Monday, Jan 27

Last time: Defined the Cayley graph $\mathcal{C}(G,S)$ of a group G generated by a (finite) set $S \subset G$ $(S=S^{-1})$

vertices = elts of G

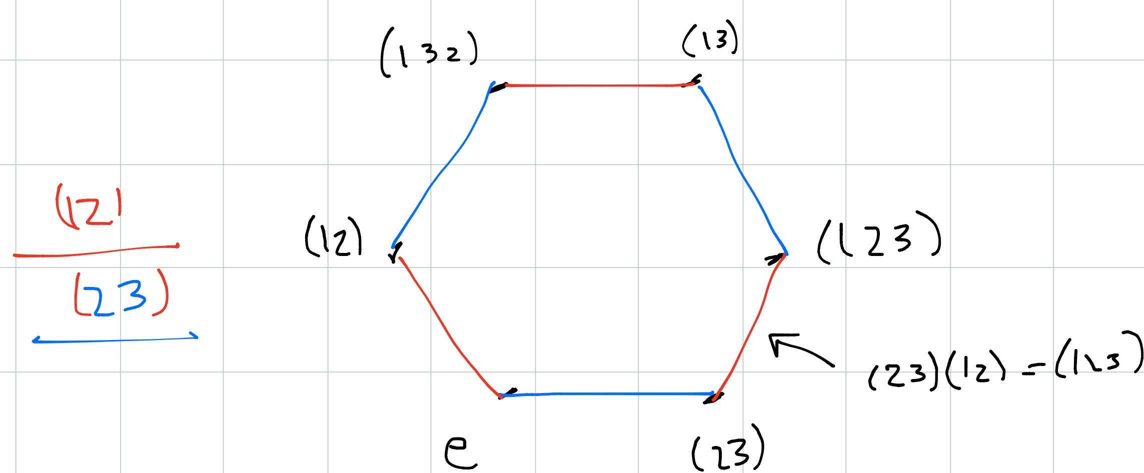
(oriented) edges: $g \longrightarrow gs$

$x \in G$ acts on $\mathcal{C}(G,S)$ by left multiplication:

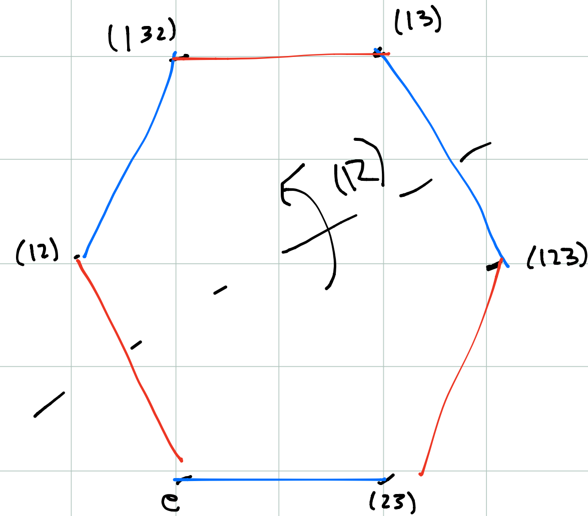
$$g \longrightarrow gs \quad \overset{x}{\mapsto} \quad (xg) \longrightarrow (xg)s$$

Example: S_3 = symmetric group on 3 letters

Generated by transpositions (12) and (23) $(S=S^{-1})$!



