Tues, Jan 21

Recall: last week we were studying the hyperbolic plane $\mathbb{H}^2$

= unit disk in $\mathbb{R}^2$

geodesics = circle arcs $\perp$ to S^1

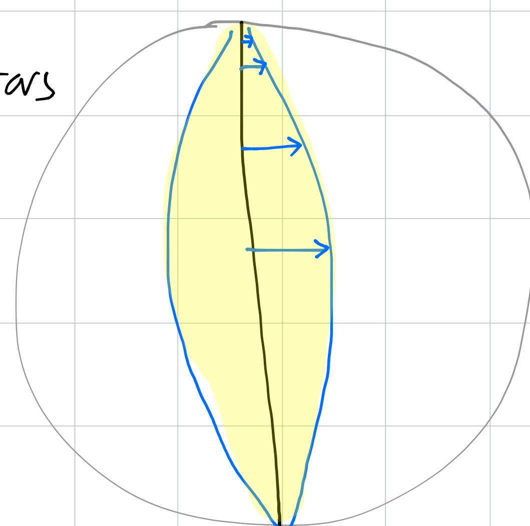
isometries = products of inversions in circles.

We just proved

Lemma: Isometries preserve $\frac{ds}{1-r^2}$

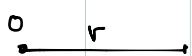
Note: ε-Nbd of a geodesic is a banana

all the blue vectors have the same length ε



Remark metric is usually normalized to $\frac{2\,ds}{1-r^2}$ to make $K=-1$

so now we can compute w/ this metric:

eg  $\sigma(t)=(t,0)$ has hyperbolic length

$$\int_0^r \frac{2\|\sigma'(t)\|}{1-t^2}\,dt = \int_0^r \frac{2\,dt}{1-t^2} = \int_0^r \left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt$$

$$= \ln\left(\frac{1+r}{1-r}\right) = 2\tanh^{-1}(r) = \rho \quad \left(\text{recall } \tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}\right)$$

$$\Rightarrow\ r = \tanh(\rho/2)$$

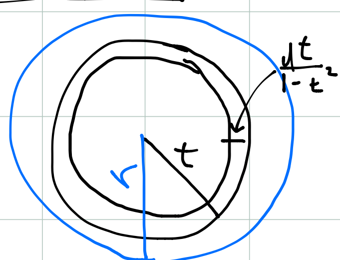
Circumference  : $\sigma(t) = (r\cos t, r\sin t)$

$$\int_0^{2\pi} \frac{2\|\sigma'(t)\|}{1-r^2}\,dt = \int_0^{2\pi} \frac{2r}{1-r^2}\,dt = \frac{4\pi r}{1-r^2}$$

$$= \frac{4\pi \tanh\frac{\rho}{2}}{1-\tanh^2\frac{\rho}{2}} = \frac{4\pi \frac{\sinh\frac{\rho}{2}}{\cosh\frac{\rho}{2}}}{1-\frac{\sinh^2 \rho/2}{\cosh^2 \rho/2}} = \frac{4\pi \sinh \rho/2 \cosh \rho/2}{\cosh^2 \rho/2 - \sinh^2 \rho/2}$$

$$= 2\pi\, 2\sinh \rho/2 \cosh \rho/2 \quad = 2\pi \sinh \rho$$

Area of circle : "integrate the circumference"



$$\int_0^r \frac{4\pi t}{1-t^2}\,\frac{2dt}{1-t^2} = 4\pi \int_0^r \frac{2t}{(1-t^2)^2}\,dt$$

$$= \frac{4\pi}{1-t^2}\Big|_0^r = 4\pi\left(\frac{1}{1-r^2}-1\right) = \frac{4\pi r^2}{1-r^2}$$

