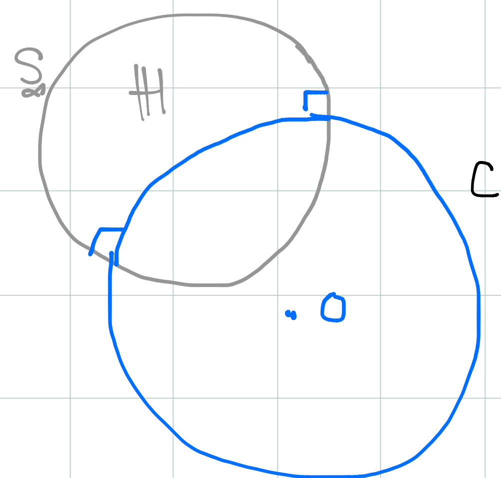
<u>Thurs</u>: Review of π_1 and covering spaces (Iezzi)

<u>Mon</u>: Support class 2-3 B1.07

Last time, we

Defined <u>reflection</u> in a geodesic $C \cap \mathbb{H}$ to be

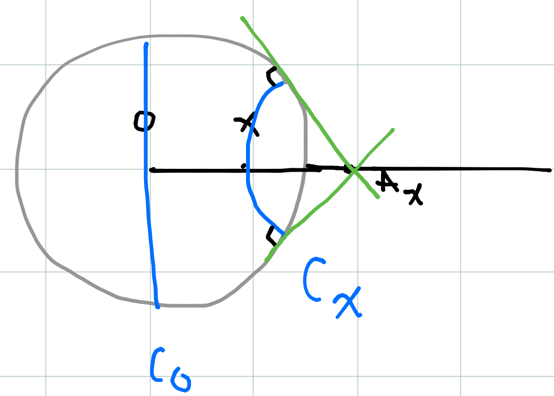
inversion in C.



<u>Def</u> An <u>isometry</u> of $\mathbb{H}$ is a product of reflections

<u>Lemma</u> $\mathrm{Isom}(\mathbb{H})$ acts transitively on $\mathbb{H}$, and the stabilizer of a point is $\cong O(2)$ (= rotations and reflections $= \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \right\}$)

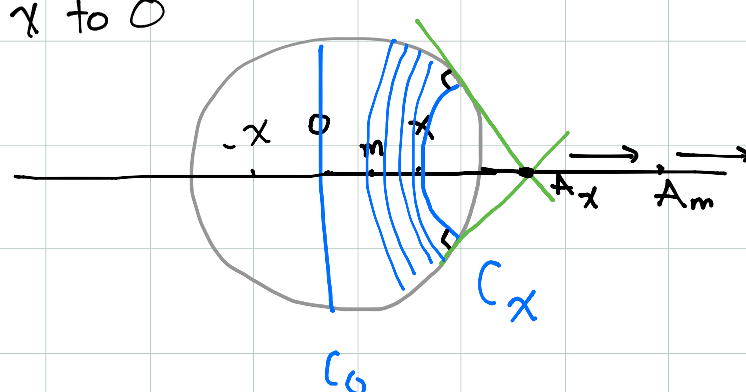
pf: First show there is a reflection taking any x to 0, then compose two to take any x to any y



r_0 = reflection in C_0 takes x to $-x$

r_x = reflection in C_x takes x to x

By the IVT, there must be m st. r_m takes x to 0



Stabilizer: Suppose r takes x to 0

Then $\mathrm{Stab}(x) = r^{-1}\,\mathrm{Stab}(0)\,r \cong \mathrm{Stab}(0)$

so it suffices to check stab(0)

(1) $O(2) \subseteq \mathrm{Stab}(0)$: $O(2)$ is generated by reflections in lines thru 0 (rotation by θ is product of 2 reflections 

)

(2) $\mathrm{Stab}(0) \subseteq O(2)$: Let $f \in \mathrm{Stab}(0) \leq \mathrm{Isom}\,\mathbb{H}^2$.

Note any reflection extends to a map of S_∞, so f does too.

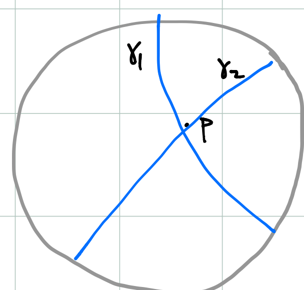
<u>Claim</u> The extension of f to S^1 is an <u>isometry</u> of $S^1 = S_\infty$:

<u>pf</u>: Fixes 0, preserves angles $\Rightarrow$ preserves arc length in S^1 $\Rightarrow$ is isometry of S^1 ✓

$\mathrm{Isom}\, S^1 = O(2)$, so we can compose f with an element $g \in O(2)$ so the composition gf fixes S^1

<u>Claim</u> $gf = id$, ie $f = g^{-1} \in O(2)$.

