+2$ points. For $a \geq p$, $\dim(V_a) \geq p+1$ By interpolation arguments, for any $p+2$ points there exists α_i such that $P(X, Y)$ interpolates these values, in particular there will exist a set of α_i for which $P(X, Y) = f$.

iii) If $a \leq p$ then every polynomial in V_a has degree less than $p + 1$, so by i) we have that the kernel is trivial, hence ev is injective.

□

A less obvious but nonetheless still true result is that the map

$$h \ : \ W_{[k]} \to V_{p-1-k}$$

is a surjective map for $0 \leq k \leq p - 2$ such that $\ker(h) = ev(V_k)$. The result of all this is the following:

i) for $a \geq p+1$, there is a SES:

$$0 \to V_{a-(p+1)} \xrightarrow{\Phi(X,Y)} V_a \xrightarrow{ev} W_{[a]} \to 0$$

So in the Groth. group: $[V_a] = [V_{a-(p+1)}] + [W_{[a]}]$

ii) for a = p we have a SES:

$$0 \to V_a \xrightarrow{ev} W_{[p]} \to 0$$

Hence: $[V_a] \cong [W_1] = [W_{[p]}]$ where the isomorphism is the identity map.

iii) for $0 \leq a \leq p-1$ we have that $[V_k]$ is irreducible.

iv) for $0 \leq k \leq p-2$, these is a SES:

$$0 \to \ V_k \xrightarrow{ev} W_k \xrightarrow{h} V_{p-1-k} \to 0$$

So in the Groth. group: $[W_k] = [V_{p-1-k}] + [V_k]$

To close this sub-section observe that combining i) and iv) gives us for $a \geq p+1$;

$$[V_a] = [V_{a-(p+1)}] + [V_{p-1-[a]}] + [V_{[a]}]$$

Note for $a \leq 2p+1$ we have actually decomposed $[V_a]$ into a sum of irreducible representations! This idea of decomposing representations into linear combinations of irreducible representations is exactly what we wish to do now with $GL_3(\mathbb{F}_p)$.

1.3 $GL_3(\mathbb{F}_p)$: Weights and terminology

In $GL_3(\mathbb{F}_p)$ we will aim to find some irreducible representations (irreps) and see if we can decompose arbitrary representations into sums of these irreps in the Grothendieck group.

To begin with, let's extend the notion of a weight we found for SL_2 into the world of GL_3. Now, rather than getting a single value for the weight as we did for $SL_2(\mathbb{F}_p)$ we will have weights being 3-tuples. (Why? well because the size of our matrices have increased so $T \subset GL_3(\mathbb{F}_p)$ has one dimension more. Also as we no longer have the determinant condition on T it has 3 degrees of freedom).

Recall from the introduction, we define the torus in $GL_3(\mathbb{F}_p)$ as

$$T := \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} : x, y, z \in \mathbb{F}_p^\times \right\}$$

Definition 1.11. Define the following terms for some prime p:

- $X(T) := \{\text{all weights } \lambda = (a, b, c) : a, b, c \in \mathbb{Z}\}$
- $X_0(T) := \{\text{weights } \lambda = (c, c, c) : c \in \mathbb{Z}\} \subset X(T)$

- $X_+(T) =$ "The dominant region" $:= \{\lambda = (a, b, c) : a \geq b \geq c\}$
- $X_p(T) =$ "The p-restricted region" $:= \{(a, b, c) : 0 \leq a - b, b - c < p\}$
- The lower alcove $:= \{(a, b, c) : -1 < a - b, b - c \ \& \ a - c < p - 2\}$
- The upper alcove $:= \{(a, b, c) : a - b, b - c < p - 1 \ \& \ a - c > p - 2\}$

Figure 1 below shows some of these definitions:

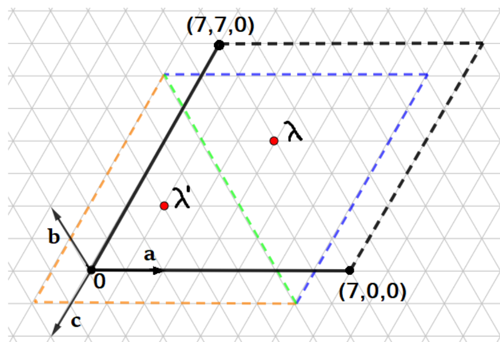


Figure 1: A weight diagram for p=7

We should think of this as looking at a slice of a 3d lattice from a bird's-eye view, this means when I talk about lines on the diagram I am referring to the 2d planes coming out of the page, and if I talk about 2d shapes such as triangles or rhombi I am really referring to the prism with that shape as the cross-section. Based on the image:

- The p-restricted region is the rhombus (really a rhombic prism) enclosed by the black lines. Note: points can live on the full lined edges and still be classed as being p-restricted, but not on the dotted lines.
- The region enclosed by the orange and green dotted lines (really a collection of planes) is the lower alcove.
- The region enclosed by the green and blue dotted lines is the upper alcove.
- If a point lies on the green line or the blue dotted line we say it lives on the boundary (of the upper alcove).
- The directions (not to scale) "a","b" and "c" have been marked on the diagram, which denote the first, second and third element in the 3-tuple we call the weight. Observe that there is more than one tuple which corresponds to any given point on the 2d-diagram, and this is why we should think of a weight diagram as a slice of a lattice rather than the whole picture (as we will soon see). This is because on these 2d diagrams we have that (a, b, c) corresponds to the same point as $(a + x, b + x, c + x)$ for any $(x, x, x) \in X_0(T)$.

- Imagine extending the bold black lines out out to infinity in the " $-c$" and "a" direction, then all points lying in the $\frac{1}{6}$-plane enclosed by these half lines is the dominant region.

For $\lambda \in X_+(T)$, there's a somewhat "natural" representation of $GL_3(\mathbb{F}_p)$ over $\mathbb{F}_p$ which we'll call W(λ) [2, Equation 3.2], this is not an irrep in general. However, for $\lambda \in X_p(T)$, there is a distinguished irreducible sub representation of $W(\lambda)$ which we will denote $F(\lambda)$.

1.4 The irreducible representations, decomposition of $\mathbf{W}(\lambda)$ for $\lambda \in$ $\mathbf{X}_p\mathbf{(T)}$ and Steinberg's tensor theorem

Theorem 1.12. *[2, Theorem 3.10]*

i) $\{F(\lambda) : \lambda \in X_p(T)\}$ *are all the actual irreps of* $GL_3(\mathbb{F}_p)$.[1]

ii) If $\lambda, \mu \in X_p(T)$ *then* $F(\lambda) \simeq F(\mu) \iff \lambda - \mu \in (p-1)X_0(T)$ *That is* $(p-1, p-1, p-1) \simeq (0,0,0)$ *and in fact:*

$$W(\lambda + (x,x,x)) \simeq W(\lambda) \otimes det^x$$
$$F(\lambda + (x,x,x)) \simeq F(\lambda) \otimes det^x$$

Figure 2 shows a section of the 3d lattice for $p = 7$. You can think of the whole lattice as an infinity-story building. Adding $(1,1,1)$ to the weight takes you up one floor directly above you, and somehow being on the $p-1^{th}$ floor "is the same" (or isomorphic) to being in the same spot but on the 0^{th}/ground floor, so the infinity tall lattice/building is just copies of level 0 to level $p-2$ stacked on top of one another. In the diagram the black floors are isomorphic, this agrees with our definition since $(20,13,6) - (14,7,0) = 6(1,1,1) \in (p-1)X_0(T)$ for $p = 7$ so $W(14,7,0) \simeq W(20,13,6)$.

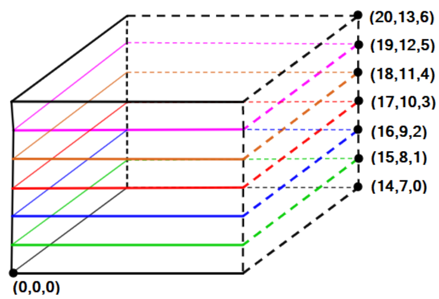


Figure 2: A section of the lattice for $p = 7$ (3d)

From this we can deduce that for any prime p there are exactly $p^2(p-1)$ irreducible representations of $GL_3(\mathbb{F}_p)$ up to isomorphism. The next question

[1] There are in fact two notions of a representation, an "actual" representation and an "algebraic" representation. We discuss what these mean very shortly

to ask is what is $F(\lambda)$ if $\lambda \notin X_p(T)$, the answer to this is given by the next theorem. Something to note is that we will be using two ideas of a representation throughout, that is, an "actual representation" and an "algebraic representation". Everything I have said so far has referred to actual representations and you can assume I am referring to actual representations unless I explicitly say otherwise (although somehow, $W(\lambda)$ and $F(\lambda)$ adopt both meanings). The point of this is that algebraic representations are defined by polynomials (this is what we done in the section discussing $SL_2(\mathbb{F}_p)$ when we defined $\mathrm{Sym}^k(\mathbb{F}_p^2)$ as homogeneous polynomials of degree k). As algebraic representations *all* the $F(\lambda)$ are irreducible and distinct. To help see this consider x^p compared with x, as polynomials these are clearly different. However if we consider them as functions mod(p) they are the same thing. For every algebraic representation (that is, it's polynomial representation) V of a group G we can ask what happens if we replace every variable x_i in the polynomial with x_i^p, the answer being that it gives us a new non-isomorphic representation denoted $V^{(p)}$, called the 1^{st} Frobenius twist, in fact this idea generalises to all powers of p as follows:

Definition 1.13. (r^{th} Frobenius twist.) Given a representation V of a group G we say $V^{(p^r)}$ is the r^{th} Frobenius twist of V and it is obtained by replacing every variable x_i in the polynomial V by it's $p^{r^{th}}$ power.

As algebraic representations these are all distinct. However by Fermat's Little Theorem we have that they are all the same as actual representations.

Theorem 1.14. *(Steinberg's theorem) [2, Theorem 3.9] Suppose $\lambda \notin X_p(T)$ and $\lambda = \sum_{i=0}^{n} p^i \lambda_i$, where all $\lambda_i \in X_p(T)$ (this can always be done), then:*

- *As algebraic representations: $F(\lambda) = \bigotimes_{i=0}^{n} F(\lambda_i)^{p^i}$*
- *As actual representations: $F(\lambda) = \bigotimes_{i=0}^{n} F(\lambda_i)$*

Now that we have seen a method to deal with $F(\lambda)$, we would like to know about $W(\lambda)$? It is currently too much to ask what happens for λ lying outside the p-restricted region, but for $\lambda \in X_p(T)$ we have the following: [2, Proposition 3.18]

- If λ is not in the upper alcove (ie it lies in the lower alcove or on the boundary) then $W(\lambda) = F(\lambda)$, that is, $W(\lambda)$ is irreducible.

- If λ is in the upper alcove then there is a SES:

  $$0 \to \ F(a, b, c) \to W(a, b, c) \to F(c + p - 2, b, a - p + 2) \to 0$$

  So in the Grothendieck group:

  $$[W(a, b, c)] = [F(a, b, c)] + [F(c + p - 2, b, a - p + 2)]$$

  The latter weight is not as strange as it first may seem, in fact $(c + p - 2, b, a - p + 2)$ is the resulting weight if (a, b, c) is reflected in the "green line" from Figure 1.

1.5 Planes and the Strong Linkage Principle

Our next task should be to give the coloured lines that we referred to when discussing Figure 1 and Figure 2 a more precise name, rather then having to refer to them by something as inconsistent as colours. We also would to like to extend this idea and think about reflections in other planes, and this is what we will do now. For $n \in \mathbb{Z}$, define the following 3 operations on (a, b, c):

$$s_{1,n} : (a, b, c) \to (a, c + np - 1, b + 1 - np)$$
$$s_{2,n} : (a, b, c) \to (c + np - 2, b, a + 2 - np)$$
$$s_{3,n} : (a, b, c) \to (b + np - 1, a + 1 - np, c)$$

Geometrically, $s_{i,n}$ is the reflection in the plane $H_{i,n}$. For $\lambda \in X_p(T)$ reflecting in this "green line" as we referred to it before as is just a reflection in the plane $H_{2,1}$ (as of now the term "green line" to mean reflection in $H_{2,1}$ is scrapped), these planes divide the lattice into alcoves, we have the lower alcove bounded by $H_{1,0}, H_{2,1}, H_{3,0}$ and the upper alcove bounded by $H_{1,1}, H_{2,1}, H_{3,1}$. It should be noted that the alcoves refer to the volume properly contained inside these boundaries so points on the planes themselves don't count.

Definition 1.15. Define $R_{i,j}$ as the rhombic prism bounded by the planes $H_{1,j}, H_{1,j+1}, H_{3,i}, H_{3,i+1}$ and say $\lambda \in R_{i,j}$ if λ is in the interior of the rhombic prism or if the weight lives on either of the planes $H_{1,j+1}$ and $H_{3,j+1}$, but not on the intersection lines of either of these with the plane $H_{2,i+j+1}$.

Further define ${R_{i,j}}^-$ as the alcove bounded by $H_{1,j}, H_{2,i+j+1}, H_{3,i}$ and say $\lambda \in {R_{i,j}}^-$ if λ is in the interior of the alcove or if λ lies on $H_{2,i+j+1}$, but not on the intersection lines with the other 2 planes.

Also define ${R_{i,j}}^+$ as the alcove bounded by $H_{1,j+1}, H_{2,i+j+1}, H_{3,i+1}$ and say $\lambda \in {R_{i,j}}^+$ if λ is in the interior of the alcove or if λ lies on $H_{1,j+1}$ or $H_{3,i+1}$, but not on the intersection lines with the plane $H_{2,i+j+1}$.

Finally, define the interior of any region to be the region minus the points on the boundaries themselves, denote this as ${R_{i,j}}^{\text{int}}$.