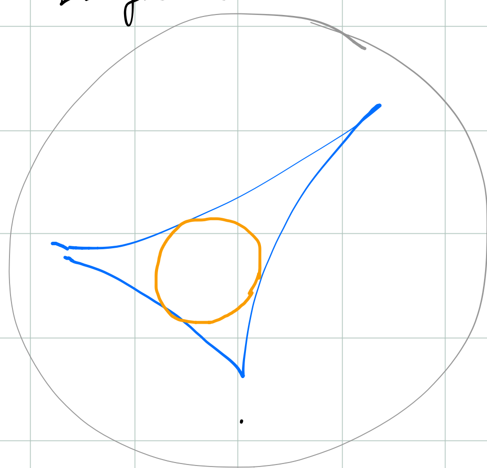
$$= 4\pi \sinh^2 \rho/2 = 2\pi(\cosh\rho - 1)$$

$$= 2\pi\left(\sqrt{1+\sinh^2\rho} - 1\right)$$

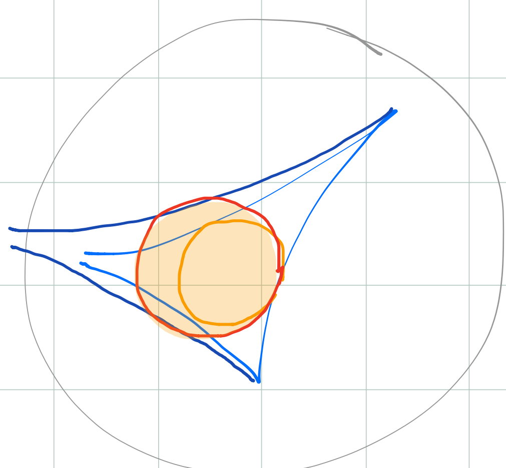
Note the hyperbolic area is approximately the same as the circumference for ρ large!

Geodesic triangles in $\mathbb{H}$:

Largest circle inscribed in a geodesic triangle?

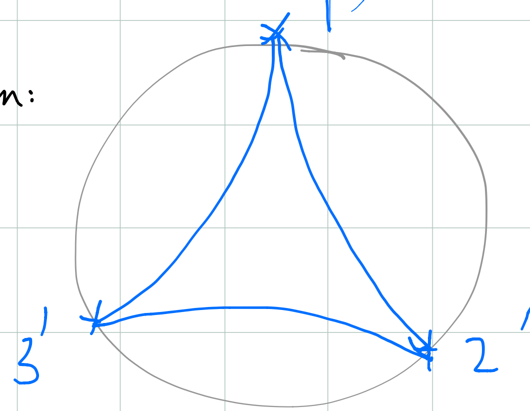


First remark: WMA the triangle is an <u>ideal</u> triangle, i.e. has vertices at ∞.

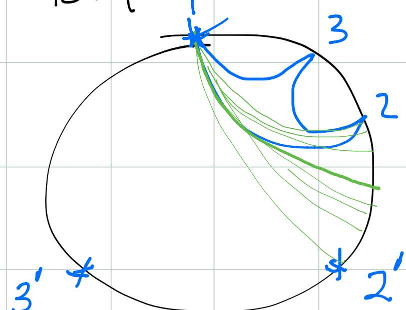


Otherwise could find a strictly larger circle by moving one vertex towards infinity (The new triangle contains a circle with strictly larger area $\Rightarrow$ larger radius)

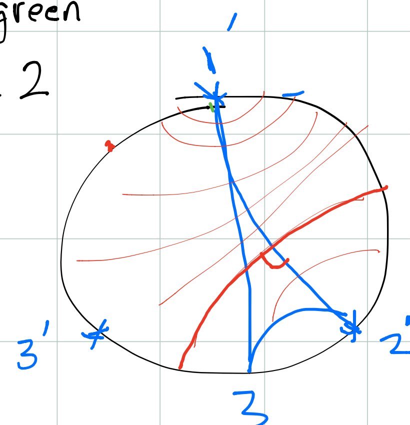
Second remark: Can use isometries to move $\triangle$ to "equilateral" position:



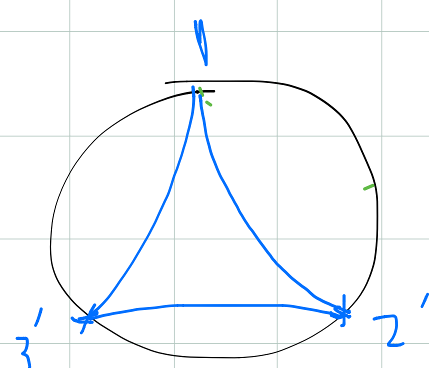
1. rotate one vertex to top:



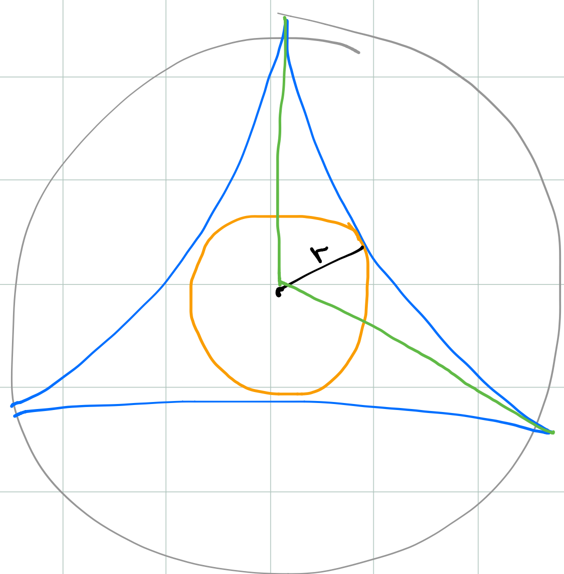
2. reflect in green arc to take 2 to 2' (IVT)



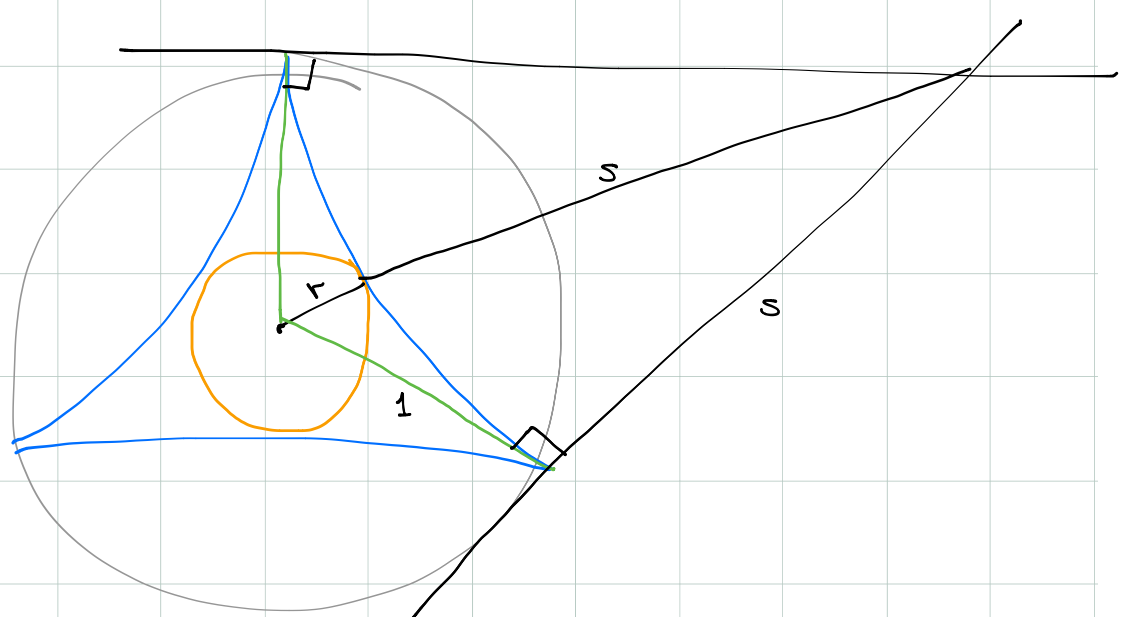
3. reflect in red arc $\perp$ to 1–2' side to take 3 to 3' (IVT)