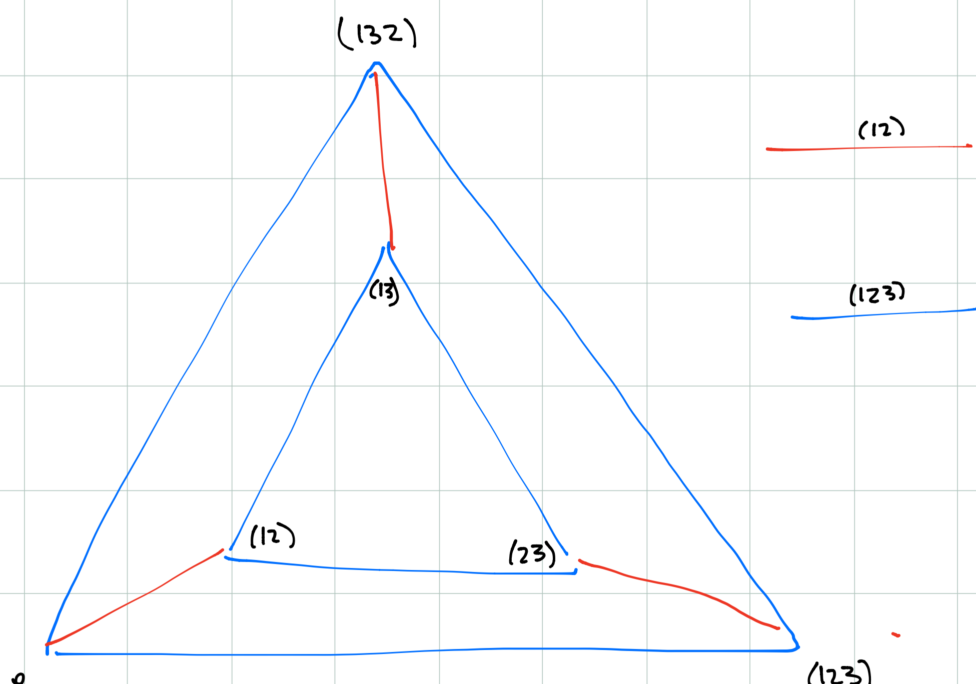
(12) acts by



(23) acts by vertical flip.

(123) acts by rotation.

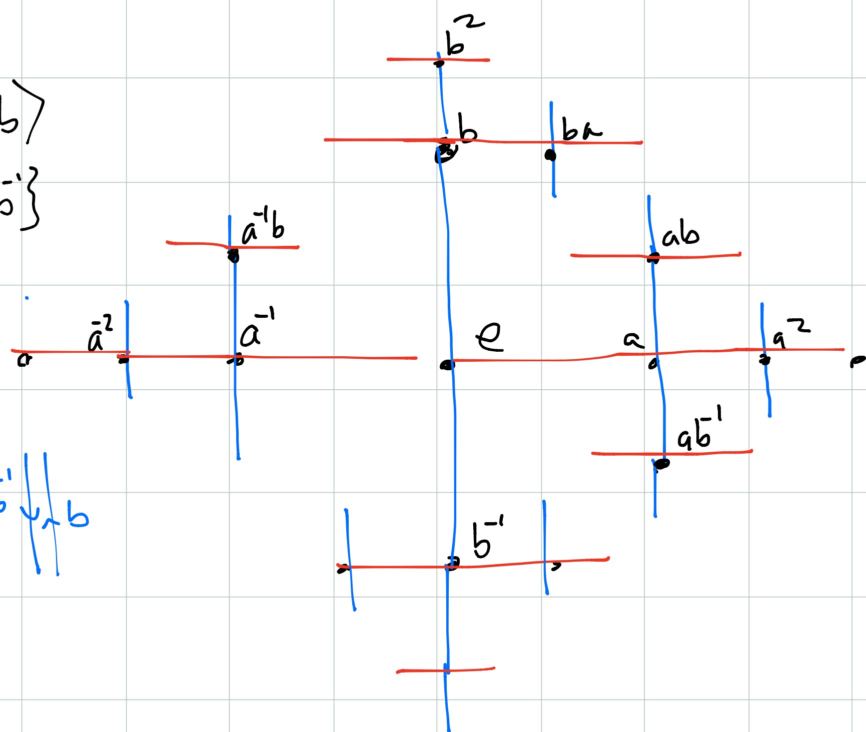
Last time did S_3 w/ generators $S = (12), (123)$ $\left(S \cup S^{-1} = \{(12), (123), (132)\} \right)$

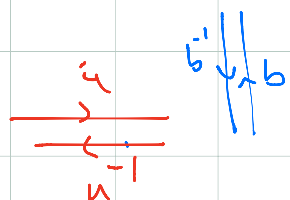


Example:

$G = F\langle a,b \rangle$

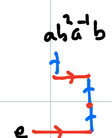
$S = \{a, b, a^{-1}, b^{-1}\}$





action of a moves tree to the right, b moves up.

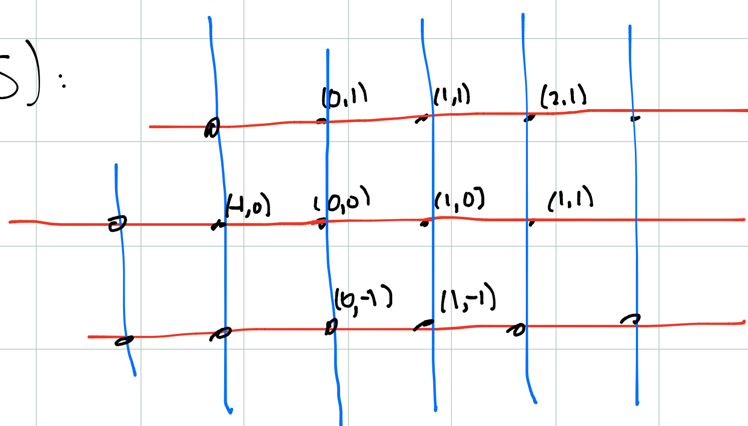
$ab^2a^{-1}b$ moves e to



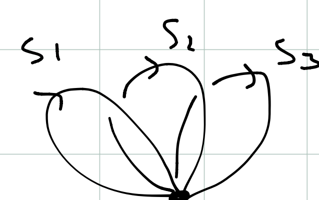
– we've seen this before!