Figure 3 shows off these new definitions. Each ${R_{i,j}}^{+/-}$ refers to a triangle on the diagram with base 5 units, (with a green, blue and red edge) and for reference the origin is the black dot near the bottom left-hand corner.

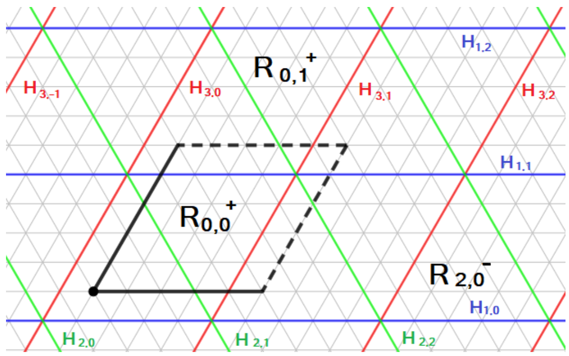


Figure 3: Planes and Rhombi

We observe that a weight $\lambda \in R_{0,0} \iff \lambda \in X_p(T)$. The other $R_{\alpha,\beta}$ can be thought of as translating $R_{0,0}$ $p\alpha$ units in the "a" direction ("a" is the direction used in Figure 1 and $p\beta$ units n the "$-$c" direction. So the (α, β) are coordinates interpreted very similarly to how you would (x,y) in the 2d-Cartesian coordinate system.

Remark 1.16. Thinking in terms of rhombi is very useful when using Steinberg's theorem as our λ_0 (ie the p^0 weight in the tensor) is just λ's position relative to the south-westmost point in the rhombus $R_{i,j}$ where λ lives (relative to $R_{i,j}$'s origin if you will). In fact, if we define:

$$R_{i,j}{}^{(n)} := \bigcup_{\alpha,\beta<p^n} R_{ip^n+\alpha\,,\,jp^n+\beta}$$

then for $k \geq 1$ the weight λ_k in the tensor decomposition is $i(1,0,0)+j(1,1,0)$ where i,j are the values such that λ lives in $R_{i,j}{}^{(k-1)}$. (Just to note: $(1,0,0)$ is a basis vector in the "a" direction, $(1,1,0)$ is a basis vector in the "$-$c" direction, that's why we are using them).

Definition 1.17. ("below": A partial ordering on $X_+(T)$). Suppose λ, $\mu \in X_+(T)$, then we say $\lambda \uparrow \mu$ if there is a sequence $\lambda = \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_i \leq ... \leq \lambda_n = \mu$ such that for each i we have that $\lambda_{i+1} = s_{a,n}\lambda_i$ for some $a \in \{1,2,3\}$, $n \in \mathbb{Z}$, where $s_{a,n}\lambda_i$ denotes the weight obtained by reflection of λ_i in the plane $H_{a,n}$.
"$\geq$" is the ordering defined by $(a,b,c) \geq (d,e,f) \iff a \geq d, c \leq f$ and $a+b+c = d+e+f$.

Figure 4 shows this definition geometrically, in terms of the diagram we say a weight is "below" the red dotted weight in the diagram if it either lies on the bold black lines beaming out of the point or if it lies origin-side of these lines.

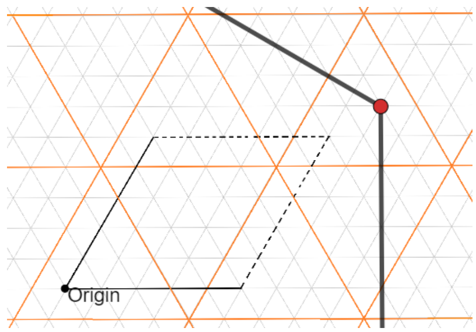


Figure 4: Geometric picture of "below"

Remark 1.18. The next definition concerns the algebraic versions of the representations F and W.

Theorem 1.19. *(The Strong Linkage Principle) [2, Theorem 3.16]. If [$F(\lambda)$] is in the decomposition of [$W(\mu)$] with some non-zero multiplicity then $\lambda \uparrow \mu$. Also $F(\mu)$ appears in the decomposition of $W(\mu)$ with multiplicity 1 always.*

1.6 Weyl dimension formula

The Weyl dimension formula[2] gives us an explicit way to calculate the dimension of any finite dimensional representation. In the case of $GL_3(\mathbb{F}_p)$ representations we have that: [4, Section 10.5]

$$\dim(W(a,b,c)) = \frac{1}{2}(a-b+1)(b-c+1)(a-c+2)$$

This formula comes in useful when trying to find the coefficients from Strong Linkage of a representation $W(\lambda)$ as it gives us a way to assign a numerical value to each representation to form an equation to involve the multiplicities. (of course, we'll need much more than this in general but it's a good start.) The next 3 subsections focus on techniques we will use to help find these coefficients.

1.7 Schur polynomials

This subsection concerns algebraic representations throughout. If V is an algebraic representation of $GL_3(\mathbb{C})$ then:

$$\text{char}_V(x,y,z) := tr(\text{diag}(x,y,z))|V$$

This is a homogeneous polynomial in x, y, z with integer coefficients (not taken mod p). Lucky for us this follows us into the world of $GL_3(\mathbb{F}_p)$ algebraic representations and in fact:

$$\text{char}(W(\lambda))(x,y,z) = S_\lambda(x,y,z)$$

[2] The Weyl-dimension formula is in fact a specific evaluation of the Weyl-character formula, which can be read about here: [3, Lecture 24]

Where S_λ is the Schur-polynomial. Expanding on this we have that:

$$\mathrm{char}(W(\lambda) \otimes W(\mu)) = S_\lambda \cdot S_\mu$$

where "$\cdot$" denotes regular multiplication of polynomials. For λ in the upper alcove we have that

$$\mathrm{char}(F(\lambda)) = \mathrm{char}\Big(W(\lambda) - W(s_{2,1}\lambda)\Big) = S_\lambda - S_{s_{2,1}\lambda}$$

Also, we have for $k = p^r$ that

$$\mathrm{char}(W(\lambda)^{(k)}) = S_\lambda(x^k, y^k, z^k).$$

Finally, note that we usually exclude the (x, y, z) and only write it if the variables are raised to a power which is not 1.

There are many formula out there for calculating these polynomials but to give an example, there is a formula given by Jacobi's bi-alternant formula [5, Section 1] which says that for x, y, z distinct:

$$S_{(a,b,c)}(x, y, z) = \frac{\det\begin{pmatrix} x^{a+2} & y^{a+2} & z^{a+2} \\ x^{b+1} & y^{b+1} & z^{b+1} \\ x^c & y^c & z^c \end{pmatrix}}{\det\begin{pmatrix} x^2 & y^2 & z^2 \\ x^1 & y^1 & z^1 \\ 1 & 1 & 1 \end{pmatrix}}$$

Remark 1.20. We in fact have that $\dim(W(\lambda)) = S_\lambda(1, 1, 1)$ so we could derive the dimension formula using this, Jacobi's bialternant formula and L'Hôpital's rule if we wanted.

Example 1.21. Using some of what we've discussed so far let's find the Schur polynomial of the representation $F(16, 8, 0)$ for $p = 5$. Firstly, F lies outside the p-restricted region so we want to use Steinberg's theorem to write F as a tensor product of Fs with weights in the p-restricted region, so either via considering rhombi or just inspection:

$$F(16, 8, 0) = F(2, 1, 0)^{(5)} \otimes F(6, 3, 0)$$

Now $F(2, 1, 0)$ is in the lower alcove so $F(2, 1, 0) = W(2, 1, 0)$. However, $F(6, 3, 0)$ lives in the upper alcove so we must use SESs to get:

$$[F(6, 3, 0)] = [W(6, 3, 0)] - [F(3, 3, 3)] = [W(6, 3, 0)] - [W(3, 3, 3)]$$

. Now;

$$[F(16, 8, 0)] = [W(2, 1, 0)^{(5)}] \otimes [(W(6, 3, 0)] - [W(3, 3, 3))]$$

Keeping in mind that we must keep our representations algebraic, that is the

Frobenius twist on $W(2,1,0)$ cannot be "dropped". Using the results above for the characters of $W(\lambda)$ we conclude:

$$\mathrm{char}(F(16,8,0)) = S_{(2,1,0)}(x^5, y^5, z^5) \cdot \Big(S_{(6,3,0)}(x,y,z) - S_{(3,3,3)}(x,y,z)\Big)$$

In theory we could, for any $\lambda \in X_+(T)$, use Strong Linkage Principle to write $W(\lambda) = F(\lambda) + \sum_{i=1}^{n} \alpha_i F(\lambda_i)$ then use the same method as in the example to find the algebraic characters of each F to get an equation involving a linear combination of Schur polynomials. Then by picking 'n' sets of actual x, y, z values we could use Jacobi's formula (or any formula for the Schur polynomials you like) to evaluate them and solve the $n \times n$ linear system for our coeffs α_i. Although this would work it seems like a very long winded method (it is!), we would like a faster and less time consuming method than this. Another way to view Schur polynomials is the following:

Definition 1.22. Let $\mu = (\mu_1, \mu_2, \mu_3)$ and define a monomial symmetric polynomial $m_\mu(x,y,z)$ to be the sum of all monomials in $x^\alpha y^\beta z^c$ where (a,b,c) ranges over all distinct permutations of (μ_1, μ_2, μ_3). Then for λ, a 3-tuple, (ie weight of a representation of GL_3) and μ ranging over all 3 tuples with entries weakly decreasing we get: [3, Appendex A, A.19]

$$S_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu}$ are these so-called Kostka numbers. So perhaps shorter method is to find the linear combination of Schur polynomials and compare coefficients of each term using Kostka numbers rather than evaluate them at specific values. This brings us to our next subsection.

1.8 Young diagrams,Young tableaux and Kostka Numbers

A Young diagram is a finite collection of boxes arranged in rows such that the row lengths are weakly decreasing as we go down. A young diagram for a weight $(a,b,c) \in X_+(T)$ is a diagram with "a" boxes in the first row, "b" boxes in the second row and "c" boxes in the third row [3, Lecture 4.1], we say the diagram has *shape* (a,b,c)
Example:

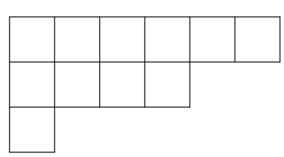


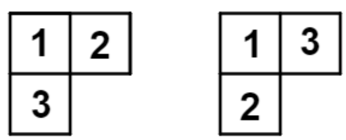
Figure 5: The Young Diagram for $\lambda = (6,4,1)$

Definition 1.23. (Young tableau) A Young tableau is obtained by filling the boxes of a Young diagram using symbols from a well ordered set (usually $\mathbb{N}$). A tableau is standard if the entries in each row and each column are increasing. A tableau is semi-standard if the entries weakly increase along each row and strictly increase down each column. The *weight* of a tableau is a tuple telling us how many times each number appears in the tableau.

Definition 1.24. (Kostka number) [3, Appendix A.1]. Given two weights λ and μ the Kostka number $K_{\lambda\mu}$ is equal to the number of ways one can fill a Young diagram of shape λ with weight μ.

From this we get the following: [3, Section 4.3]. $K_{\lambda\mu} \neq 0 \iff \lambda_1 \geq \mu_1$, $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$ and $\lambda_1 + \lambda_2 + \lambda_3 = \mu_1 + \mu_2 + \mu_3$. We will denote this relation by $\lambda \trianglerighteq \mu$, it is known as dominance ordering. We also see that $K_{\lambda\lambda} = 1$

Example 1.25. Let us find $K_{(2,1,0),(1,1,1)}$. That is, how many ways can we fill a Young diagram of shape $(2,1,0)$ with weight $(1,1,1)$ ie full it with one "1", one "2" and one "3". We see below there are 2 ways so $K_{(2,1,0),(1,1,1)} = 2$.



To put this in context for our situation, this would tell us that the coefficient of xyz in the Schur polynomial $S_{2,1,0}(x,y,z)$ is 2.

Proposition 1.26. *Assume $K(a,b,0,c,d,e) \neq 0$, that is it satisfies the iff statement above (where I will write $K(a,b,0,c,d,e)$ throughout instead of $K_{(a,b,0),(c,d,e)}$ for ease of reading since it's clear what is meant), then:*

i) if $c \geq d \geq e \geq b$ we have $K(a,b,0,c,d,e) = b+1$

ii) if $c \geq d \geq b \geq e$ we have $K(a,b,0,c,d,e) = e+1$

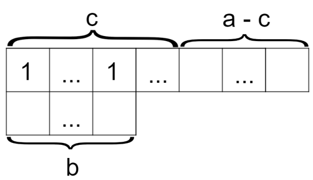
iii) if $c \geq b \geq d \geq e$ we have $K(a,b,0,c,d,e) = a-c+1$

iv) if $b \geq c \geq d \geq e$ we have $K(a,b,0,c,d,e) = a-b+1$

v) if we have $K(a,b,n,c,d,e)$ with $c \geq d \geq e \geq n$ we get the equality $K(a,b,n,c,d,e) = K(a-n,b-n,0,c-n,d-n,e-n)$ which can be found using i) - iv).

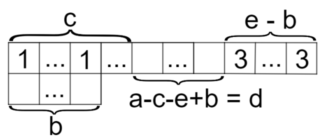
Proof. We know for all parts of the proof that the 1s must fill the leftmost 'c' boxes on the first row since if they didn't there would be some row and/or column that didn't satisfy the increasing conditions.

i) We have c (number of 1s) $\geq b$ (boxes on second row) so after filling in all our 1s the Young diagram is:



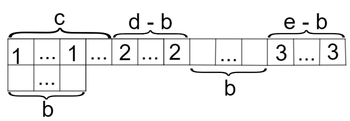
we have no 1s left, d 2s left, e 3s left.