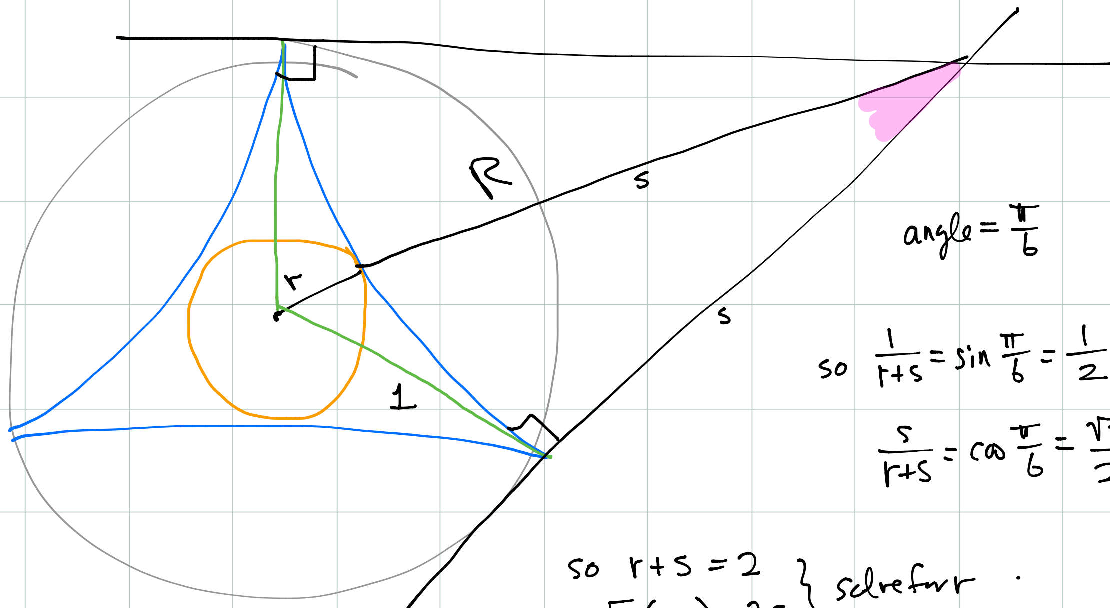


Now can calculate:



want to find r





angle $= \frac{\pi}{6}$

so $\frac{1}{r+s} = \sin\frac{\pi}{6} = \frac{1}{2}$

$\frac{s}{r+s} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

so $\left.\begin{array}{l} r+s = 2 \\ \sqrt{3}(r+s) = 2s \end{array}\right\}$ solve for r.

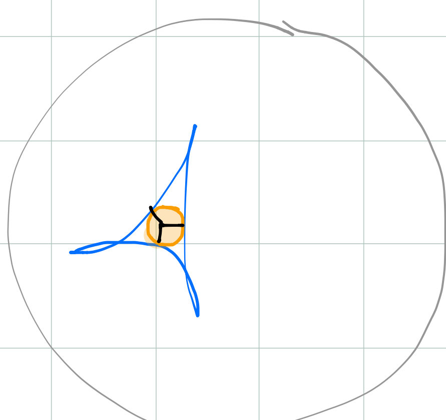
get $s = 2 - r$

$2\sqrt{3} = 2(2-r)$

$\sqrt{3} = 2 - r$, $r = 2 - \sqrt{3}$

So hyperbolic radius is $\rho = \ln\left(\frac{1+r}{1-r}\right) = \ln\left(\frac{3-\sqrt{3}}{\sqrt{3}-1}\right) = \ln\frac{(\sqrt{3}-3)(\sqrt{3}+1)}{2} = \ln\sqrt{3} = \frac{\ln 3}{2}$

Another way to say this:



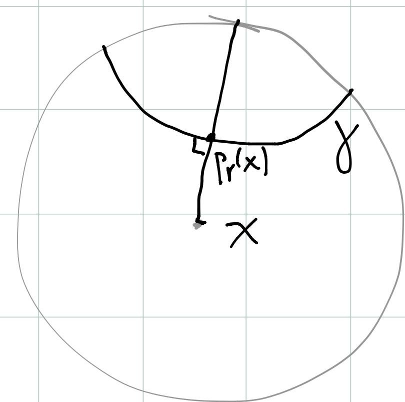
Cor Geodesic Δ's have centers that are within $\ln\sqrt{3}$ of all sides.