Example $\mathbb{Z}^2 = \{(a,b) \mid a,b \in \mathbb{Z}\}$ $\quad S = \{(1,0), (0,1)\}$

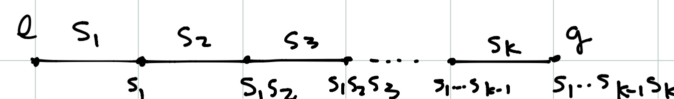
$\mathcal{C}(\mathbb{Z}^2, S)$:



The action of G on $\mathcal{C}(G,S)$ is free, and the quotient is just a finite wedge of circles, a "rose" with one petal for each generator.



$\mathcal{C}(G,S)$ is connected because S generates G: to get a path from e to g, write $g = s_1 \cdots s_k$, then



To describe other examples, useful to have the notion of a <u>group presentation</u>

<u>Def</u> A group H is <u>normally generated</u> by $R \subset H$ if H is generated by R and all of its conjugates gRg^{-1}. Sometimes write $H = \langle\langle R \rangle\rangle$

<u>eg</u> S_3 is normally generated by (12)

(it's <u>generated</u> by (12) and $(23) = (13)(12)(13)^{-1}$.
We don't need the other conjugate $(13) = (23)(12)(23)$ but it doesn't hurt to include it)

Recall G <u>generated</u> by S means $\exists$ homomorphism $F(S) \twoheadrightarrow G$. The kernel of this map is a (normal, free) subgroup of $F(S)$. If the kernel is normally generated by R, we write $G = \langle S \mid R \rangle$; this is called a <u>presentation</u> of G.

eg $F\langle S\rangle = \langle S \mid \ \rangle$ (no relations)

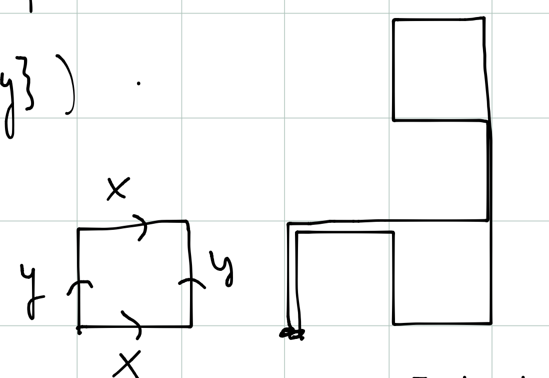
$\mathbb{Z}/n\mathbb{Z} = \langle x \mid x^n \rangle$

$\mathbb{Z}^2 = \langle x, y \mid xyx^{-1}y^{-1} \rangle$ (exercise)

A <u>relation</u> is any element of $\ker F\langle S\rangle \twoheadrightarrow G$

It corresponds to a closed loop in $\mathcal{C}(G,S)$:

eg in $\mathcal{C}(\mathbb{Z}^2, \{x,y\})$



$yx^2yx^{-1}yxy^{-3}x^{-1}yx^{-1}y^{-1}$

$xyx^{-1}y^{-1} = 1$ $\quad = yyyy^{-3}yy^{-1}x^2x^{-1}xx^{-1}x^{-1}$

$= \emptyset$

<u>Def</u> A group G is <u>finitely presented</u> if there are subsets $S, R \subseteq G$ with $G = \langle S \mid R \rangle$, ie

$F\langle S\rangle \twoheadrightarrow G$ with krnel $\langle\langle R \rangle\rangle$, i.e. $G \cong F\langle S\rangle / \langle\langle R\rangle\rangle$

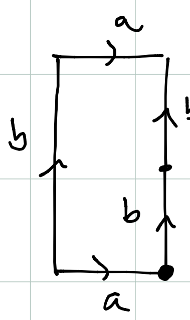
(this should be finitely present<u>able</u>, but...)

Using presentations it is easy to write down lots of groups — just write a finite set of generators and some words in the generators to use for relations.

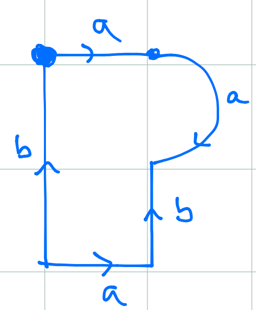
But: It's clear that the same group can have many different presentations so there is the problem of deciding when two presentations define isomorphic groups. (The isomorphism problem)

It's not even clear from looking at a presentation whether a given word in $F\langle S\rangle$ is in $\langle\langle R\rangle\rangle$ (The word problem) or whether two words are conjugate in the group (the conjugacy problem). All of these problems are unsolvable in general, meaning there is no algorithm for solving them that will work for any presentation(s).

