Tues, Jan. 28

Last time: Gave examples of Cayley graphs.

We now have a metric space associated to any pair (G,S) with S finite. This is not a smooth manifold, but still want to think of it as having curvature: We're especially interested in negative curvature.

Let X be a metric space. A geodesic is a "shortest path between points."

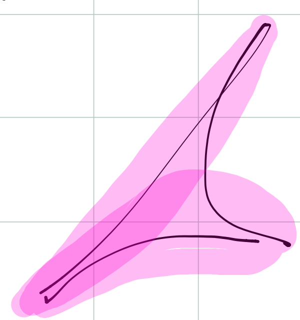
Def A geodesic in X is a map $\gamma : [0,d] \longrightarrow X$ s.t.

$$d(\gamma(s),\gamma(t)) = |s-t| \text{ for all } s,t \in [0,d]$$

Note this is a global def'n, unlike the def in Riemannian geometry. In this def a great circle arc on S^2 is not a geodesic if it goes more than half-way around S^2.

Def X is a geodesic metric space if every two points are connected by a geodesic: $\exists$ geodesic $\gamma : [0, d(x,y)]$ w/ $\gamma(0) = x$ and $\gamma(d(x,y)) = y$

Def A geodesic metric space X is <u>δ-hyperbolic</u> if geodesic triangles are δ-thin, ie



each side of Δ is contained in a δ-nbd of the union of the other two sides.

Def X is (Gromov) hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

That's for a metric space. What about a <u>group</u> G?

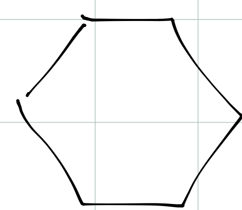
Def G is hyperbolic if there is some generating set S st. $\mathcal{C}(G,S)$ is hyperbolic

There is an obvious question here: Does this depend on the choice of S? We know Cayley graphs for different S are different — can one be hyperbolic and one not?

Before we answer this, let's look at our examples

$\underline{G = S_3}$

or



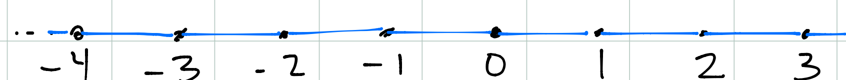
or ...

With $\underline{\text{any}}$ generating set $S \subseteq S_3$, the entire graph is finite, with diameter $< 6 = |S_3|$

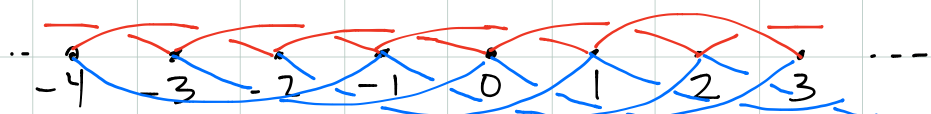
So all triangles are 6-thin...

— $\boxed{\text{All finite graps are hyperbolic}}$

$\underline{G = \mathbb{Z}}$

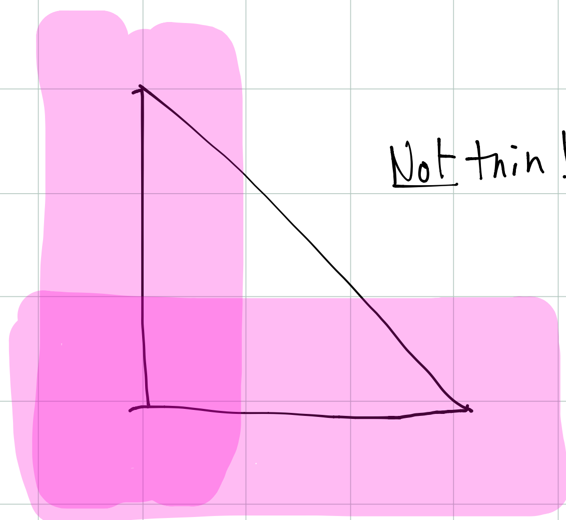


Triangles are 0-thin.