<u>Pf</u> $p \in \mathbb{H}$. Choose 2 geodosics thru p:



gf fixes endpts $\Rightarrow$

fixes geodosics

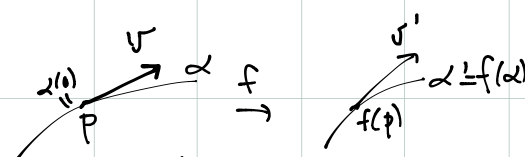
$\Rightarrow$ fixes $\cap$ pt p ✓

So, we have geodosics and isometries

I claim these determine the metric

<u>Lemma</u>: Isometries preserve $\dfrac{ds}{1-r^2}$ $\quad ds =$ distance in $\mathbb{R}^2$
$r =$ distance to 0

What does this mean?



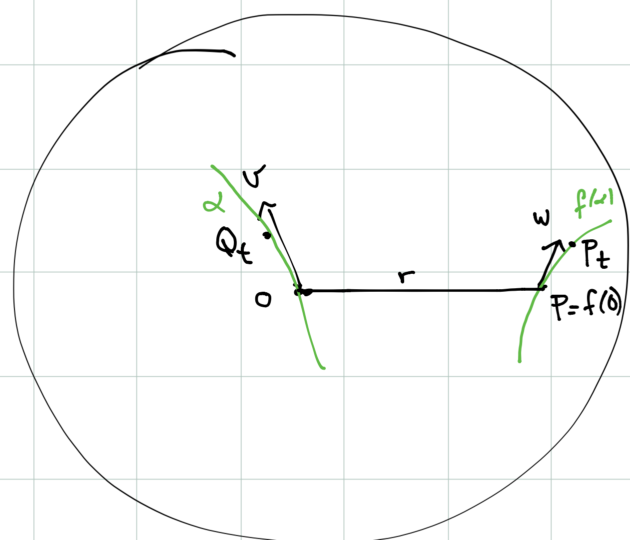
$v =$ tang vector at p $\qquad v' =$ tang at p'
(bad notation!)

Then $\dfrac{\|v\|}{1-r^2} = \dfrac{\|v'\|}{1-r'^2}$ $\qquad$ ($\|v\| =$ Euclidean length)

$$\lim_{\varepsilon \to 0} \frac{\|\alpha(\varepsilon) - \alpha(0)\|}{\varepsilon}$$