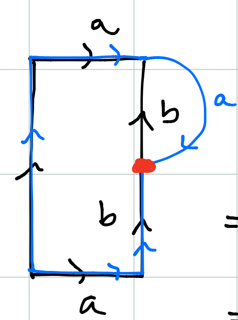
Example $\langle a,b \mid a^{-1}bab^{-2},\ b^{-1}aba^{-2}\rangle$ is a presentation of the trivial group.



$a^{-1}bab^{-2}$



$b^{-1}aba^{-2}$



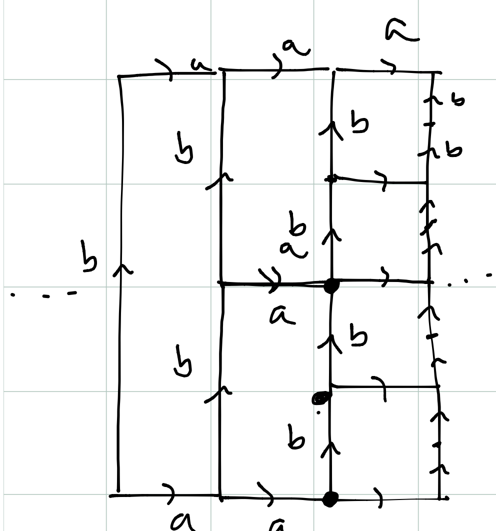
$$ba^{-1}b^{-1}ab = a^{-1}a^{-1}b^{-1}ab$$

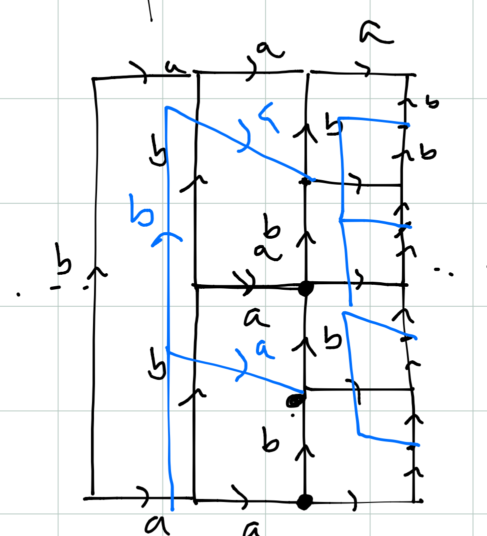
$$\Rightarrow b = a^{-1}$$

$$\Rightarrow b = 1 = a$$

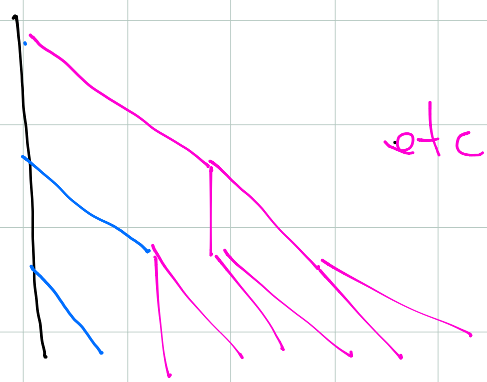
Example $\langle a,b \mid a^{-1}ba = b^2\rangle$ is called a Baumslag–Solitar group – has many interesting properties (eg it's a proper quotient of itself)

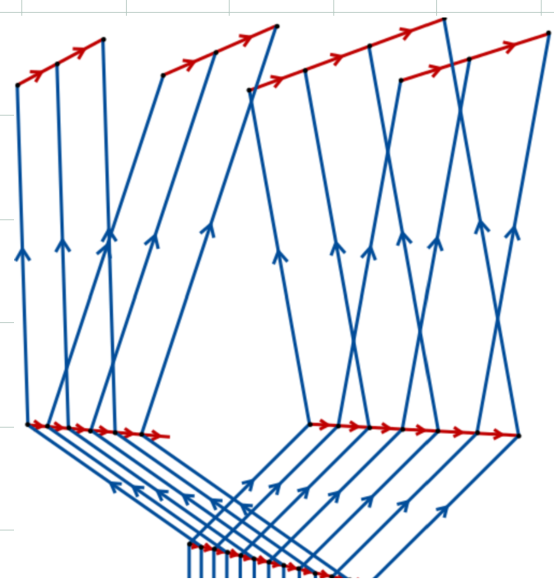
Its Cayley graph doesn't collapse





sideview





Cayley graph of Baumslag-Solitar group B(1,2)