Since e (number of 3s) $\geq b$ (boxes on second row) the rightmost $e - b$ spaces of the first row must be 3s, so after filling these in our diagram becomes:



we have no 1s left, d 2s left, b 3s left.

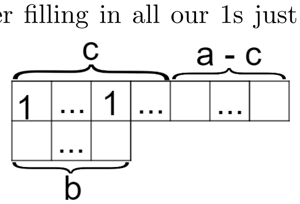
d (number of 2s) $\geq b$ (number of boxes in the second row) so the leftmost $d - b$ spaces of the first row after the list of 1s must be 2s, filling these in gives us:



we have no 1s left, b 2s left, b 3s left.

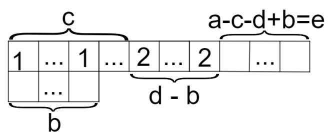
The scenario we are now in is a recurring one throughout these proofs so I will explain it clearly here: we're just left to fill in b boxes along the top and 'b' boxes along the bottom with 2s and 3s such that the list is weakly increasing when reading from left to right with the top and bottom row empty boxes completely vertically disjoint so we need not worry about columns increasing anymore, this is guaranteed to happen no matter where we place the remaining numbered boxes. Now, for every way of labelling the top row with 2s and 3s there is exactly one way to label the bottom row, that is just to write the remaining 2s down followed by the remaining 3s, so the answer boils down to finding out how many ways we can fill a b-tuple with 2s and 3s such that the elements weakly increase, of course, there is $b + 1$ ways to do this ie. $(2, 2, .., 2)$, $(2, ..., 2, 3)$, $(2, ..., 2, 3, 3)$, ..., $(2, 3, ..., 3)$, $(3, ..., 3)$.

ii) we have c (number of 1s) $\geq b$ (boxes on second row) once again so after filling in all our 1s just like before our Young diagram looks like so:



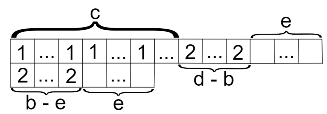
we have no 1s left, d 2s left, e 3s left.

The number of 3s left, 'e' $\leq$ 'b', the number of spaces on the second row, so no worries about overflow of 3s this time. We have d (number of 2s) $\geq$ b (number of boxes in second row) so the leftmost $d - b$ boxes on the first row directly after the list of 1s must be 2s. On the diagram this gives:



we have no 1s left, b 2s left, e 3s left.

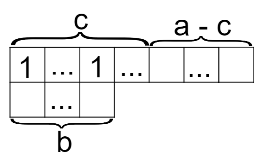
Now, b (number of 2s left) $\geq$ e (boxes left on first row) so the leftmost $b - e$ entries on the 2^{nd} row must be 2s. The diagram is:



we have no 1s left, e 2s left, e 3s left

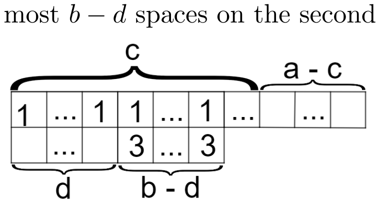
Now we're back to the familiar game of filling 'e' spaces in each row with 2s and 3s such that it's weakly increasing when reading from left to right. Using the same argument as i), the number of ways to do this is $e + 1$.

iii) Once again c (number of 1s) $\geq$ b (boxes on second row) so the diagram starts as usual as:



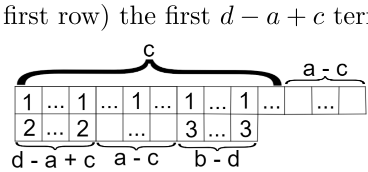
we have no 1s left, d 2s left, e 3s left.

Now, d (number of 2s) $\leq$ b (number of boxes in second row) so the rightmost $b - d$ spaces on the second row must be 3s. The diagram is now:



with no 1s left, d 2s left, $a - c$ 3s left.

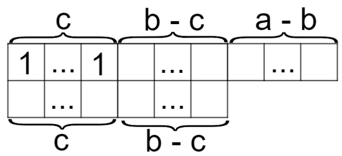
Observe, $a + b = c + d + e \iff a - c = d + e - b \implies a - c \leq d$ since $e \leq b$.
So since d (number of 2s left) $\geq$ $a - c$ (the number of boxes left on the first row) the first $d - a + c$ terms of the 2^{nd} row are 2, hence we get:



no 1s left, $a - c$ 2s left, $a - c$ 3s left

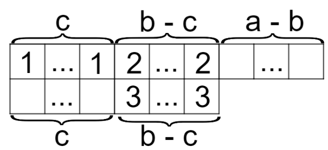
Once again we play the same game but with $a - c$ boxes, the number of ways to do this is $a - c + 1$.

iv) If c (number of 1s) $\leq$ b (boxes on second row) our diagram looks like so:



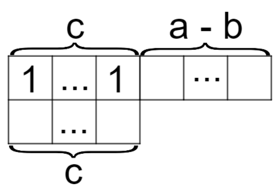
we have no 1s left, d 2s left, e 3s left.

There are $b - c$ columns where we are forced to place a 2 in the top row and a 3 directly below it, seen below:



no 1s, $d - b + c$ 2s left, $e - b + c$ 3s left.

Ignoring these $b - c$ columns we get:

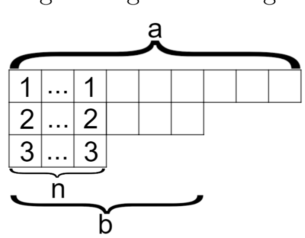


no 1s, $d - b + c$ 2s left, $e - b + c$ 3s left.

But this is exactly the diagram we would draw if we wanted to calculate $K(a-b+c, c, 0, c, d-b+c, e-b+c)$. Furthermore, $c \geq c \geq d-b+c, e-b+c$ (since $e \leq d \leq b$) so conditions are met for iii) to be used. So

$$K_{(a,b,0,c,d,e)} = K_{(a-b+c,c,0,c,d-b+c,e-b+c)} = a - b + 1$$

v) We go straight to a diagram and the explanation follows underneath:



$c - n$ 1s left, $d - n$ 2s left, $e - n$ 3s left.

It's clear that the 1,2,3s in the leftmost n columns are forced there in order for the columns to be increasing. So we can effectively ignore these columns and we immediately get that

$$K(a, b, n, c, d, e) = K(a - n, b - n, 0, c - n, d - n, e - n)$$

□

If we check the iff statement and indeed find $\lambda \trianglerighteq \mu$, then it must be the case that $\mu_3 \geq \lambda_3$. Therefore for any weights $\lambda, \mu \in X_+(T)$ we must fall into one of the five cases in the proposition. Using this Proposition and comparing coefficients of Schur functions seems to be a nicer way to find the Strong Linkage Principle coefficients than having to evaluate the polynomials at multiple different values if doing this by hand.

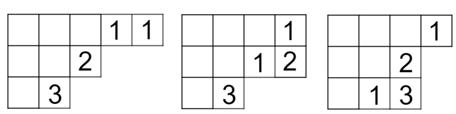
1.9 The Littlewood-Richardson rule.

To finish this section off we will introduce a combinatorical rule which will allow us to decompose the product of Schur functions (or in our case a tensor product of W representations) into a linear combination of other Schur functions (W representations). This will be used by us to re-express the tensor products that arise from Steinberg's tensor theorem as a sum of representations.

Given two Schur functions $S_\lambda \cdot S_\mu$ the Littlewood Richardson rule says the coefficient of S_ν (the co-efficient is usually denoted as $N_{\lambda\mu\nu}$) is the number of ways we can form the Young diagram with shape ν starting from a shape λ diagram via a strict μ-expansion [3, Appendix A.1]. This means if $\mu = (\mu_1, \mu_2, \mu_3)$ then we must add μ_1 1 boxes to the diagram, μ_2 2 boxes and μ_3 3 boxes. From this we see if $\nu_1+\nu_2+\nu_3 \neq \lambda_1+\lambda_2+\lambda_3+\mu_1+\mu_2+\mu_3 \implies N_{\lambda\mu v} = 0$. We also have the condition that when reading along each row from left to right the numbers weakly increase and when reading down each column the numbers strictly increase. As well as this we have the condition that the rows cannot increase in length (this is good as otherwise the weight wouldn't lie in the dominant region.) Lastly, we need the expansion to be *strict*, what this means is if we read the numbers in the boxes from right to left starting from the top row we require that for any $1 \leq \alpha \leq \mu_1 + \mu_2 + \mu_3$ the first α boxes have the property that the number of 1s appearing $\geq$ number of 2s appearing $\geq$ number of 3s appearing. We say that the concatenation of the boxes reading right to left and starting in the top row forms a *lattice word*. This definitely warrants an example.

Example 1.27. Express $S_{(3,2,1)} \cdot S_{(2,1,1)}$ as a linear combination of other Schur functions.

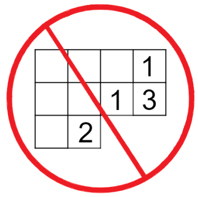
The possible strict $(2,1,1)$-expansions of the Young diagram of shape $(3,2,1)$ with 3 or less rows are as follows:



So we have that $S_{(3,2,1)} \cdot S_{(2,1,1)} = S_{(5,3,2)} + S_{(4,4,2)} + S_{(4,3,3)}$ + some other Schur functions with partition size 4 or more. In the case of representations of GL_3 we have that W(λ) where λ is a partition of length 4 or more is 0 so relating this example back to our situation we have:

$$W(3,2,1) \otimes W(2,1,1) = W(5,3,2) + W(4,4,2) + W(4,3,3)$$

Finally, before going onto the next section here is a non-example as I think it's important to see one of these too:



This doesn't work, as although it satisfies the increasing condition put on the rows and columns, it's entries when put together reading right to left and starting at the top, ie 1312, doesn't form a lattice word since the prefix "13" contains more 3s than 2s..

2 Tensor products involving irreps for small primes

This section involves the explicit calculations involving the irreps of $GL_3(\mathbb{F}_p)$ for small p, in particular, for each $\lambda = (a, b, c) \in X_p(T)$ we will calculate the following:

i) $[F(\lambda) \otimes F(p-1, 0, 0)]$

ii) $[F(\lambda) \otimes F(p-1, p-1, 0)]$

iii) $[F(\lambda) \otimes F(2(p-1), p-1, 0)]$

We also define the following in the Grothendieck group:

- $\sigma_0 = F(0, 0, 0)$
- $\sigma_1 = F(p-1, p-1, 0) + F(p-1, 0, 0) - 2F(0, 0, 0)$
- $\sigma_2 = F(0, 0, 0) - \Big(F(p-1, p-1, 0) + F(p-1, 0, 0)\Big) + F(2(p-1), p-1, 0)$

We will then look at how each of these σ_i tensor with the irreps and have some discussion about that.

2.1 For $p = 2$

Note: please see Appendix A (if you wish) for the full working out for each of the 12 tensor products. The results are as follows:

1. $[F(0, 0, 0)]$:

   i) $[F(0, 0, 0) \otimes F(1, 0, 0)] = [F(1, 0, 0)]$

   ii) $[F(0, 0, 0) \otimes F(1, 1, 0)] = [F(1, 1, 0)]$

   iii) $[F(0, 0, 0) \otimes F(2, 1, 0)] = [F(2, 1, 0)]$

2. $[F(1, 0, 0)]$:

   i) $[F(1, 0, 0) \otimes F(1, 0, 0)] = [F(1, 0, 0)] + 2[F(1, 1, 0)]$

   ii) $[F(1, 0, 0) \otimes F(1, 1, 0)] = [F(2, 1, 0)] + [F(0, 0, 0)]$

iii) $[F(1,0,0) \otimes F(2,1,0)] = [F(2,1,0)] + 3[F(1,0,0)] + 2[F(1,1,0)] + [F(0,0,0)]$

3. $[F(1,1,0)]$:

   i) $[F(1,1,0) \otimes F(1,0,0)] = [F(2,1,0)] + [F(0,0,0)]$

   ii) $[F(1,1,0) \otimes F(1,1,0)] = [F(1,1,0)] + 2[F(1,0,0)]$

   iii) $[F(1,1,0) \otimes F(2,1,0)] = [F(2,1,0)] + 3[F(1,1,0)] + 2[F(1,0,0)] + [F(0,0,0)]$

4. $[F(2,1,0)]$:

   i) $[F(2,1,0) \otimes F(1,0,0)] = [F(2,1,0)] + 3[F(1,0,0)] + 2[F(1,1,0)] + [F(0,0,0)]$

   ii) $[F(2,1,0) \otimes F(1,1,0)] = [F(2,1,0)] + 3[F(1,1,0)] + 2[F(1,0,0)] + [F(0,0,0)]$

   iii) $[F(2,1,0] \otimes F(2,1,0)] = 3[F(2,1,0)] + 6[F(1,1,0)] + 6[F(1,0,0)] + 4[F(0,0,0)]$

For $p = 2$ we have: $\sigma_0 = F(0,0,0)$
$\sigma_1 = F(1,1,0) + F(1,0,0) - 2F(0,0,0)$
$\sigma_2 = F(0,0,0) - \big(F(1,1,0) + F(1,0,0)\big) + F(2,1,0)$

We want to calculate $\sigma_i \otimes F(\lambda_j)$ $\forall$ $i \in \{1,2,3\}$, $\lambda_j \in X_2(T)$ and put all this information in a table.
We will not explicitly write out the results of the sigma tensors since they can be read off the information in the table by taking the correct linear combination of values from our above results, but they can be seen on Table 1:

Definition 2.1. If $\mathbf{a} = \sum_{\lambda \in X_p(T)} a_\lambda F(\lambda)$ and $\mathbf{b} = \sum_{\lambda \in X_p(T)} b_\lambda F(\lambda)$ in the Grothendieck group K, then define the inner product on a, b as:

$$< \mathbf{a}, \mathbf{b} >= \sum_{\lambda \in X_p(T)} a_\lambda b_\lambda$$

Definition 2.2. If $\mathbf{a} = \sum_{\lambda \in X_p(T)} a_\lambda F(\lambda) \in K$ then we say $\mathbf{a} \trianglerighteq 0$ if the inner product satisies the following:

$$< \mathbf{a}, F(\sigma_i) \otimes F(\lambda) > \ \geq 0 \ \forall \ i \in \{1,2,3\}, \lambda \in X_p(T)$$

That is, the linear combinations of the a_i are all non-negative where the coefficents in these linear combinations are the rows from the table. Two things to note, firstly:

$$< \mathbf{a}, F(\sigma_0) \otimes F(\lambda) > \quad = a_\lambda$$

F(weight) $\otimes$ F(weight)	$(0,0,0)$	$(1,0,0)$	$(1,1,0)$	$(2,1,0)$
$(0,0,0)\otimes(0,0,0)$	1	0	0	0
$(0,0,0)\otimes(1,0,0)$	0	1	0	0
$(0,0,0)\otimes(1,1,0)$	0	0	1	0
$(0,0,0)\otimes(2,1,0)$	0	0	0	1
$(1,0,0)\otimes(1,0,0)$	0	1	2	0
$(1,0,0)\otimes(1,1,0)$	1	0	0	1
$(1,0,0)\otimes(2,1,0)$	1	3	2	1
$(1,1,0)\otimes(1,1,0)$	0	2	1	0
$(1,1,0)\otimes(2,1,0)$	1	2	3	1
$(2,1,0)\otimes(2,1,0)$	4	6	6	3
$\sigma_1\otimes(0,0,0)$	-2	1	1	0
$\sigma_2\otimes(0,0,0)$	1	-1	-1	1
$\sigma_1\otimes(1,0,0)$	1	-1	2	1
$\sigma_2\otimes(1,0,0)$	0	3	0	0
$\sigma_1\otimes(1,1,0)$	1	2	-1	1
$\sigma_2\otimes(1,1,0)$	0	0	3	0
$\sigma_1\otimes(2,1,0)$	2	5	5	0
$\sigma_2\otimes(2,1,0)$	2	1	1	2

Table 1: Table displaying multiplicity of irreps in tensor calculations for $p = 2$.

so if $\mathbf{a} \trianglerighteq 0$ then all co-effs are non-negative. Also, the rows with all positive entries don't tell us anything about what the a_i can be, so we only extract the rows from the table with at least one negative entry.

Denote the co-efficents of each $F(\lambda)$ as follows:
co-eff of $F(0,0,0) = a_0$
co-eff of $F(1,0,0) = a_1$
co-eff of $F(1,1,0) = \overline{a_1}$ (to preserve some symmetry)
co-eff of $F(2,1,0) = a_2$

Then the four "interesting" rows of the table with at least one negative value give us the following 4 inequalities:

1. $a_1 + \overline{a_1} - 2a_0 \geq 0$
2. $a_0 - a_1 - \overline{a_1} + a_2 \geq 0$
3. $a_0 - a_1 + 2\overline{a_1} + a_2 \geq 0$
4. $a_0 + 2a_1 - \overline{a_1} + a_2 \geq 0$

Now, if we could find four different 4-dimensional vectors (v_1, v_2, v_3, v_4) with each $v_i >= 0$such that for each inequality above if we set $(a_0, a_1, \overline{a_1}, a_2) = (v_1, v_2, v_3, v_4)$ it gives us the chosen inequality > 0 and the others $= 0$ then we can express every a s.t. $a \trianglerighteq 0$ as a positive linear combination of these vectors and their associated representations. One such way to attempt to find 4 such vectors is to set each inequality equal to b_i and realise it's just a 4×4 non-degenerate linear system. So after inverting the coefficient matrix and multiplying it with $(b_1, b_2, b_3, b_4)^T$ that;

$$\begin{pmatrix} a_0 \\ a_1 \\ \overline{a_1} \\ a_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}b_1 - \frac{1}{3}b_2 + \frac{1}{6}b_3 + \frac{1}{6}b_4 \\ -\frac{1}{3}b_2 + \frac{1}{3}b_4 \\ -\frac{1}{3}b_2 + \frac{1}{3}b_3 \\ \frac{1}{2}b_1 + \frac{2}{3}b_2 + \frac{1}{6}b_3 + \frac{1}{6}b_4 \end{pmatrix}$$

We see that setting $b_1 = b_2 = b_4 = 0$ and $b_3 = 6$, that our required vector is (1,0,2,1). Similarly, setting $b_1 = b_2 = b_3 = 0$ and $b_4 = 6$ gives us $(1, 2, 0, 1)$. This is as far as we can get since setting $b_2 = b_3 = b_4 = 0$ and $b_1 > 0$ we get that $a_0 < 0$ which isn't allowed.

However, all this working out was not for nothing, as there is something else we can ask about these tensors with the σ_i, namely, for which $\lambda \in X_2(T)$ is it the case that it's coefficient, α_i, never appears with a negative sign in any tensor product $\sigma_i \otimes F(\mu) >$, $\sigma \in \{0, 1, 2\}$, $\mu \in X_2(T)$?

Reading this off the table, the only λ with this property is $(2, 1, 0)$, with knowledge of what's to come, notice how all components of this weight are distinct.

2.2 For $p = 3$

We will do one more explicit example as it's clear the number of tensors increases extremely quickly. For $p = 3$ we wish to carry out the following three tensors of each $\lambda \in X_3(T)$:

i) $[F(\lambda) \otimes F(2, 0, 0)]$

ii) $[F(\lambda) \otimes F(2, 2, 0)]$

iii) $[F(\lambda) \otimes F(4, 2, 0)]$

Remark 2.3. Observe that $W(\lambda + (x, x, x)) \otimes W(\mu + (y, y, y)) = W(\lambda) \otimes det^x \otimes W(\mu) \otimes det^y = W(\lambda) \otimes W(\mu) \otimes det^{x+y}$ since the tensors of the 1 dimensional representations commute as multiplication in $\mathbb{F}_p$ commutes. So if $W(\lambda) \otimes W(\mu) = \sum_i W(\Gamma_i)$ then $W(\lambda + (x, x, x)) \otimes W(\mu + (y, y, y)) = \sum_i W(\Gamma_i + (x+y, x+y, x+y))$

Alternatively we can argue it as follows. Given $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ then the coefficient of a weight $W(\pi)$ in the sum expression of $W(\lambda) \otimes W(\mu)$ is just asking how many ways π can be attained by strict μ-expanding a Young diagram of shape λ. Notice that if $\lambda_3 > 0$ then the leftmost λ_3 columns don't affect our game so they can be ignored for now.

Then we are left with $\lambda^* := (\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0)$ and we will add $(\lambda_3, \lambda_3, \lambda_3)$ to the weights of whatever answers we get. Now consider the μ-expansion, the argument is that if we wish to place '2' or '3' in the top row we definitely must place a '1' first if the 1-character prefix stands any chance of being a lattice

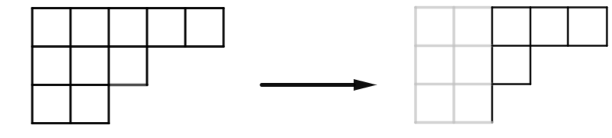


Figure 6: Supporting diagram 1

word, however rows cannot be decreasing when reading from left to right so only 1s are allowed in row 1. Now with this in mind let us try place a '3' in row 2, again we must place a '2' first so we get a lattice word, but the increasing row argument means we cant place a '3', finally we have only row 3 left, and all our 3s are forced there. Now if we put μ_3 3s in the 3rd row we must place at least μ_3 2s in the second row and at least μ_3 1s in the first row so as to get a lattice word. In conclusion these elements are fixed, so let's ignore the new columns formed by the placement of the 3s, as seen on Figure 7: So now we have to

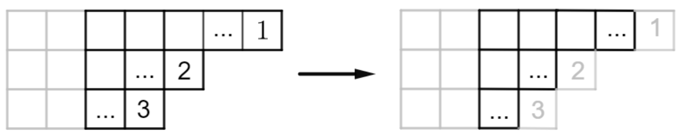


Figure 7: Supporting diagram 2

$(\mu_1 - \mu_3, \mu_2 - \mu_3, 0)$−expand a Young diagram of shape $(\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0)$ and then add $(\lambda_3 + \mu_3, \lambda_3 + \mu_3, \lambda_3 + \mu_3)$ to any weights of diagrams we get. This is exactly the same result as we got above.

Therefore we only need to explicitly carry out these calculations for 9 of the irreps (9 of them with unique positions on a diagram), as the row of the table for each corresponding irrep with twist $(1, 1, 1)$ will be given by swapping the values to the left and right of the forward-slashes in the table (Table 2). Let

$$A(\mu) = \sum_1^9 a_i F(\lambda_i) + \sum_1^9 b_i F(\lambda_i + (1, 1, 1))$$

where the λ_i are the irreducible weights, labelled such that if we plotted them on a diagram we would start in the bottom left with $\lambda_1 = (0, 0, 0)$ and read across the weights from left to right then go up to the next row and start read it from left to right etc, ie we're setting $\lambda_1 = (0, 0, 0), \lambda_2 = (1, 0, 0), ..., \lambda_4 = (1, 1, 0), ..., \lambda_7 = (2, 2, 0), ..., \lambda_9 = (4, 2, 0)$. We would like to ask the same questions as we did for $p = 2$. Firstly, let's see if we can find find a set of 18-dimensional vectors such that if we want any one of the inequalities formed from "interesting" rows in the table greater than 0 and the others $= 0$, there is a vector in the set which will do that.

$F(weight) \otimes$ $F(weight)$	$(0,0,0)/$ $(1,1,1)$	$(1,0,0)/$ $(2,1,1)$	$(2,0,0)/$ $(3,1,1)$	$(1,1,0)/$ $(2,2,1)$	$(2,1,0)/$ $(3,2,1)$	$(3,1,0)/$ $(4,2,1)$	$(2,2,0)/$ $(3,3,1)$	$(3,2,0)/$ $(4,3,1)$	$(4,2,0)/$ $(5,3,1)$
$(1,0,0) \otimes (2,0,0)$	0/1	1/0	0	0	2/0	0	0	0	0
$(2,0,0) \otimes (2,0,0)$	0	0	1/0	1/0	0	1/0	2/0	0	0
$(1,1,0) \otimes (2,0,0)$	0	0/1	0	0	0	1/0	0	0	0
$(2,1,0) \otimes (2,0,0)$	0/2	0	0	0/1	1/0	0	0	2/0	0
$(3,1,0) \otimes (2,0,0)$	2/0	0/2	0	2/0	0/4	1/0	1/0	0	1/0
$(2,2,0) \otimes (2,0,0)$	2/0	0	0	0	0/1	0	0	0	1/0
$(3,2,0) \otimes (2,0,0)$	0/4	1/0	0	0/1	2/0	0/3	0/1	1/0	0
$(4,2,0) \otimes (2,0,0)$	5/0	0/3	3/0	1/0	0/4	1/0	2/0	0/3	1/0
$(1,0,0) \otimes (2,2,0)$	0	0	0	0/1	0	0	0	1/0	0
$(1,1,0) \otimes (2,2,0)$	1/0	0	0	1/0	0/2	0	0	0	0
$(2,1,0) \otimes (2,2,0)$	0/2	1/0	0	0	1/0	0/2	0	0	0
$(3,1,0) \otimes (2,2,0)$	4/0	0/1	1/0	1/0	0/2	1/0	0	0/3	0
$(2,2,0) \otimes (2,2,0)$	0	0/1	2/0	0	0	0	1/0	0/1	0
$(3,2,0) \otimes (2,2,0)$	0/2	2/0	0/1	0/2	4/0	0	0	1/0	0/1
$(4,2,0) \otimes (2,2,0)$	5/0	0/1	2/0	3/0	0/4	3/0	3/0	0/1	1/0
$(1,0,0) \otimes (4,2,0)$	0/4	0	0	0/1	2/0	0/3	0	1/0	0
$(1,1,0) \otimes (4,2,0)$	4/0	0/1	0	0	0/2	1/0	0	0/3	0
$(2,1,0) \otimes (4,2,0)$	0/5	4/0	0/2	0/4	7/0	0/2	0/2	2/0	0/1
$(3,1,0) \otimes (4,2,0)$	15/0	0/6	4/0	6/0	0/12	7/0	4/0	0/6	1/0
$(3,2,0) \otimes (4,2,0)$	0/15	6/0	0/4	0/6	12/0	0/6	0/4	7/0	0/1
$(4,2,0) \otimes (4,2,0)$	25/0	0/10	8/0	10/0	0/20	10/0	8/0	0/10	4/0
$\sigma_1 \otimes (0,0,0)$	-2/0	0	1/0	0	0	0	1/0	0	0
$\sigma_2 \otimes (0,0,0)$	1/0	0	$-1/0$	0	0	0	$-1/0$	0	1/0
$\sigma_1 \otimes (1,0,0)$	0/1	$-1/0$	0	0/1	2/0	0	0	1/0	0
$\sigma_2 \otimes (1,0,0)$	0/3	0	0	0	0	0/3	0	0	0
$\sigma_1 \otimes (2,0,0)$	2/0	0	$-1/0$	1/0	0/1	1/0	2/0	0	1/0
$\sigma_2 \otimes (2,0,0)$	3/0	0/3	3/0	0	0/3	0	0	0/3	0
$\sigma_1 \otimes (1,1,0)$	1/0	0/1	0	$-1/0$	0/2	1/0	0	0	0
$\sigma_2 \otimes (1,1,0)$	3/0	0	0	0	0	0	0	0/3	0
$\sigma_1 \otimes (2,1,0)$	0/4	1/0	0	0/1	0	0/2	0	2/0	0
$\sigma_2 \otimes (2,1,0)$	0/1	3/0	0/2	0/3	6/0	0	0/2	0	0/1
$\sigma_1 \otimes (3,1,0)$	6/0	0/3	1/0	3/0	0/6	0	1/0	0/3	1/0
$\sigma_2 \otimes (3,1,0)$	9/0	0/3	3/0	3/0	0/6	6/0	3/0	0/3	0
$\sigma_1 \otimes (2,2,0)$	2/0	0/1	2/0	0	0/1	0	$-1/0$	0/1	1/0
$\sigma_2 \otimes (2,2,0)$	3/0	0	0	3/0	0/3	3/0	3/0	0	0
$\sigma_1 \otimes (3,2,0)$	0/6	3/0	0/1	0/3	6/0	0/3	0/1	0	0/1
$\sigma_2 \otimes (3,2,0)$	0/9	3/0	0/3	0/3	6/0	0/3	0/3	6/0	0
$\sigma_1 \otimes (4,2,0)$	10/0	0/4	5/0	4/0	0/8	4/0	5/0	0/4	0
$\sigma_2 \otimes (4,2,0)$	15/0	0/6	3/0	6/0	0/12	6/0	3/0	0/6	3/0

Table 2: Multiplicity of irreps in decomposition of $F(\lambda) \otimes F(\mu)$, for the various λ and μ, and $\sigma_i \otimes F(\lambda)$, $\lambda \in X_3(T)$ for p=3.

