GGT - Lecture 2

Last time: defined free group using a universal property: $\exists S \subset F$, any set map $S \longrightarrow G$ extends to a unique homomorphism

$$\begin{tikzcd} S \arrow[r] \arrow[d, phantom, "\cap" description] & G \\ F \arrow[ur, dotted] & \end{tikzcd}$$

Claimed: This $\Rightarrow$ every $x \in F$ is a product of elts of S and S^{-1}.

pf Suppose not. Then I Claim there are 2 different homomorphisms $F \longrightarrow F$ extending the inclusion $S \hookrightarrow F$

clearly $id: F \to F$ extends it

Let G be the subgroup of F formed by all products of elts of S and S^{-1}. We have $S \hookrightarrow G$, so by the universal property, $\exists f: F \longrightarrow G$ extending the inclusion.

Then $id: F \to F$

and $f: F \to G \subsetneq F$ both extend

the inclusion $S \subset F$, but $f \neq id$.

(it's not surjective), contradicting uniqueness.

Next we constructed a group $F(S)$ from a set S

elts = reduced words in $S \cup S'$ plus $\emptyset$

operation = adjunction + cancellation

id = $\emptyset$

inverse $(x_1 \dots x_k)^{-1} = \overline{x}_k \dots \overline{x}_1$

($x \to \overline{x}$ involution on $S \amalg S'$)

associative law

$$\left.\begin{array}{l} X = x_1 \dots x_k \\ Y = y_1 \dots y_\ell \\ Z = z_1 \dots z_m \end{array}\right\}$$

want to show:
$(XY)Z = X(YZ)$

$$\overbrace{x_1 \cdots \underbrace{\cdots x_k}_{r}}^{X}\ \overbrace{\underbrace{y_1 \cdots}_{r} \cdots \underbrace{\cdots y_l}_{s}}^{Y}\ \overbrace{\underbrace{z_1 \cdots}_{s} \cdots z_m}^{Z}$$

r letters cancel between X and Y, $0 \le r \le \min(k, l)$

s letters cancel between Y and Z, $0 \le s \le \min(l, m)$

If $r+s<l$

$$\overbrace{\boxed{a}\ \underbrace{\boxed{\bar{u}}}_{r}}^{X}\ \overbrace{\underbrace{\boxed{u}}_{r}\ \boxed{v}\ \underbrace{\boxed{w}}_{s}}^{Y}\ \overbrace{\underbrace{\boxed{\bar{w}}}_{s}\ \boxed{b}}^{Z}$$

So answer on either side is

$x_1 \cdots x_{k-r} y_{r+1} \cdots y_{l-s} z_{s+1} \cdots z_m = avb$ which is reduced!

If $r+s=l$, both sides are equal to $(x_1 \cdots x_{k-r})(z_{s+1} \cdots z_m)$

$= ab$, which is well-defined

If $r+s>\ell$

$$\overbrace{\boxed{\;a\;}\,\boxed{\bar{v}\,\bar{u}}}^{X}\;\overbrace{\underbrace{\boxed{\;u\;}\,\boxed{\;v\;}}_{r}\,\boxed{\;w\;}}^{Y}\;\overbrace{\boxed{\bar{w}\,\bar{v}}\,\boxed{\;b\;}}^{Z} \qquad \underbrace{v\,w}_{s}$$

$(XY)Z = (aw)\bar{w}\bar{v}b$. Note 1st letter of $\bar{v}$ is not equal to last letter of a, so

$= a\bar{v}b$ <u>reduced</u> if $\bar{v} \neq \emptyset$, otherwise $= ab$

$X(YZ) = (a\bar{v}\bar{u})ub = a\bar{v}b$ as above ✓

So we have a group. Is it a free group?

Claim it's free on S.

Have $S \subset F(S)$

Suppose have a set map $S \to G$ a group. Need to show $\exists!$ homomorphism $f: F \to G$ extending φ.

Given $w \in F\langle S\rangle$, $w = s_1 \dots s_n$, $s_i \in S \cup S'$

Define $f(\emptyset) = 1$

$$\left.\begin{array}{l} f(s_i) = \varphi(s_i) \\ f(\bar{s}_i) = \varphi(s_i)^{-1} \end{array}\right\} \text{ for } s_i \in S$$

$$f(w) = f(s_1) \dots f(s_k)$$

This is a homomorphism:

$$u = x_1 \dots x_k \quad v = y_1 \dots y_\ell \Rightarrow uv = x_1 \dots x_{k-r} y_{r+1} \dots y_\ell$$

$$f(u)f(v) = f(x_1 \dots x_k)\, f(y_1 \dots y_\ell)$$

$$= f(x_1) \dots f(x_k)\, f(y_1) \dots f(y_k)$$

$$= f(x_1) \dots \underbrace{f(y_r)^{-1} \dots f(y_1)^{-1}}_{r}\, \underbrace{f(y_1) \dots f(y_r)}_{r}\, f(y_{r+1}) \dots f(y_\ell)$$