Pf: largest inscribed circle is tangent to all 3 sides

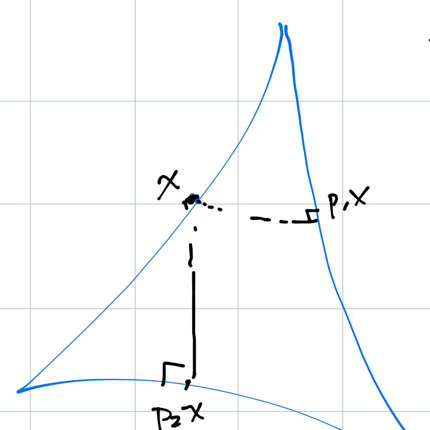
Projections

$x \in \mathbb{H}^2$ and $\gamma \subset \mathbb{H}^2$ a geodesic. Then there is a unique point $p_\gamma(x)$ on γ closest to x, called the <u>projection</u> of x onto γ

(exercise). Hint:



<u>Now claim</u>: <u>All</u> geodesic triangles in $\mathbb{H}$ are $\ln 3$-<u>thin</u>:

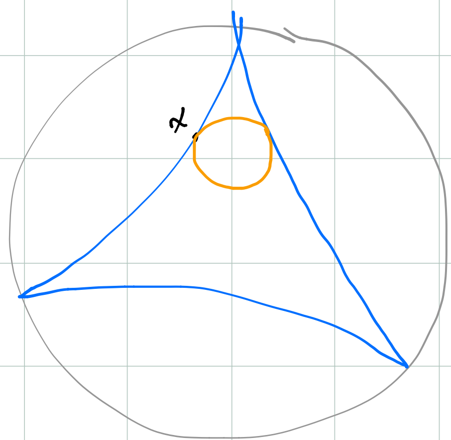


<u>Def</u> A Δ is <u>C-thin</u> if for all x the projections $p_1 x$ and $p_2 x$ to the sides s_1, s_2 not containing x satisfy

$$\min(d(x, p_1 x), d(x, p_2 x)) \leq C$$

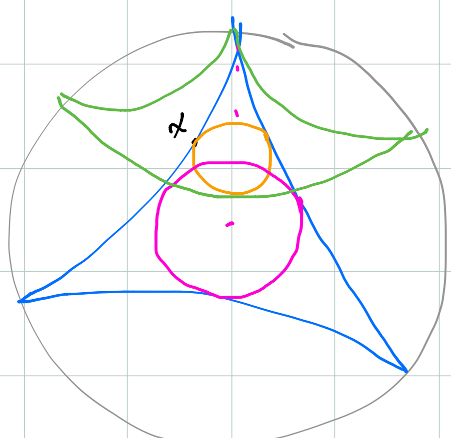
Any Δ is contained in ideal Δ (exercise), so

Pf: wlog we may assume Δ is an ideal triangle

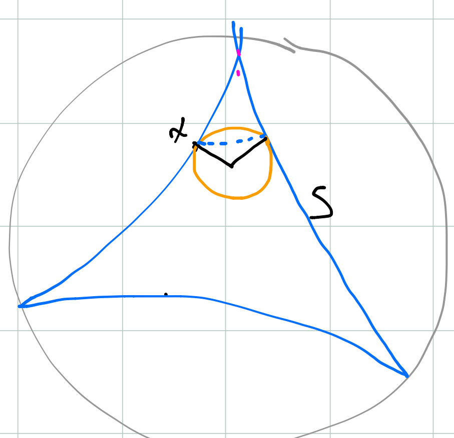


(in fact our standard ideal Δ)

An inscribed circle tangent to x has hyperbolic radius $< \frac{\ln 3}{2}$



Pf: The pink one has radius $\frac{\ln 3}{2}$. The green Δ is isometric to the blue one, and the image of the pink circle contains the orange one. ✓



So $d(x,s) < \ln 3$ ✓