From the table we get:

$$-2a_1 + a_3 + a_7 \geq 0 \quad (1)$$
$$a_1 - a_3 - a_7 + a_9 \geq 0 \quad (2)$$
$$b_1 - a_2 + b_4 + 2a_5 + a_8 \geq 0 \quad (3)$$
$$2a_1 - a_3 + a_4 + b_5 + a_6 + 2a_7 + a_9 \geq 0 \quad (4)$$
$$a_1 + b_2 - a_4 + 2b_5 + a_6 \geq 0 \quad (5)$$
$$2a_1 + b_2 + 2a_3 + b_5 - a_7 + b_8 + a_9 \geq 0 \quad (6)$$

We also have the 6 inequalities given by the σ_i tensored with the terms with twist $(1, 1, 1)$, these are just the above inequalities with 'a' and 'b' swapped so I'll not explicity write them out. Finally, of course, we have that all 18 coefficients must be ≥ 0.

Now let's firstly set $(1) = C$, some positive constant, and require the other 11 to be equal to 0. Then $(4)-(2) \implies a_1 = a_4 = b_5 = a_6 = 3a_7 = 0$ and also from the unwritten "twist inequalities" that $b_1 = b_4 = a_5 = b_6 = 3b_7 = 0$ Then (6) and it's twist equivalent $\implies$ the remaining terms $a_2, a_3, a_8, a_9, b_2, b_3, b_8, b_9 = 0$ also. So once again it's not possible to find a tuple to make one (1) positive and the others negative.

Now to look at the second question, namely for which $\lambda_i \in X_3(T)$ does the corresponding coefficient α_i/β_i (depending on twist) never appear with a minus in the result of the tensor products with the σ_i.

From reading the columns of the table we get that the coefficients of $(2, 1, 0)$, $(3, 1, 0)$, $(3, 2, 0)$, $(4, 2, 0)$ never appear with a minus. Just like with $p = 2$ notice how the components of each weight are distinct.

It sadly seems to be the case, solely based off the explicit examples when $p = 2$ and $p = 3$, that finding a set of vectors with positive entries satisfying 1 inequality positive and the others equal to 0 doesn't happen, and even if it somehow did happen for some random prime it's perhaps not worth the effort to find it. So with that in mind we shall leave this section with a single open question for general p.

Question 2.4. *Is it the case for every prime p that if the components of some weight $\mu \in X_p(T)$ are all different then the corresponding coefficient, α_i, never appears with a minus after taking the tensor product with any σ_i, $i \in \{0, 1, 2\}$? Further, is this an iff statement?*

3 Decomposition of some $W(\lambda)$ terms for $p = 5$

3.1 Decomposition diagrams and comments

It became obvious when working through the $p = 2$ example by hand and even working through the $p = 3$ examle by code that explicitly calculating these tensors takes a long quite, even for small p. So I decided to look into the decomposition of various $W(\lambda)$ that might appear after Littlewood-Richardson to see if there's any patterns which could speed up the calculating process or in a perfect world give us a general formula for how (certain) $W(\lambda)$ decompose.

One thing to note is that in the tensor calculations $W(\lambda)\otimes W(p-1,0,0)$ for $\lambda_i \in X_p(T)$, we never get resulting terms outside $X_{2p-1}(T)$ and similarly for tensoring with $(p-1,p-1,0)$. For $W(\lambda)\otimes(2(p-1),p-1,0)$ we now have to look as far as $X_{3p-1}(T)$, but we shouldn't concern ourselves with anything outside that. So, for $p=5$ I picked various weights in $X_{2p}(T)$ and began decomposing their associated W representation. This section will be dedicated to some of the more interesting ones and some comments and thoughts about them.[3]

Proposition 3.1. *Let $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ and suppose the sum of its components is n, ie $\sum_{i=1}^{3}\lambda_i=n$. Also suppose $W(\lambda)=\sum\alpha_i F(\mu_i)$ where $\mu_i=(\mu_{i_1},\mu_{i_2},\mu_{i_3})\in X_p(T)$, then for all i we have:*

$$\sum_{j=1}^{3}\mu_{i_j}\cong n(\textit{mod p-1})$$

Proof. To begin, consider the center of $GL_3(\mathbb{F}_p)$ which is given by

$$Z=\left\{\begin{pmatrix}x&0&0\\0&x&0\\0&0&x\end{pmatrix}:x\in\mathbb{F}_p^\times\right\}$$

Restriction of the $GL_3(\mathbb{F}_p)$ representation $W(\lambda)$ to Z gives us that Z acts as the character $x^{\sum\lambda_i}$, that is $Z\cdot W(\lambda)=x^{\sum\lambda_i}W(\lambda)$, therefore we have that

$$Z\cdot\sum\alpha_iF(\mu_i)=x^{\sum\lambda_i}\sum\alpha_iF(\mu_i)=\sum\alpha_ix^{\sum\lambda_i}F(\mu_i)$$

where $\cdot$ denotes the action, the first equals holds since we have just substituted in what we know from the Proposition itself, and the second by linearly. By Fermat's Little Theorem we have

$$x^{\sum\lambda_i}=x^{(\sum\lambda_i)+n_i(p-1)}$$

for $n_i\in\mathbb{Z}$. To conclude, we have that $Z\cdot F(\mu_i)=x^{\sum_j\mu_{i_j}}F(\mu_i)$ for all i, so:
$\sum_{j=1}^{3}\mu_{i_j}=(\sum\lambda_i)+n_i(p-1)$ for $n_i\in\mathbb{Z}\iff\sum_{j=1}^{3}\mu_{i_j}\cong\sum_{i=1}^{3}\lambda_i$ mod$(p-1)$. □

Remark 3.2. Observe that if $W(\lambda)=\sum\alpha_iF(\lambda_i)$ in the Grothendieck group (I won't write the square brackets anymore but it will be clear from context whether or not we are in it) with $\lambda\in X_p(T)$ then $W(\lambda+(x,x,x))=\sum\alpha_iF(\lambda_i+(x,x,x))$ with $\lambda\in X_p(T)$. This can be seen by writing $W(\lambda+(x,x,x))=W(\lambda)\otimes det^x$, substituting in our decomposition and using that the tensor product is distributive to get the result.

[3]see Appendix B for some of the steps of some of the following decompositions.

For this reason I decomposed weights $\lambda = (\lambda_1, \lambda_2, 0)$, we can just "add the twist" to get the others for free. In the diagrams to follow the red dot denotes the W representation and the blue dots represent the irreps appearing the full decomposition. Concentric circles and/or numbers beside the dots are used to signify the multiplicity with which the irrep appears.Without further ado, let's discuss some $p = 5$ examples.

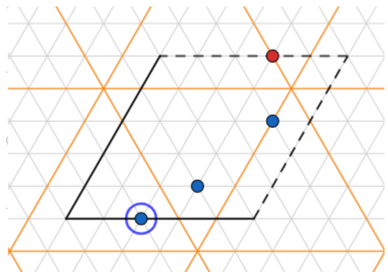


Figure 8: decomposition diagram for W(8,5,0)

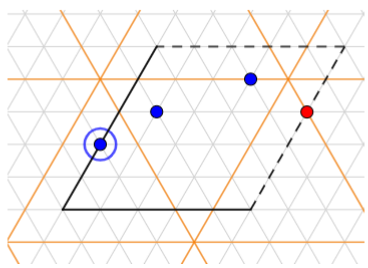


Figure 9: decomposition diagram for W(8,3,0)

Remarks: Using the previous proposition we can work out the weights of the F terms on the diagram (up to isomorphism) by choosing any tuple which lands on the dot representing F and then adding some multiple of $(1, 1, 1)$ until we are congruent to the weight of W mod$(p-1)$. We can also work out the decomposition of any $W((8, 5, 0) + (x, x, x))$ by the exact same idea.

Definition 3.3. On a 2d scale, define $\mathcal{L}$ to be the line passing through the weights $(0, 0, 0)$, $(a, \frac{a}{2}, 0)$ $\forall$ $a \cong 0 \pmod 2$. Also define ℓ to be the operation on all weights sending $(a, b, c) \to (-c, -b, -a)$. Note that ℓ incorporates some amount of twist which is not a multiple of $(p-1, p-1, p-1)$ in general, so in the language we used in section 1.4 when describing Figure 2, ℓ will cause weights to "jump floors".

Observe that the 2d dot diagrams are identical up to reflection in this line.

Remark 3.4. (a, b, c) lies on $\mathcal{L}$ $\iff$ $a + c = 2b$. To see this notice that if (a, b, c) lies on the line then the dot representing the weight won't move under reflection in ℓ, that is $(a, b, c) - (-c, -b, -a) \in X_0(T) \implies a + c = 2b$

Perhaps it is worth noting that thus far we have said nothing about how this ℓ interacts with the representations W and F associated to each weight, and it's important to keep these things separate. But with that in mind here are two more examples which show off this "symmetry of the dots" phenomenon, followed by 3 examples where the weight of W lies on the line $\mathcal{L}$ itself.

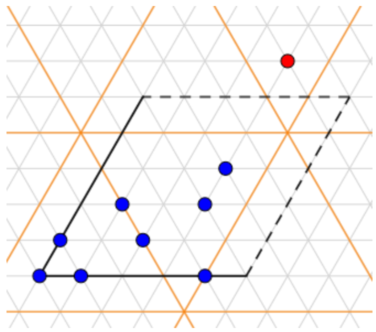


Figure 10: decomposition diagram for W(9,6,0)

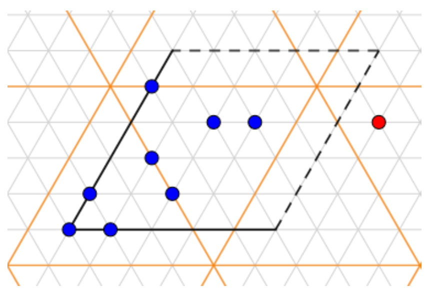


Figure 11: decomposition diagram for W(9,3,0)

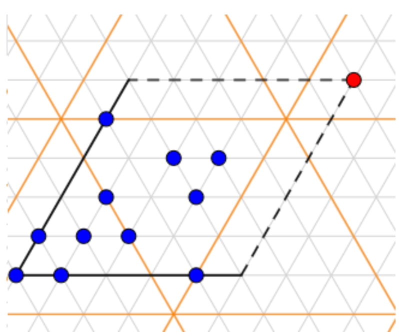


Figure 12: decomposition diagram for W(10,5,0)

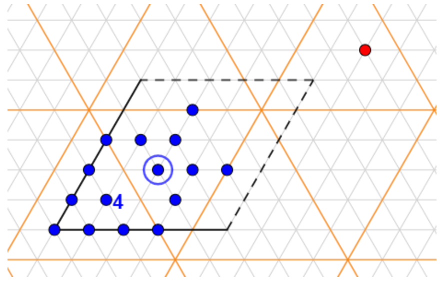


Figure 13: decomposition diagram for W(12,6,0)

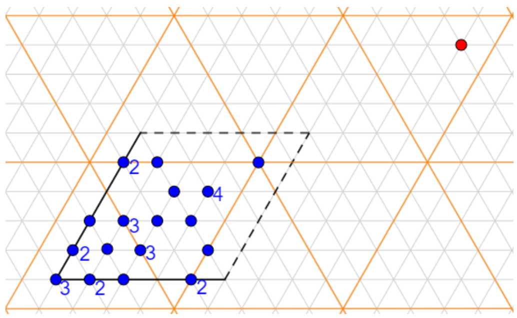


Figure 14: decomposition diagram for W(16,8,0)

Remark 3.5. Notice that the latter three diagrams also support the idea that symmetry is at play. Moreover notice that $W(10,5,0)$ has a lot more distinct terms in it's decomposition (11 distinct terms) than both of $W(9,5,0)$ and $W(9,3,0)$, even though they all lie on the dotted line. A similar thing happens with $W(12,6,0)$ compared to $W(9,6,0)$ and $W(9,3,0)$. This trend seems intuitive since the closer a point is to lying on $\mathcal{L}$ the more terms are available to it in the Strong Linkage Principle decomposition, but perhaps it says more, since there is no guarantee the coefficients in SLP have to be non-zero, and a term in the decomposition could have a high multiplicity and give lots of repeated terms, so maybe it's giving us an idea of how the coefficients arising from the strong linkage decomposition behave and also a rough idea of multiplicity (maybe there's a lower bound for how many co-efficients must be non-zero, or perhaps an upper bound on the multiplicities all depending on where $W(\lambda)$ is positioned.)