Proof. Suffices to show it for reflections f

First look at v = tang vector at O, $P = f(0)$



Take Q_t on α,

$P_t = f(Q_t)$

we want to show

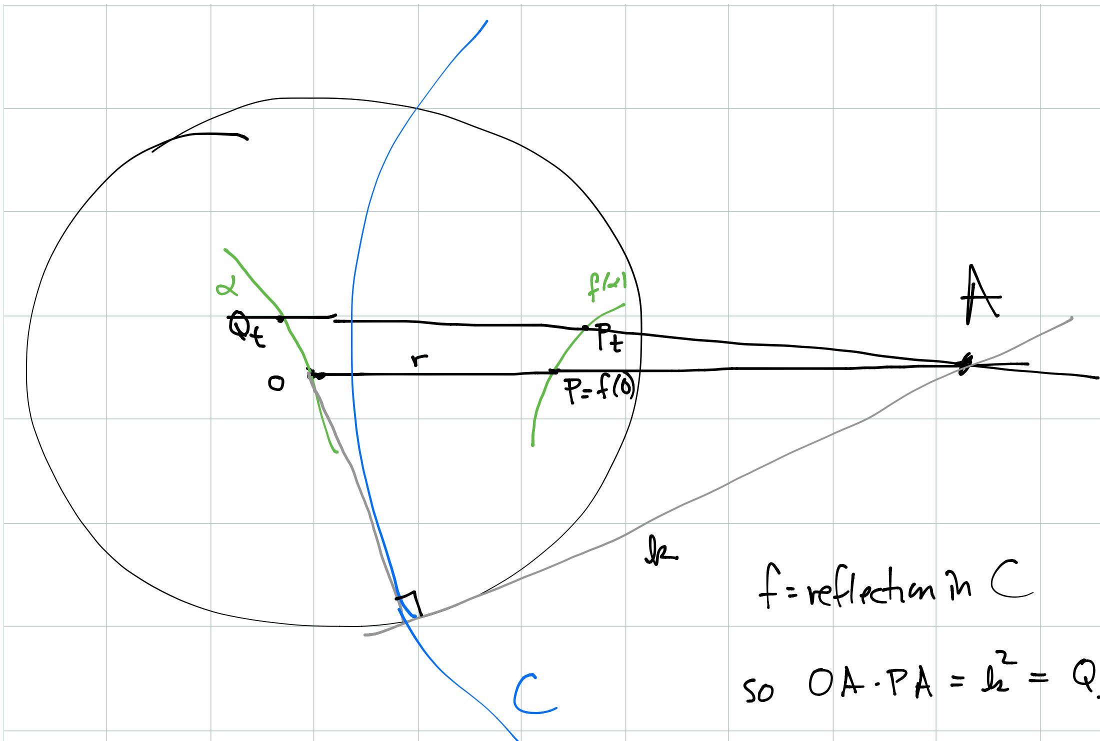
$$\|v\| = \frac{\|w\|}{1-r^2}$$

ie as $t \to 0$

$$\frac{\frac{Q_t O}{t}}{\frac{P_t P}{t} \cdot \frac{1}{1-r^2}} \to 1$$

ie $\frac{Q_t O}{P_t P} \to 1 - r^2$

So want to estimate $\frac{Q_t O}{P_t P}$



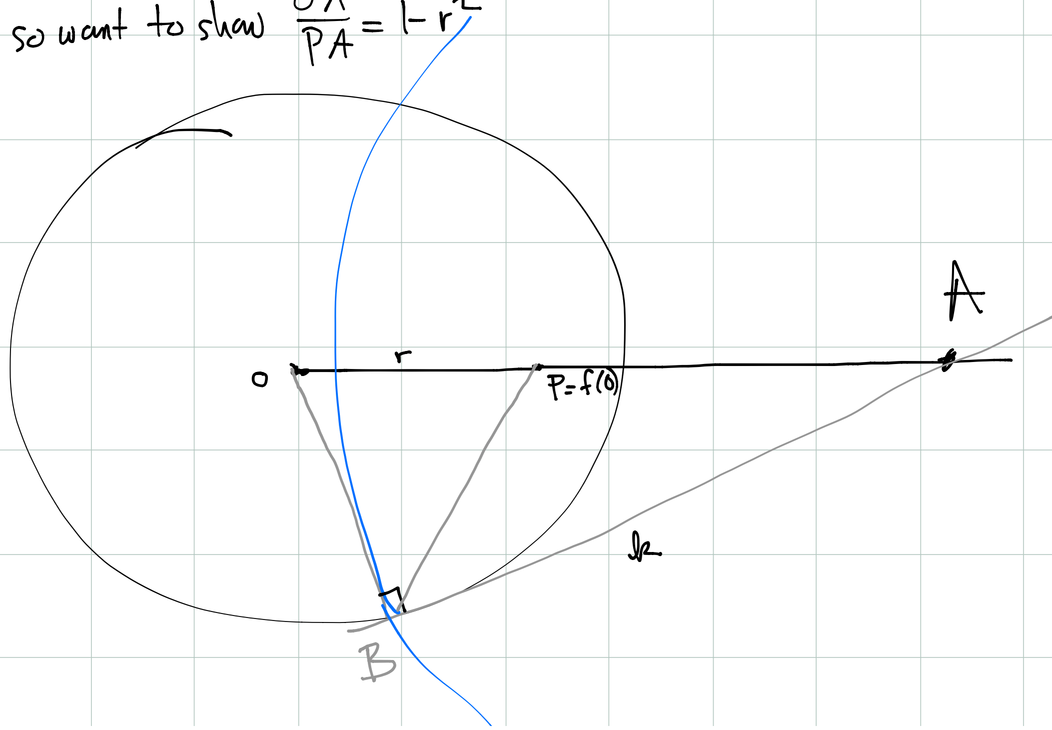
f = reflection in C

so $OA \cdot PA = k^2 = Q_tA \cdot P_tA$

so $OQ_tA \sim P_tPA$

so $\dfrac{OQ_t}{PP_t} = \dfrac{OA}{P_tA} \longrightarrow \dfrac{OA}{PA}$ as $t \to 0$