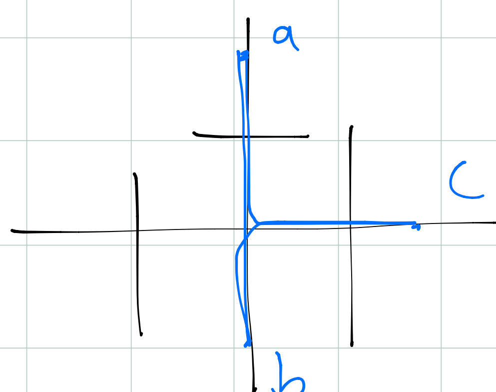
Triangles are 2-thin $(d(n, n+1) = 2)$

$G = \mathbb{Z}^2$:



Not thin!

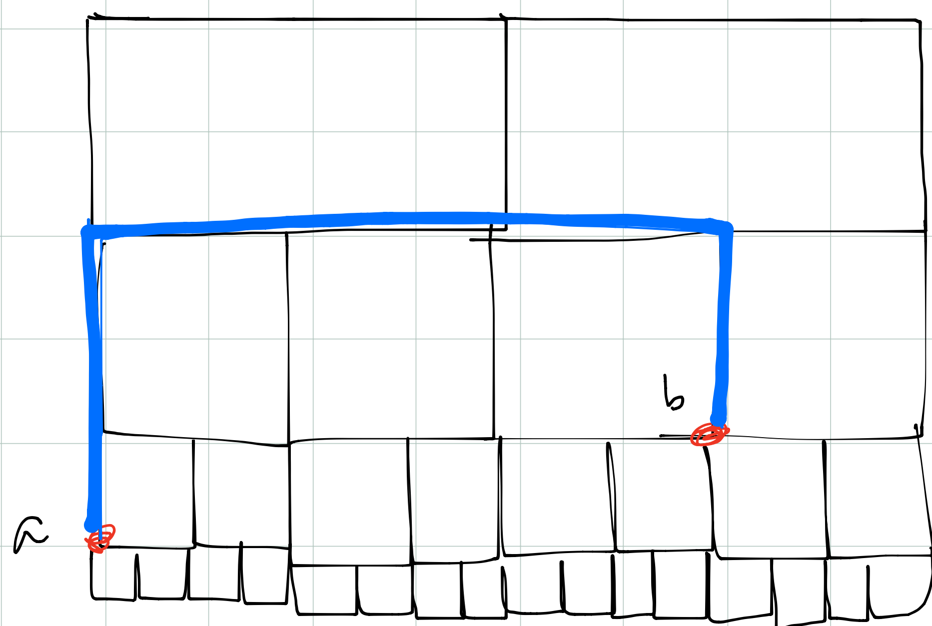
$G = F_2$:

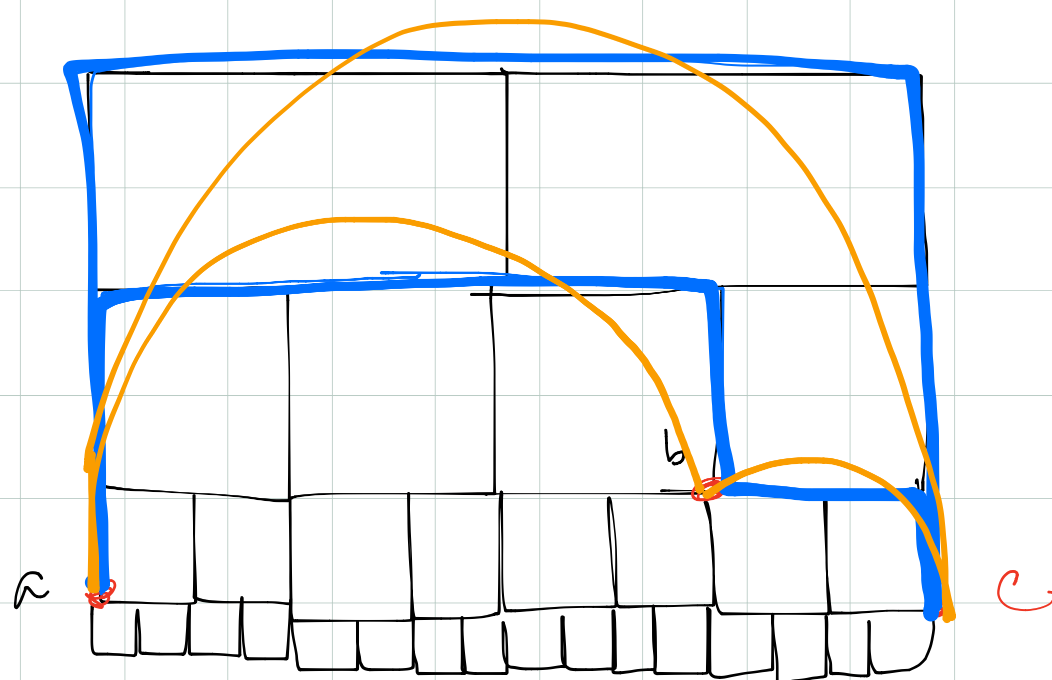


Thin!
$\delta = 0$

$G = B(1,2)$ If a Δ has all 3 vertices in one "sheet"