3.2 Some work regarding $W(\lambda)$ decomposition for general p

This subsection goes into more detail on some patterns and possible results that exist in regards to representation decomposition. Recall: $\ell \cdot (a,b,c) = (-c,-b,-a)$. Now extend the definition of Schur polynomials to the space of rational functions as follows:

Definition 3.6. Let $\lambda = (a, b, c)$ be a dominant weight with $c < 0$, then for $x, y, z \neq 0$ define

$$S_{(a,b,c)}(x, y, z) := S_{(a-c,b-c,0)}(x, y, z) \times (xyz)^c$$

Definition 3.7. If $\lambda \in X_+(T)$, let

$$\ell \cdot [W(\lambda)] := [W(\ell \cdot \lambda)]$$

If $R = \sum \alpha_i [W(\lambda)]$ then define

$$\ell \cdot R := \sum \alpha_i \ell \cdot [W(\lambda)]$$

As $\{[W(\lambda)] : \lambda \in X_+(T)\}$ is a basis for the Grothendieck group since it's free abelian, this is well defined.

Definition 3.8. If $\lambda \in X_+(T)$, define:

$$\ell \cdot S_\lambda(x, y, z) := S_{\ell \cdot \lambda}(x, y, z)$$

If $S_\mu = \sum \alpha_i S_{\lambda_i}$ then also define:

$$\ell \cdot S_\mu := \sum \alpha_i \ell \cdot S_{\lambda_i}$$

As the Schur polynomials form a basis of the space of Symmetric Polynomials this is also well defined. [3, Appendix A.1]

We have the following lemma:

Lemma 3.9. *Let $\lambda \in X_+(T)$, then for $x, y, z \neq 0$:*

$$\ell \cdot S_\lambda(x, y, z) := S_{(\ell \cdot \lambda)}(x, y, z) = S_\lambda(x^{-1}, y^{-1}, z^{-1})$$

Proof. (for $x \neq y \neq z$). Let $\lambda = (a, b, c)$, then using Jacobi's bialternant formula we have:

$$S_{(\ell\cdot\lambda)}(x,y,z) = S_{(-c,-b,-a)}(x,y,z) = \frac{\det\begin{pmatrix} x^{2-c} & y^{2-c} & z^{2-c} \\ x^{1-b} & y^{1-b} & z^{1-b} \\ x^{-a} & y^{-a} & z^{-a} \end{pmatrix}}{\det\begin{pmatrix} x^{2} & y^{2} & z^{2} \\ x^{1} & y^{1} & z^{1} \\ 1 & 1 & 1 \end{pmatrix}}$$

$$S_{\lambda}(x^{-1},y^{-1},z^{-1}) = S_{(a,b,c)}(x^{-1},y^{-1},z^{-1}) = \frac{\det\begin{pmatrix} x^{-a-2} & y^{-a-2} & z^{-a-2} \\ x^{-b-1} & y^{-b-1} & z^{-b-1} \\ x^{-c} & y^{-c} & z^{-c} \end{pmatrix}}{\det\begin{pmatrix} x^{-2} & y^{-2} & z^{-2} \\ x^{-1} & y^{-1} & z^{-1} \\ 1 & 1 & 1 \end{pmatrix}}$$

$$= \frac{(xyz)^{-2}\det\begin{pmatrix} x^{-a} & y^{-a} & z^{-a} \\ x^{-b+1} & y^{-b+1} & z^{-b+1} \\ x^{-c+2} & y^{-c+2} & z^{-c+2} \end{pmatrix}}{(xyz)^{-2}\det\begin{pmatrix} 1 & 1 & 1 \\ x^{1} & y^{1} & z^{1} \\ x^{2} & y^{2} & z^{2} \end{pmatrix}}$$

$$= \frac{\det\begin{pmatrix} x^{2-c} & y^{2-c} & z^{2-c} \\ x^{1-b} & y^{1-b} & z^{1-b} \\ x^{-a} & y^{-a} & z^{-a} \end{pmatrix}}{\det\begin{pmatrix} x^{2} & y^{2} & z^{2} \\ x^{1} & y^{1} & z^{1} \\ 1 & 1 & 1 \end{pmatrix}}$$

To see it also holds for some of x, y, z equal to eachother either use a different definition of the Schur Polynomials or if say $x = y$, we could take the limit as $\epsilon \to 0$ of $S_{\ell\cdot\lambda}(x+\epsilon, y, z)$ and $S_\lambda((x+\epsilon)^{-1}, y^{-1}, z^{-1})$. As Schur polynomials are continuous as functions of x, y, z this works out (you could evaluate the limits using repeated instances of L'Hôpital's rule, for example.) ☐

Lemma 3.10. $\ell \cdot (S_\lambda \cdot S_\mu) = S_{(\ell\cdot\lambda)} \cdot S_{(\ell\cdot\mu)}$

Proof. By Littlewood-Richardson suppose we get

$$S_\lambda \cdot S_\mu = \sum_i \alpha_i S_{\gamma_i}(x,y,z)$$

where since we are evaluating at (x, y, z) the Schur functions of size 4 and above partitions are zero since we interpret that as $S_{a,b,c,d,e,...}(x, y, z, 0, 0, ...)$ which is zero for any of $d, e, ...$ non zero. Then by definition and the above lemma:

$$\ell \cdot \big(S_\lambda \cdot S_\mu\big) = \sum_i \alpha_i S_{\gamma_i}(x^{-1},y^{-1},z^{-1})$$

Also observe:

$$\begin{aligned} S_{\ell\cdot\lambda}(x,y,z)\cdot S_{\ell\cdot\mu}(x,y,z) &= S_\lambda(x^{-1},y^{-1},z^{-1})\cdot S_\mu(x^{-1},y^{-1},z^{-1}) \\ &= \sum_i \alpha_i S_{\gamma_i}(x^{-1},y^{-1},z^{-1}) \end{aligned}$$

□

Proposition 3.11. $\ell\cdot\Big(W(\lambda)\otimes W(\mu)\Big) = \ell\cdot W(\lambda)\otimes\ell\cdot W(\lambda)$

Proof. Suppose by the Littlewood-Ricardson rule or otherwise we obtain

$$W(\lambda)\otimes W(\mu) = \sum_i^n \alpha_i W(\gamma_i)$$

Then by definition.

$$\ell\Big(W(\lambda)\otimes W(\mu)\Big) = \ell\cdot\Big(\sum_i \alpha_i W(\gamma_i)\Big) = \sum_i^n \alpha_i W(\ell\cdot\gamma_i)$$

Now similarly, suppose we have

$$\ell\cdot W(\lambda)\otimes\ell\cdot W(\lambda) = W(\ell\cdot\lambda)\otimes W(\ell\cdot\lambda) = \sum_j \beta_j W(\pi_j)$$

For any weight ϕ, let $\phi^\Sigma := \sum_{i=1}^3 \phi_i$, ie the sum of the components. Then by Remark 2.5 we can "pull out" the minus components of our weights as powers of the determinant representation and use Littlewood-Richardson to obtain the W terms in the sum expression, then "put the minuses back in" to our resulting terms to arrive at the $W(\pi_j)$ terms given above. The upshot of this being that

$$\pi_j^\Sigma = (\ell\cdot\lambda)^\Sigma + (\ell\cdot\mu)^\Sigma$$

But we know that

$$\begin{aligned} \gamma_i^\Sigma &= \lambda^\Sigma + \mu^\Sigma \\ \implies (\ell\cdot\gamma_i)^\Sigma &= (\ell\cdot\lambda)^\Sigma + (\ell\cdot\mu)^\Sigma \end{aligned}$$

Therefore we have that

$$W(\ell\cdot\lambda)\otimes W(\ell\cdot\mu) = \sum_{i=1}^n c_i W(\ell\cdot\gamma) + \sum_{k=1}^N d_k W(\tau_k)$$

That is, the $\ell\cdot\gamma$ could *potentially* show up in the sum resulting from the tensor. Note there is no guarantee yet that the $c_i = \alpha_i$ match, it's just suggestive rewriting. We could of course have some extra terms also, and these are given in the second sum. We now want to show that in fact $\alpha_i = c_i$ for all i, and that every d_k is zero.

We will do this via Schur polynomials (or rather their extension to Schur rational functions), taking the character of all the above we have:

$$S_\lambda \cdot S_\mu = \sum_i^n \alpha_i S_{\gamma_i}$$

$$\ell \cdot \Big(S_\lambda \cdot S_\mu\Big) = \sum_i^n \alpha_i S_{(\ell \cdot \gamma_i)}$$

$$S_{(\ell\cdot\lambda)} \cdot S_{(\ell\cdot\mu)} = \sum_i^n c_i S_{(\ell\cdot\gamma_i)} + \sum_k^N d_k S_{\gamma_k}$$

But we know from Lemma 3.9 that $\ell \cdot (S_\lambda \cdot S_\mu) = S_{\ell\cdot\lambda} \cdot S_{\ell\cdot\mu}$ so in fact:

$$\sum_i^n (c_i - \alpha_i) S_{(\ell\cdot\gamma_i)} + \sum_k^N d_k S_{\gamma_k} = 0$$

so by comparing co-efficients (or the lack of) of terms on the LHS and RHS or by realising that for this to evaluate to 0 for all (x, y, z) we must have $a_i = c_i$ and $d_k = 0$ we finish the proof. □

Proposition 3.12. *The Symmetric decomposition proposition.*

$$\ell \cdot (F(a, b, c)) = F(\ell \cdot (a, b, c)) = F(-c, -b, -a)$$

Also if

$$W(\lambda) = \sum \alpha_i F(\lambda_i), \;\; \lambda_i \in X_1(T)$$

then;

$$W(\ell \cdot \lambda) \cong \sum \alpha_i F(\ell \cdot \lambda_i)$$

Proof. Suppose $\mu \in$ Lower alcove or the boundary, then $\mu' := \ell \cdot \mu \in$ Lower alcove or the boundary also. Using relation between F and W we get:

$$\ell \cdot F(\mu) = \ell \cdot W(\mu) = W(\ell \cdot \mu) = W(\mu') = F(\mu')$$

For $\mu \in$ Upper alcove, $\mu' = \ell \cdot \mu \in$ Upper alcove also. Using relation between F and W we get:

$$\begin{aligned}\ell \cdot F(\mu) = \ell \cdot (W(\mu) - W(s_{2,1} \cdot \mu)) &= W(\mu') - W((s_{2,1} \cdot \mu)') \\ &= F(\mu') + F(s_{2,1} \cdot \mu') - F((s_{2,1} \cdot \mu)') = F(\mu')\end{aligned}$$

where the last equals sign is true because $s_{2,1}$ and ℓ commute as operations on weights. Observe:

$$\begin{aligned}\ell \cdot (s_{2,1} \cdot (a, b, c)) = \ell \cdot (c + p - 2, b, a - p + 2) &= (-a + p - 2, -b, -c - p + 2) \\ &= s_{2,1} \cdot (-c, -b, -a) \\ &= s_{2,1} \cdot (\ell \cdot (a, b, c)).\end{aligned}$$

For $\lambda \notin X_p(T)$, we want to use Steinberg's to write

$$F(\lambda) = \bigotimes_i F(\lambda_i)^{p^i}$$

So as actual representations we have:

$$F(\lambda) = \Big(\bigotimes_j \big(W(\lambda_j) - W(s_{2,1} \cdot \lambda j)\big) \Big) \bigotimes \Big(\bigotimes_k W(\lambda_k) \Big)$$

where j runs over all the λ_j in the upper alcove and k runs over all the λ_k in the lower alcove. By an induction on proposition 3.11 we get that

$$\ell \cdot F(\lambda) = \Big(\bigotimes_j \big(\ell \cdot W(\lambda_j) - \ell \cdot W(s_{2,1} \cdot \lambda j)\big) \Big) \bigotimes \Big(\bigotimes_k \ell \cdot W(\lambda_k) \Big)$$

Now, if $\lambda \notin X_p(T)$ then $\ell \cdot \lambda \notin X_p(T)$ also since the operation ℓ doesn't change the values of differences of the components, just permutes them. We know how ℓ behaves on the weights themselves, and this is all we need to know to be able to use steinbergs on $F(\ell \cdot \lambda)$.

$$\begin{aligned} (a, b, c) &= \sum (a_i p_i, b_i p_i, c_i p_i) \\ \iff (-c, -b, -a) &= \sum (-c_i p_i, -b_i p_i, -a_i p_i) \\ \iff \ell \cdot (a, b, c) &= \sum \ell \cdot (a_i p_i, b_i p_i, c_i p_i) \end{aligned}$$

So Steinberg's for $F(\ell \cdot \lambda)$ reads

$$F(\ell \cdot \lambda) = \bigotimes F(\ell \cdot (a_i, b_i, c_i))^{p^i}$$

With all $(a_i, b_i, c_i) \in X_p(T)$,if μ is the upper alcove then so to is $\ell \cdot \mu$, so finally

$$F(\ell \cdot \lambda) = \Big(\bigotimes_j \big(W(\ell \cdot \lambda_j) - W(\ell \cdot (s_{2,1} \cdot \lambda j))\big) \Big) \bigotimes \Big(\bigotimes_k W(\ell \cdot \lambda_k) \Big)$$

With j running over $\ell \cdot \lambda_j$ in upper alcove $\iff$ λ_j is in upper alcove and k running over $\ell \cdot \lambda_k$ in lower alcove $\iff$ λ_k is in lower alcove, so with this we conclude that $\ell \cdot F(\lambda) = F(\ell \cdot \lambda)$.