$$= f(x_1 \dots x_{k-r} y_{r+1} \dots y_\ell) = f(uv) \checkmark$$

<u>Uniqueness</u>

Suppose $f' : F \longrightarrow G$ also extends φ

$w \in F \Rightarrow w = s_1 \dots s_k$, $s_i \in S \cup S'$

f' a homomorphism $\Rightarrow$ $f'(\phi) = 1 \Rightarrow f'(s_i \bar{s_i}) = 1$

$$\Rightarrow f'(\bar{s_i}) = f'(s_i)^{-1}$$

$w \in F \Rightarrow w = s_1 \dots s_k$, $s_i \in S \cup S'$

$$\Rightarrow f'(w) = f'(s_1) \dots f'(s_k)$$

$$= \varphi(s_1)^{\pm 1} \dots \varphi(s_k)^{\pm 1}$$

$$= f(s_1 \dots s_k) = f(w) \checkmark$$

Some facts about free groups:

$F = F\langle S \rangle$, $S' \subset F$ a different set with $F = F\langle S' \rangle$

Then $|S| = |S'|$, ie there is a bijection $S \leftrightarrow S'$

Pf: Any map $S \to \mathbb{Z}/2$ extends to a unique homomorphism $F \longrightarrow \mathbb{Z}/2$. There are $2^{|S|}$ such maps, so $|\mathrm{Hom}(F, \mathbb{Z}/2)| = 2^{|S|}$

Similarly, $|\mathrm{Hom}(F, \mathbb{Z}/2)| = 2^{|S'|}$.

$2^{|S|} = 2^{|S'|} \Rightarrow \exists$ bijection $S \to S'$, ip $|S| = |S'|$

(note: we will be almost exclusively concerned with the case $|S| < \infty$ or S countable)

So now we can make

Def If $F = F\langle S \rangle$, then $|S|$ is the rank of S

Thm $F \cong F'$ iff they have the same rank pf If $F = F\langle S\rangle$, $F' = F\langle S'\rangle$ and $|S| = |S'|$, choose a bijection $S \longrightarrow S'$ and extend it to $\rho: F \longrightarrow F'$ (uniquely)

Sim, get $\rho': F' \longrightarrow F$. The composition $\rho' \circ \rho$ is the identity on S, so $=$ id on F. Similarly, $\rho \circ \rho' =$ id on F', ie ρ is an $\cong$, with inverse ρ'.

Conversely, if $\rho: F \longrightarrow F'$ is an isomorphism, and $F = F\langle S\rangle$, let $S' = \rho(S)$. Since ρ is 1-1, $|S'| = |S|$. Claim F' is free on S'.

pf: $\varphi: S' \longrightarrow G \Rightarrow \varphi \circ \rho: S \longrightarrow G$ extends to $!\ f: F \longrightarrow G$. Then $f' = f \circ \rho^{-1}$ extends φ

Uniqueness also follows immediately ✓

More facts as exercises?

(when are two elts conjugate?)

So, we've defined free groups, shown they exist

proved they are isomorphic $\Leftrightarrow$ have same rank.

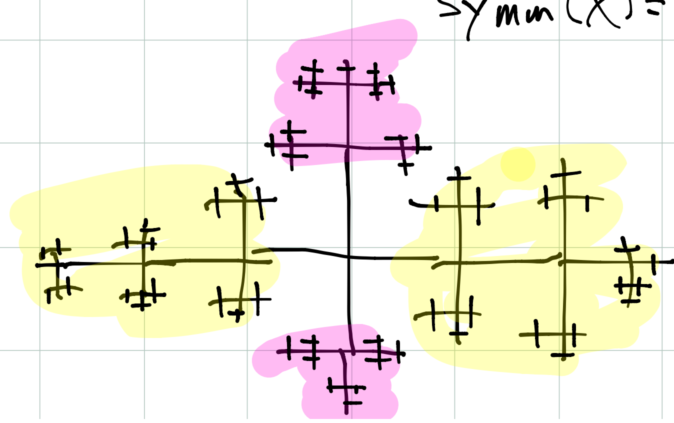
To do GGT, want to make them *act* on something. For free groups this is easy:

X = space $\quad$ $Sym(X)$ is a group

$F = F(S)$. Any set map $S \to Symm(X)$ extends to a homomorphism $F \longrightarrow Symm(X)$, i.e. an *action* of F on X. But we want *interesting* actions, from which we can deduce properties of F

eg $F = F\langle a, b \rangle$, X = "∞ telephone pole" = metric space

$Symm(X) = Isom(X)$



all edges have the same length

<u>Claim</u> Action of F is <u>free</u>: point stabilizers are trivial.

<u>Pf</u>: p = central pt never comes back by ping-pong

Each vertex $= wp$ for some $w \in F$

If $u(wp) = wp$, then $w^{-1}uwp = p$

so $w^{-1}uw = 1$ so $u = 1$.

(If x is not a vertex it's on some translate of the central ...)

$=$

This "picture" of F is the key to proving that a given group is free ...