so want to show $\dfrac{OA}{PA} = 1 - r^2$

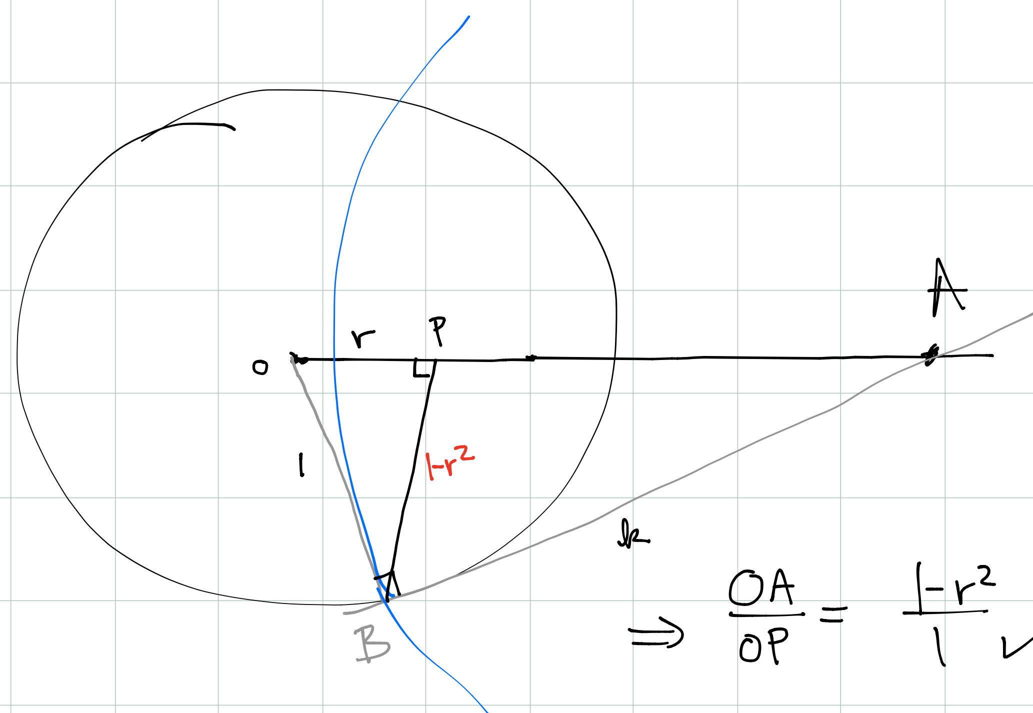


$OA \cdot PA = k^2$

so $\dfrac{OA^2}{k^2} = \dfrac{OA}{PA} \cdot \dfrac{OA}{OA} = \dfrac{OA}{PA} \cdot \dfrac{PA}{PA} = \dfrac{k^2}{PA^2}$

$\Rightarrow \quad \dfrac{OA}{k} = \dfrac{k}{PA} \Rightarrow OAB \sim ABP$

$\Rightarrow OA \perp BP$

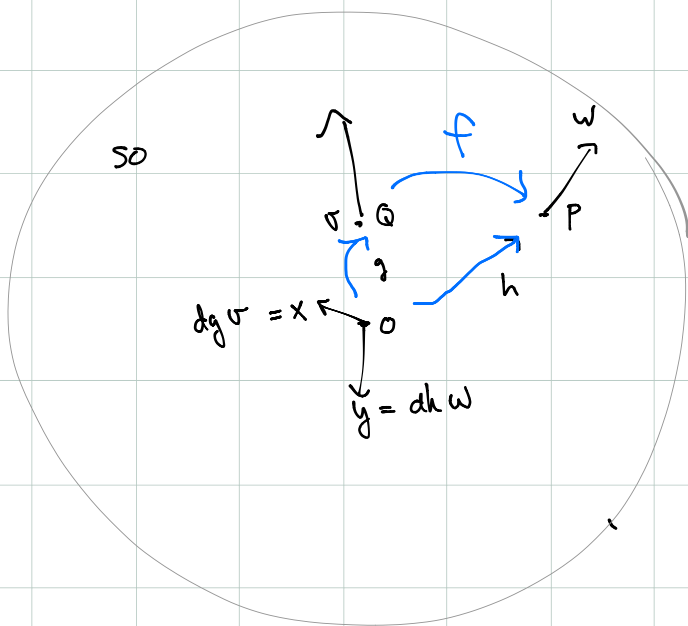


$\Rightarrow \dfrac{OA}{OP} = \dfrac{1-r^2}{1}$ ✓

What about v at $Q \neq 0$? $\quad w = df\,v$ at $P = f(Q)$

Choose reflections g taking O to Q, h taking O to P

Then $h^{-1} f g(0) = 0$, so $h^{-1} f g \in O(2)$, preserves lengths of vectors



$$\|x\| = \|y\|$$

$$\| \qquad\qquad \|$$

$$\frac{\|v\|}{1-r_Q} \qquad \frac{\|w\|}{1-r_P}$$

✓

So now we can compute w/ this metric:

<u>Remark</u> metric is usually normalized to $\frac{2ds}{1-r^2}$ to make $K=-1$