To finish, suppose $W(\lambda) = \sum_i \alpha_i F(\lambda_i)$, then in the Grothendieck group;

$$\begin{aligned}
W(\ell \cdot \lambda) &= \ell \cdot W(\lambda) = \ell \cdot \big(\sum_i \alpha_i F(\lambda_i)\big) \\
&= \ell \cdot \Bigg(\Big(\sum_j \alpha_j \Big(W(\lambda_j) - W(s_{2,1} \cdot \lambda_j)\Big)\Big) + \Big(\sum_k \alpha_k W(\lambda_k)\Big)\Bigg) \\
&= \Bigg(\sum_j \alpha_j \Big(W(\ell \cdot \lambda_j) - W\Big(\ell \cdot (s_{2,1} \cdot \lambda_j)\Big)\Big)\Bigg) + \Big(\sum_k \alpha_k W(\ell \cdot \lambda_k)\Big) \\
&= \Big(\sum_j \alpha_j F(\ell \cdot \lambda_j)\Big) + \Big(\sum_k \alpha_k F(\ell \cdot \lambda_k)\Big) \\
&= \sum_i \alpha_i F(\ell \cdot \lambda_i)
\end{aligned}$$

where once again, j and k run over the upper alcove and lower alcove λ_i respectively. □

Remark 3.13. It should be noted that this result could have been proven much quicker and less painfully using a duality based proof. The reason why I went to all the effort to prove it using ℓ only was to keep the pattern of the section that is:
"see a possible result based on the dots → test it for some concrete examples → write it up as a guess if it holds for these few examples and seems quite reasonable → remove it if I find a counter-example or leave it as a guess if it keeps holding for more concrete examples I try → if I'm able, get a proof and promote it to a proposition."

Guess 3.14. *For some $\mu \in X_+(T)$ but not in $X_p(T)$ or on a plane let μ' be μ reflected in the closest plane $H_{2,n}$ such that $\mu' \uparrow \mu$, ie $\mu' \in ({R_{i,j}}^-)^{int}$ and $\mu \in ({R_{i,j}}^+)^{int}$, so they both live in ${R_{i,j}}^{int} - H_{2,i+j+1}$. Then, if μ, μ' appear in the strong linkage decomposition of some $W(\lambda)$ with coefficients α and β respectively we have;*

$$\beta \geq \alpha \text{ if } \mu' \notin X_p(T)$$

This held for every example I did for primes 3,5,7, for λ in the $2p$ restricted region at least. I think this could extend to all of $X_+(T)$ because of the state a decomposition is in after both Steinbergs and rewriting F terms as W terms. If $\beta - \alpha < 0$ we would have a part of the character formula of this decomposition state looking like so:

$$S_\lambda = \sum ... + \prod_{i=1}^{n} S_{\mu_i}(x^{p^i}, y^{p^i}, z^{p^i})\Big(\alpha S_{\overline{\mu}} + (\beta - \alpha) S_{(s_{2,1}\overline{\mu})}\Big)$$

where $\overline{\mu}$ is μ's relative position to $R_{i,j}$'s origin where it lives, ie the p^0 in Steinberg's tensor theorem, and the μ_i are the p^i powered terms in the tensor. Note that both μ and μ' share these terms as they live in the same $R_{i,j}$. By the examples it seems to be the case that for some reason $S_{(s_{2,1}\overline{\mu})}$ doesn't appear with negative coefficient, and in fact in all the examples I tried $\beta = \alpha$ and the $S_{(s_{2,1}\overline{\mu})}$ term vanishes completely. However, if μ and μ' were in the upper and lower alcove to start with, that is we didn't need Steinberg's, then it didn't always hold in examples I did for small primes. So I'm unsure what's going on but my guess is that $\beta \geq \alpha$ if there is a $\prod_{i=1}^{n} S_{\mu_i}(x^{p^i}, y^{p^i}, z^{p^i})$ term in front of the difference of two Schur functions. I tried some general Kostka number things but nothing of any value immediately fell out, this could of course just be a bad guess.

4 Further work for tensor products of irreps for general p.

The main thing we can take from the previous section is that these decompositions have these nice symmetric properties about the line $\mathcal{L}$. So in order for us to work out the following two tensors for $\lambda \in X_p(T)$

i) $F(\lambda) \otimes F(p-1, 0, 0)$

ii) $F(\lambda) \otimes F(p-1, p-1, 0)$

we only actually have to work out one of i) or ii) and we get the other for free via reflection. Also, due to twist, we actually only have to work these out for weights of the form $(a, b, 0)$, then to get any other weight we just need to tensor by some power of the det representation. The results for a few cases where $a \leq p-1$ are as follows:

$$F(a,a,0) \otimes F(p-1,0,0) = \begin{cases} \sum_{i=0}^{\frac{a}{2}} F(a+p-1-i, a-i) \\ + \sum_{i=\frac{a}{2}+1}^{a} 2F(a+p-1-i, a, i) & \textbf{if a is even} \\ \\ \sum_{i=0}^{\frac{a+1}{2}} F(a+p-1-i, a-i) \\ + \sum_{i=\frac{a+3}{2}}^{a} 2F(a+p-1-i, a, i) & \textbf{if a is odd} \end{cases}$$

The next one holds for $a \geq 2$ and $a \neq p-1$, if $a = 0$ the weight isnt in the p-restricted region and if $a = 1$ use the formula below this one instead. With that in mind we have:

$$F(a,a-1,0)\otimes F(p-1,0,0)$$

$$=\begin{cases} F(a,a-1,0)+F(a-1,a-1,1)+F(p-2,a,a) \\ +2F(a+p-2,a,0)+F(a+p-2,a-1,1)+F(p-2+\frac{a}{2},a,\frac{a}{2}) \\ +\sum_{i=2}^{\frac{a}{2}}\Big(F(a+p-1-i,a-1,i)+2F(i+p-2,a-1,a+1-i)\Big) \\ +\sum_{i=1}^{\frac{a}{2}-1}\Big(F(a+p-2-i,a,i)+2F(i+p-2,a,a-i)\Big) & \textbf{if a even} \\ \\ F(a,a-1,0)+F(a-1,a-1,1)+2F(a+p-2,a,0) \\ +F(a+p-2,a-1,1)+F(p-2+\frac{a+1}{2},a-1,\frac{a+1}{2}) \\ +\sum_{i=2}^{\frac{a-1}{2}}\Big(F(a+p-1-i,a-1,i)+2F(i+p-2,a-1,a+1-i)\Big) \\ +\sum_{i=1}^{\frac{a-1}{2}}\Big(F(a+p-2-i,a,i)+2F(i+p-2,a,a-i)\Big) & \textbf{if a odd} \end{cases}$$

If $a=p-1$ then we can use the if a is even case with slight modification. We must include an extra $F(a-1,a-1,1)$ (really $F(p-2,p-2,1)$ in this specific case) term to account for the fact $F(p-1,p-2,0)$ (first term in the expansion) lies in the upper alcove. We also must take this answer and subtract $W(p-2,p-2,1)\otimes W(p-1,0,0)$ since, once again, $(p-1,p-2,0)$ is in the upper alcove. This second tensor can be found using twist and the first result which I will not explicitly write out. The final explicit result I will give is the following, it works fine for any $0\leq a\leq p-1$:

$$F(a,0,0)\otimes F(p-1,0,0)$$

$$=\begin{cases} \sum_{i=0}^{\frac{a}{2}}(2-\delta_{i,0}-\delta_{i,\frac{a}{2}})F(a-i,i,0)+2\sum_{i=1}^{\frac{a}{2}-1}F(i+p-1,a-i,0) \\ +\sum_{i=1}^{\frac{a}{2}-1}F(a+1-i,i,1)+\sum_{i=\frac{a}{2}+1}^{\min\{a,p-2\}}F(p-2,i,a+1-i) \\ +F(p-1+\frac{a}{2},\frac{a}{2},0 & \textbf{if a even} \\ \\ \sum_{i=0}^{\frac{a-1}{2}}\Big((2-\delta_{i,0})F(a-i,i,0)+2F(i+p-1,a-i,0)\Big) \\ +\sum_{i=1}^{\frac{a-1}{2}}F(a-1-i,i,1)+\sum_{i=\frac{a+1}{2}}^{a}F(p-2,i,a+1-i) & \textbf{if a odd} \end{cases}$$

Question 4.1. *For $0\leq a\leq p-1$ and $0\leq n\leq a$ is it feasible to write down a "simple" general sum decomposition for $W(a,a-n,0)\otimes W(p-1,0,0)$ in a case by case form like above?*

Question 4.2. *For $(a, b, 0) \in X_p(T)$ is it feasible to write down a "simple" sum decomposition for $W(a, b, 0) \otimes W(p-1, 0, 0)$?*

Where we call a sum decomposition "simple" if it can be coded as a function such that the function will return the full decomposition in some reasonable time-frame having only taken a, n, p (or a, b, p in the more general version of the question) as arguments and nothing else.

5 A Python program.

Although it was worthwhile to decompose some $W(\lambda)$ representations by hand to get a feel for how things worked, there were still a lot of error-prone calculations to do even for λ close to but not quite inside the p-restricted region, there were a couple of patterns I noticed, especially whilst working on the content from Section 3, but even these took a while to test out on only a few different weights by hand. For that reason I decided to write some Python code to carry out these decompositions for small primes. At the time of writing this code is able to do the following:

- Decompose W into a linear combination of (not necessarily irreducible) Fs. (ie perform Strong Linkage and find the coefficients)
- Carry out Steinberg's theorem on a F rep (ie write any F rep as a tensor product of F irreps)
- Perform the Littlewood-Richardson rule on two W reps
- Write W as a linear combination of irreducible representations.
- Write $F \otimes F$ as a linear combination of irreducible representations.

Disclaimer: This code currently makes very little effort to be "Pythonic", although it has been well commented on. Functions have not been written in the most efficient way either. This is partly because the code (at least short term) is only going to be used to test ideas/compute tensors for "small" weights and primes and partly due to my own limitations as a programmer.

URL: https://github.com/DylanJohnston/GL3RepsToolkit

A The complete workings out for Section 2

The method will be similar throughout, first all $F(\lambda)$ terms will be expressed in terms of W terms. Then the Littlewood-Richardson rule will be used to calculate the tensor product as a sum of W terms. From here, results and methods such as strong linkage principle, Weyl dimension formula, Kostka numbers and others will enable us to express all W terms back as combinations of F terms.

Remark A.1. Throughout I will use α, β, c etc to represent the coefficients of the F terms involved in the strong linkage. Anytime you see these the strong linkage principle is being used, hopefully this will make it a little easier to follow the working out.

- $\boldsymbol{\lambda = (0,0,0)}$ **:**

a) $[F(0,0,0) \otimes F(1,0,0)] = [W(0,0,0) \otimes W(1,0,0)] = [W(1,0,0)]$
$= [F(1,0,0)]$

b) $[F(0,0,0) \otimes F(1,1,0)] = [W(0,0,0) \otimes W(1,1,0)] = [W(1,1,0)]$
$= [F(1,1,0)]$

c) $[F(0,0,0) \otimes F(2,1,0)] = [W(0,0,0) \otimes W(2,1,0)] = [W(2,1,0)]$
$= [F(2,1,0)]$

- $\boldsymbol{\lambda = (1,0,0)}$ **:**

a) $[F(1,0,0) \otimes F(1,0,0)] = [W(1,0,0) \otimes W(1,0,0)] = [W(2,0,0)] + [W(1,1,0)]$

- $[W(2,0,0)] = [F(2,0,0)] + \alpha[F(1,1,0)] = [F(1,0,0)^{(2)} \otimes F(0,0,0)]$
$+ \alpha[F(1,1,0)]$

  Weyl dimension formula $6 = 3 + 3\alpha \implies \alpha = 1$

  So as actual reps, $[W(2,0,0)] = [F(1,0,0)] + [F(1,1,0)]$

- $[W(1,1,0)] = [F(1,1,0)]$ since $(1,1,0)$ lies on the boundary of the upper alcove.

$\implies [F(1,0,0) \otimes F(1,0,0)] = [F(1,0,0)] + 2[F(1,1,0)]$

b) $[F(1,0,0) \otimes F(1,1,0)] = [W(1,0,0) \otimes W(1,1,0)] = [W(2,1,0)] + [W(1,1,1)] =$
$[F(2,1,0)] + [F(1,1,1)] \cong [F(2,1,0)] + [F(0,0,0)]$

c) $[F(1,0,0) \otimes F(2,1,0)] = [W(1,0,0) \otimes W(2,1,0)] = [W(3,1,0)] + [W(2,2,0)]$
$+ [W(2,1,1)]$

- $[W(3,1,0)] = [F(3,1,0)] + \alpha[F(2,1,1)] + \beta[F(2,2,0)]$
$= [F(1,0,0)^{(2)} \otimes F(1,1,0)] + \alpha[F(2,1,1)] + \beta[F(1,1,0)^{(2)} \otimes F(0,0,0)]$

  Weyl Dimension formula: $15 = 9 + 3\alpha + 3\beta \implies \alpha + \beta = 2$

  Schur and Kostka:

  As Schur polynomials we have:

  $S_{(3,1,0)} = (x^2 + y^2 + z^2)S_{(1,1,0)} + \alpha S_{(2,1,1)} + \beta(x^2y^2 + x^2z^2 + y^2z^2)$

- coefficient of x^2yz term ie $\mu = (2,1,1)$:
  In $S_{(3,1,0)}$: $= K_{(3,1,0,2,1,1)} = 2$
  RHS = 1 + α + 0
  $\implies \alpha = 1 \implies \beta = 1$ (from Weyl dim formula)

So as actual reps: $[W(3,1,0)] = [F(1,0,0) \otimes F(1,1,0)] + [F(2,1,1)] + [F(1,1,0) \otimes [F(0,0,0)] = [F(2,1,0)] + [F(0,0,0)] + [F(2,1,1)] + [F(1,1,0)] \cong [F(2,1,0)] + [F(0,0,0)] + [F(1,0,0)] + [F(1,1,0)]$

- $[W(2,2,0)] = [F(2,2,0)] + \alpha[F(2,1,1)] = [F(1,1,0)^{(2)} \otimes F(0,0,0)] + \alpha[F(2,1,1)]$

  Weyl Dimension formula: $6 = 3 + 3\alpha \implies \alpha = 1$

  So as actual reps: $[W(2,2,0)] = [F(1,1,0) \otimes F(0,0,0)] + [F(2,1,1)] \cong [F(1,1,0)] + [F(1,0,0)]$

- $[W(2,1,1] = [F(2,1,1)]$ since $(2,1,1)$ lies on the boundary of the alcoves.