Geodesics go "up and over"





These triangles <u>are</u> thin

<u>But</u>: Triangles which do <u>not</u> lie in the same sheet need <u>not</u> be thin (exercise?)

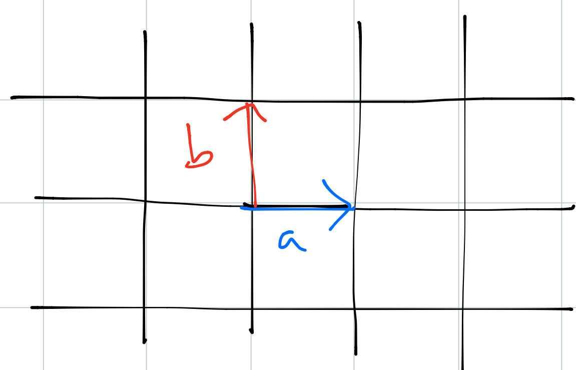
Surface groups

$\exists!$ oriented surface of genus g for $g = 0, 1, 2, \dots$ called S_g

$S_0 = $ 2-sphere, $S_1 = T = $ torus, $S_2 = $  etc

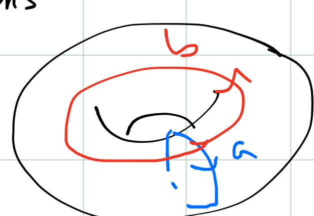
$\pi_1(S_g)$ is called a surface group (though often only for $g > 1$)

$\underline{g=1}$: Picture of T, its universal cover $\widetilde{T} = \mathbb{R}^2$ and action of $\pi_1 T = \mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid [a,b] \rangle$ on $\mathbb{R}^2$:

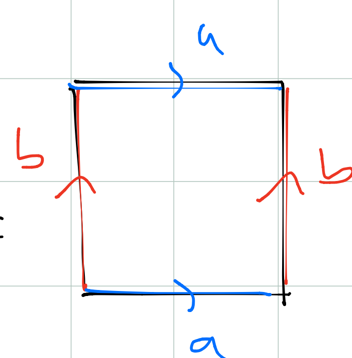


$= \mathbb{R}^2$, tiled by squares

quotient by action of $\pi_1 T$ by deck transformations $\downarrow$



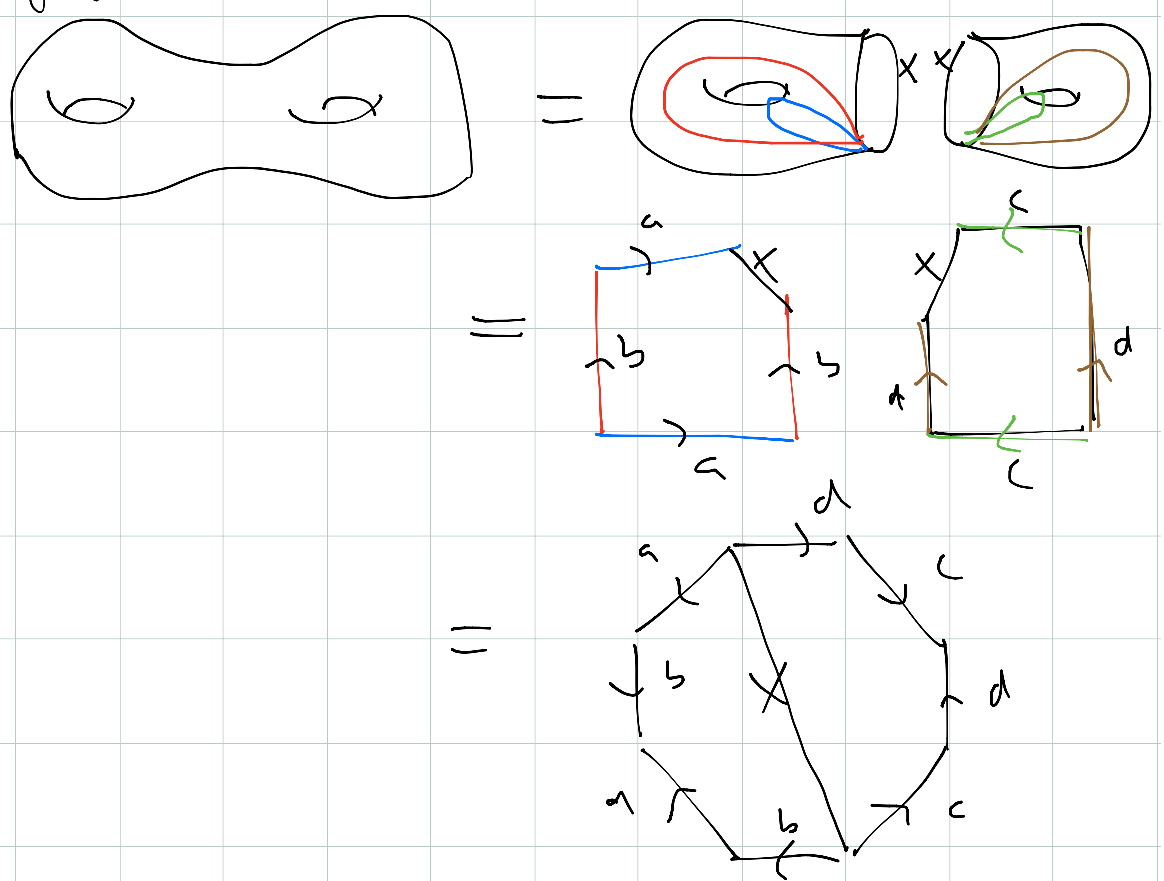
$=$



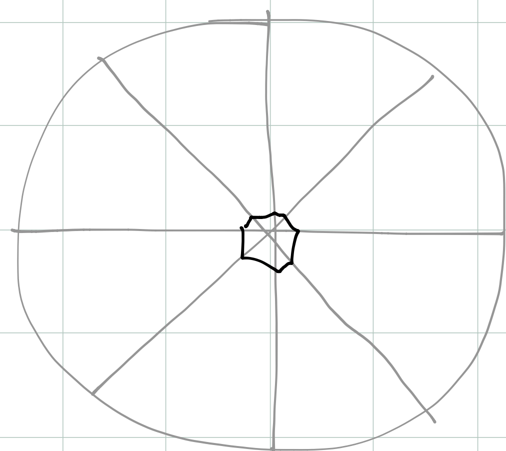
There is a very similar picture for S_g, $g>1$

But instead of tiling $\mathbb{R}^2$ by squares, we tile $\mathbb{H}^2$ by regular $4g$-gons

eg $g=2$



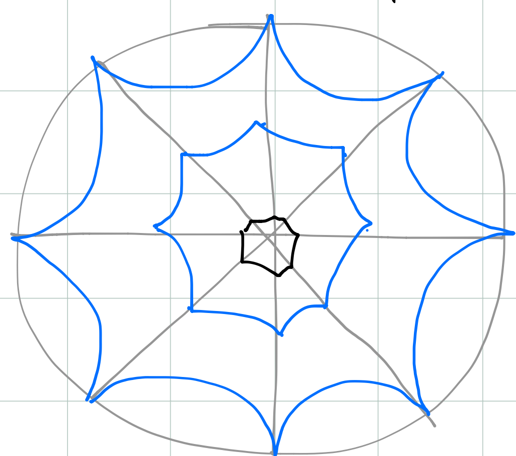
You can't tile $\mathbb{R}^2$ by regular 8-gons, why can you tile $\mathbb{H}^2$? A: IVT



$$\text{metric} = \frac{ds}{1-r^2}$$

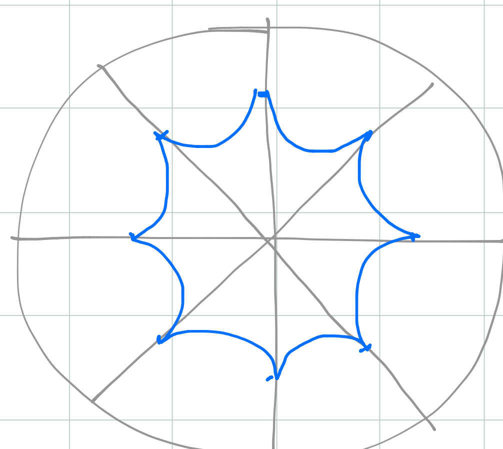
For r close to 0, metric is $\approx$ Euclidean, so angles in a regular 8-gon are $\approx \frac{3\pi}{4}$

Now let the radius of the 8-gon grow

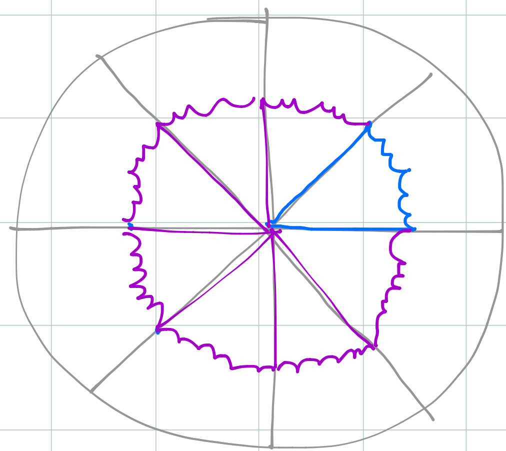


When they get to S_∞, the angles are 0

Somewhere in between the angles are $\frac{\pi}{4}$.

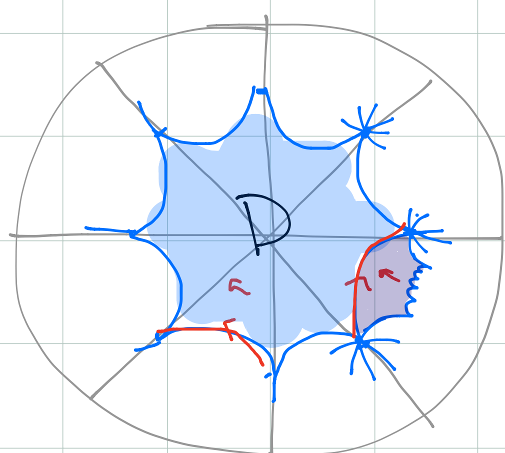


Move a vertex to 0 by an isometry to see better



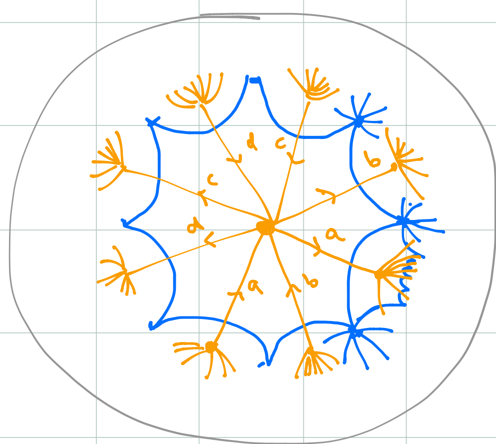
You can fit 8 of these together exactly at each vertex.

<u>Claim</u>: If you continue, this tiles $\mathbb{H}^2$.



There is an orientation-preserving isometry f_a of $\mathbb{H}^2$ taking $\overset{a}{\frown}$ to $\overset{a}{\frown}$

It moves D to an adjacent tile



Let Γ be the dual graph to this tiling of $\mathbb{H}^2$

At each vertex

There is one edge for each generator of $\pi_1 S_g$ (including inverses)

$\pi_1 S_g$ acts freely by isometries: $\mathbb{H}^2 = \tilde{S}_g$ and f_a is the deck transformation associated with a.

So the vertices $\longleftrightarrow$ elements of $\pi_1 S_g$

In other words, this is the Cayley graph

$\mathcal{G}(\pi_1 S_g, \{a_1, b_1, \ldots, a_n, b_n\})$

It is embedded isometrically in $\mathbb{H}^2$.

<u>Claim</u> distance in $\mathcal{G}$ is $\approx$ distance in $\mathbb{H}^2$

$\therefore$ triangles in $\mathcal{G}$ are δ-thin for some $\delta \approx \ln 3$.