$\implies [F(1,0,0) \otimes F(2,1,0)]$
$= [F(2,1,0)] + [F(0,0,0)] + 2[F(1,0,0)] + [F(1,1,0)] + [F(1,0,0)] + [F(2,1,1)]$
$\cong [F(2,1,0)] + 3[F(1,0,0)] + 2[F(1,1,0)] + [F(0,0,0)]$

- $\boldsymbol{\lambda = (1,1,0)}$ **:**

a) $[F(1,1,0) \otimes F(1,0,0)]$ = (same as part b) above) = $[F(2,1,0)] + [F(0,0,0)]$

b) $[F(1,1,0) \otimes F(1,1,0)] = [W(1,1,0) \otimes W(1,1,0)] = [W(2,2,0)] + [W(2,1,1)]$ = (from working in part c) above) = $[F(1,1,0)] + [F(1,0,0)] + [F(2,1,1)] \cong [F(1,1,0)] + 2[F(1,0,0)]$

c) $[F(1,1,0) \otimes F(2,1,0)] = [W(1,1,0) \otimes W(2,1,0)] = [W(3,2,0)] + [W(3,1,1)] + [W(2,2,1)]$

- $[W(3,2,0)] = [F(3,2,0)] + \alpha[F(3,1,1) + \beta[F(2,2,1)]$
  $= [F(1,1,0)^{(2)} \otimes F(1,0,0)] + \alpha[F(1,0,0)^{(2)} \otimes F(1,1,1)] + \beta[F(2,2,1)]$

  Weyl dimension formula:

  $15 = 9 + 3\alpha + 3\beta \implies (\alpha, \beta) = (2,0)$ or (1,1) or (0,2)

  Schur and Kostka:

  As Schur polynomials we have:

  $S_{(3,2,0)} = S_{(1,1,0)}(x^2,y^2,z^2) \cdot S_{(1,0,0)} + \alpha S_{(1,0,0)}(x^2,y^2,z^2) \cdot S_{(1,1,1)} + \beta S_{(2,2,1)}$
  $S_{(3,2,0)} = (x^2y^2 + x^2z^2 + y^2z^2)(x+y+z) + \alpha(x^2+y^2+z^2)(xyz) + \beta S_{(2,2,1)}$

  - coefficient of x^2y^2z term ie $\mu = (2,2,1)$:
    In $S_{(3,2,0)}$: $K_{(3,2,0,2,2,1)} = 2$
    RHS, by inspection, $= 1 + 0\alpha + \beta K_{(2,2,1,2,2,1)}$
    $\implies 2 = 1 + \beta \implies \beta = 1$

- coefficient of x^3yz term ie $\mu = (3,1,1)$:
  In $S_{(3,2,0)}$: $K_{(3,2,0,3,1,1)} = 1$
  RHS $= 0 + 1\alpha + \beta K_{(2,2,1,3,1,1)} = 0$
  $\implies \alpha = 1$

So as actual reps $[W(3,2,0)] = [F(1,1,0) \otimes F(1,0,0)]$
$+ [F(1,0,0) \otimes F(1,1,1)] + [F(2,2,1)] \cong$ (by previous calcs)
$\cong [F(2,1,0)] + [F(1,0,0)] + [F(1,1,0)] + [F(0,0,0)]$

- $[W(3,1,1)] = [F(3,1,1)] + \alpha[F(2,2,1)] = [F(1,0,0)^{(2)} \otimes F(1,1,1)] + \alpha[F(2,2,1)]$
  Weyl dimension formula:
  $6 = 3 + 3\alpha \implies \alpha = 1$
  So as actual reps $[W(3,1,1) = [F(1,0,0) \otimes F(1,1,1)] + [F(2,2,1)] = [F(2,1,1)] + [F(2,2,1)] \cong [F(1,0,0)] + [F(1,1,0)]$
- $[W(2,2,1)] = [F(2,2,1)] \cong [F(1,1,0)]$ since $(2,2,1)$ lies on boundary of upper alcove.

$\implies [F(1,1,0) \otimes F(2,1,0)] = [F(2,1,0)] + 3[F(1,1,0)] + 2[F(1,0,0)] + [F(0,0,0)]$

- $\boldsymbol{\lambda = (2,1,0)}$ **:**

a) $[F(2,1,0) \otimes F(1,0,0)] = [W(2,1,0) \otimes W(1,0,0)] =$ (by a previous part) $= [F(2,1,0)] + 3[F(1,0,0)] + 2[F(1,1,0)] + [F(0,0,0)]$

b) $[F(2,1,0) \otimes F(1,1,0)] = [W(2,1,0) \otimes W(1,1,0)] =$ (by a previous part) $= [F(2,1,0)] + 3[F(1,1,0)] + 2[F(1,0,0)] + [F(0,0,0)]$

c) $[F(2,1,0) \otimes F(2,1,0)] = [W(2,1,0) \otimes W(2,1,0)] = [W(4,2,0)] + [W(4,1,1)] + [W(3,3,0)] + 2[W(3,2,1)] + [W(2,2,2)]$

  - $[W(4,2,0)] = [F(4,2,0)] + \alpha[F(4,1,1)] + \beta[F(3,3,0)] + c[F(2,2,2)]$
    $= [F(2,1,0)^{(2)} \otimes F(0,0,0)] + \alpha[F(1,0,0)^{(2)} \otimes F(2,1,1)]$
    $+ \beta[F(1,1,0)^{(2)} \otimes F(1,1,0)] + c[F(2,2,2)]$
    Weyl dimension formula: $27 = 8 + 9\alpha + 9\beta + c$
    Schur and Kostka: As Schur polynomials we have:
    $S_{(4,2,0)} = S_{(2,1,0)}(x^2,y^2,z^2) + \alpha(x^2+y^2+z^2) \cdot S_{(2,1,1)} + \beta(x^2y^2 + x^2z^2 + y^2z^2)(xy+xz+yz) + cS_{(2,2,2)}$
    - coefficient of $x^2y^2z^2$ ie $\mu = (2,2,2)$:
      In $S_{(4,2,0)} : K_{(4,2,0,2,2,2)} = 3$
      RHS $= K_{(2,1,0,1,1,1)} + 0\alpha + cK_{(2,2,2,2,2,2)} = 2 + c$
      $\implies c = 1$

    - coefficient of x^3y^3 ie $\mu = (3,3,0)$:
      In $S_{(4,2,0)} : K_{(4,2,0,3,3,0)} = 1$
      $RHS = 0 + 0\alpha + \beta + 0c = \beta$
      $\implies \beta = 1 \implies c = 1$ (Weyl dim formula).

$\therefore$ as actual reps; $[W(4,2,0)] = [F(2,1,0) \otimes F(0,0,0)]$
$+ [F(1,0,0) \otimes F(2,1,1)] + [F(1,1,0) \otimes F(1,1,0)] + [F(2,2,2)]$
$= [F(2,1,0)] + [W(3,1,1)] + [F(2,2,1)] + [F(1,1,0)] + 2[F(1,0,0)] +$
$[F(2,2,2)] \cong [F(2,1,0)] + 3[F(1,0,0)] + 3[F(1,1,0)] + [F(2,2,2)]$

- $[W(4,1,1)] = [F(4,1,1)] + \alpha[F(2,2,2)] = \alpha[F(1,0,0)^{(2)} \otimes F(2,1,1)]$ $+ \alpha[F(2,2,2)]$

  Weyl dimension formula $10 = 9\alpha + \beta \implies \alpha = 1$

  $\therefore$ as actual reps: $[W(4,1,1) \cong [F(1,0,0) \otimes F(1,0,0)] + [F(0,0,0] =$ $[F(1,0,0)] + 2[F(1,1,0)] + [F(0,0,0)]$

- $[W(3,3,0)] = [F(3,3,0)] + \alpha[F(2,2,2)] = [F(1,1,0)^{(2)} \otimes F(1,1,0)] +$ $\alpha[F(2,2,2)]$

  Weyl dimension formula: $10 = 9\alpha + \beta \implies \alpha = 1$

  $\therefore$ as actual reps; $[W(3,3,0)] = F(1,1,0) \otimes F(1,1,0)] + [F(0,0,0)]$
  $= [F(1,1,0)] + 2[F(1,0,0)] + [F(0,0,0)]$

- $[W(3,2,1)] = [F(3,2,1)] \cong [F(2,1,0)]$ (remember: we have two of these!)

- $[W(2,2,2] = [F(2,2,2)] \cong [F(0,0,0)]$

$\implies [F(2,1,0] \otimes F(2,1,0)] = 3[F(2,1,0)] + 6[F(1,1,0)] + 6[F(1,0,0)] +$ $4[F(0,0,0)]$

A.1 Table of dimensions for $W(\lambda)$ & $F(\lambda)$

Remark A.2. if $\lambda - \mu \in X_0(T)$ then $\dim(W(\lambda)) = \dim(W(\mu))$. I will fill out the table for λ values with at least one 0 term, for other λ terms just use this rule.

Remark A.3. Recall that the Weyl dimension formula is:

$$\dim(W(a,b,c)) = \frac{1}{2}(a-b+1)(b-c+1)(a-c+2)$$

λ	dim(W(λ))	dim(F(λ))	λ	dim(W(λ))	dim(F(λ))
(0,0,0)	1	1	(3,1,0)	15	not needed
(1,0,0)	3	3	(3,2,0)	15	not needed
(1,1,0)	3	3	(3,3,0)	10	not needed
(2,0,0)	6	not needed	(4,0,0)	15	not needed
(2,1,0)	8	8	(4,1,0)	24	not needed
(2,2,0)	6	not needed	(4,2,0)	27	not needed
(3,0,0)	10	not needed	(4,3,0)	24	not needed

B Some summary steps regarding working out the $W(\lambda)$ decomposition for $p = 5$

- $W(8,5,0) = F(8,5,0) + \alpha F(8,4,1)$ by Strong Linkage Principle.

  $W(8,5,0) = (1,1,0)^{(5)} \otimes F(3,0,0) + \alpha F(8,4,1)$

  Then by Schur/Kostka/Weyl dim formula we get $\alpha = 1$ and after Littlewood-Richardson we get the full decomposition to be

  $W(8,5,0) = F(4,1,0) + 2F(3,1,1) + F(8,4,1)$

- $W(8,3,0) = F(8,3,0) + \alpha F(7,4,0)$

  $W(8,3,0) = F(1,0,0)^{(5)} \otimes F(3,3,0) + \alpha F(7,4,0)$

  by Weyl dim formula we get $\alpha = 1$ so after Littlewood-Richardson we get the full decomposition to be

  $W(8,3,0) = F(4,3,0) + 2F(3,3,1) + F(7,4,0)$

- $W(9,6,0) = F(9,6,0) + \alpha F(9,4,2) + \beta F(8,6,1) + cF(8,5,2) + dF(5,5,5)$

  $W(9,6,0) = F(1,1,0)^{(5)} \otimes F(4,1,0) + \alpha F(1,0,0) \otimes F(4,4,2) + \beta F(1,1,0)^{(5)} \otimes F(3,1,1) + cF(8,5,2) + dF(5,5,5)$

  Schur/Kostka gives us $(a,b,c,d) = (1,1,1,1)$. Result follows from Littlewood-Richardson which we can we read off the diagram.

- $W(9,3,0)$ is very similar in style to the above. We get 4 strong linkage principle coefficients, all of which are 1.

- $W(12,6,0) = F(12,6,0) + (10,8,0) + \beta F(8,8,2) + cF(8,6,4) + dF(10,4,4) + eF(12,4,2) + gF(7,6,5)$

  $W(12,6,0) = F(2,1,0)^{(5)} \otimes F(2,1,0) + \alpha F(1,1,0)^{(5)} \otimes F(5,3,0) +$

  $\beta F(1,1,0)^{(5)} \otimes F(3,3,2) + cF(8,6,4) + dF(1,0,0) + eF(1,0,0)^{(5)} \otimes F(7,4,2) + gF(7,6,5)$.

  Schur/Kostka give us $(\alpha, \beta, c, d, e, g) = (1,1,1,1,1,1)$

  The result follows from Littlewood-Richardson again, and can be better read straight off the diagram.

- $W(10,5,0)$ is very similar to above.

- $W(16,8,0)$ again has a similar style to the 2 previous, although $F(11,8,5)$ appears in the Strong linkage decomposition and ends up having coefficient 0, so not all SLP coefficients are 1 in this case.

References

[1] Andrea Pasquali. representations of sl2 and gl2 in defining characteristic.

[2] Florian Herzig. The weight in a Serre-type conjecture for tame n-dimensional Galois representations. *Duke Math. J.*, 149(1):37–116, 2009.

[3] W. Fulton and J. Harris. *Representation Theory: A First Course*. Graduate texts in mathematics. Springer, 1991.

[4] B. Hall and B.C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer, 2003.

[5] Wikipedia. Schur polynomial — Wikipedia, the free encyclopedia, 2019.