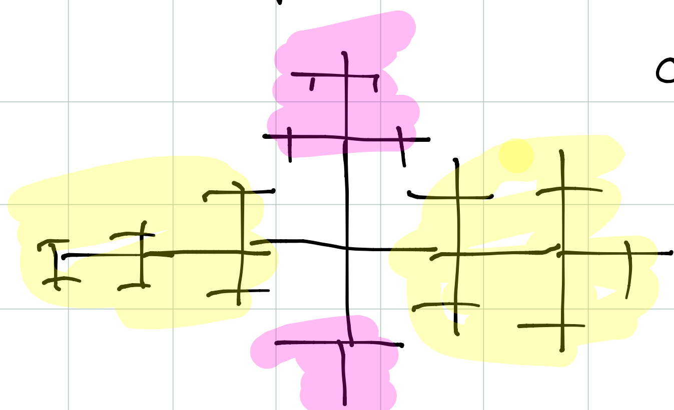
GGT – Lecture 3

Last time: constructed a metric space "∞ telephone pole", showed $F\langle a,b\rangle$ acts freely on it by isometries



all edges have the same length

Note action preserves horizontal and vertical edges – so could make them different lengths, thus <u>deforming</u> the action. Later will talk about more ways to deform the action – make an entire <u>space</u> of actions...

__Shmeel__: Action of F is __free__: point stabilizers are trivial.

This "picture" of F is the key to proving that a given group is free...

<u>Def</u> G <u>generated</u> by $S \subset G$ if $\exists$ homomorphism $F(S) \twoheadrightarrow G$

(ie all elts of G are products of elts of S and S^{-1})

<u>Ping-Pong Lemma</u> (for $|S| = 2$)

Suppose G is generated by $S = \{a, b\}$ and acts on X

If $X = A \amalg B \amalg C$ s.t. $\forall\, k \in \mathbb{Z}$, $k \neq 0$

(1) $a^k(B) \subseteq A$

$b^k(A) \subseteq B$

and (2) $\exists\, x \in C$, $a^k x, a^{-k} x \in A$ and

$b^k x, b^{-k} x \in B$

Then $G \cong F\langle a, b\rangle$.

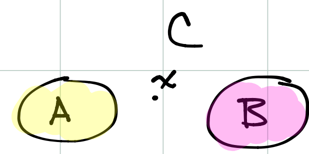
<u>Pf:</u> w a non-$\emptyset$ reduced word in $F\langle a, b\rangle$

$\Rightarrow w = a^{k_1} b^{k_2} a^{k_3} \cdots$

OR $= b^{k_1} a^{k_2} b^{k_1} \cdots$

Ends in a^k, b^k, a^{-k} or b^{-k}, so w plays ping-pong with x

eg $w = a^2 b^{-1} a^{-1}$



$$x \xrightarrow{a^{-1}} \underset{\substack{\cap \\ A}}{a^{-1}x} \xrightarrow{b^{-1}} \underset{\substack{\cap \\ B}}{b^{-1}a^{-1}x} \xrightarrow{a^{2}} \underset{\substack{\cap \\ A}}{a^{2}b^{-1}a^{-1}x}$$

So $wx \in A$ if $w = a^{k_1} \cdots$

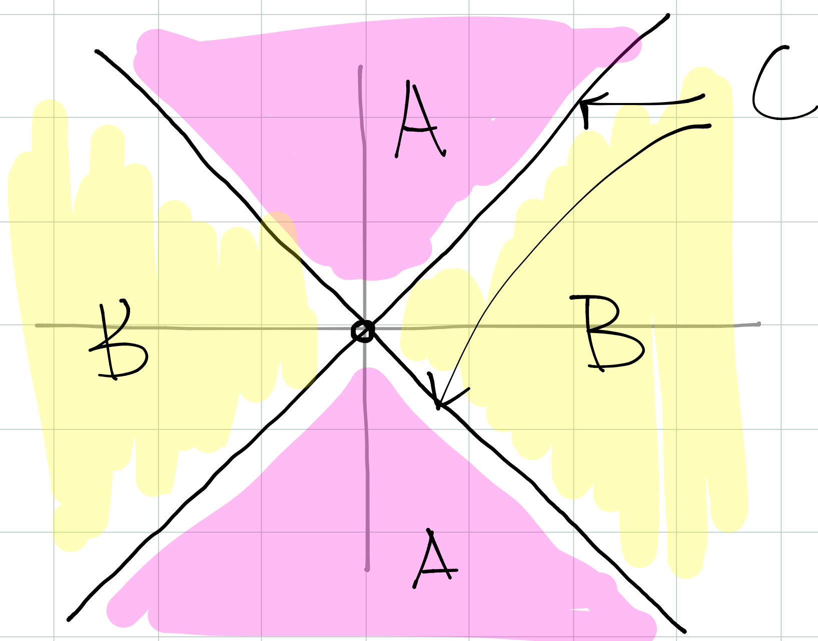
$\in B$ if $w = b^{k_1} \cdots$

in either case, $wx \neq x$, so $w \neq 1$, ("no relations")

This says $F\langle a,b\rangle \longrightarrow G$ is injective.

as well as surjective, ie $F\langle a,b\rangle \cong G$.

eg $\left\langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\rangle \leq SL(2,\mathbb{Z})$



$X = \mathbb{R}^2 - \{0\}$

Symm = Homeo(X)

(exercise: modify arg to show $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ is free)

From here - where to go?

Want to show a subgp of a free group is free, so can say what a presentation is ($\ker F(S) \to G$ is a free gp, normally generated by R.)

So need to review π_1, deck transfs, covering spaces.

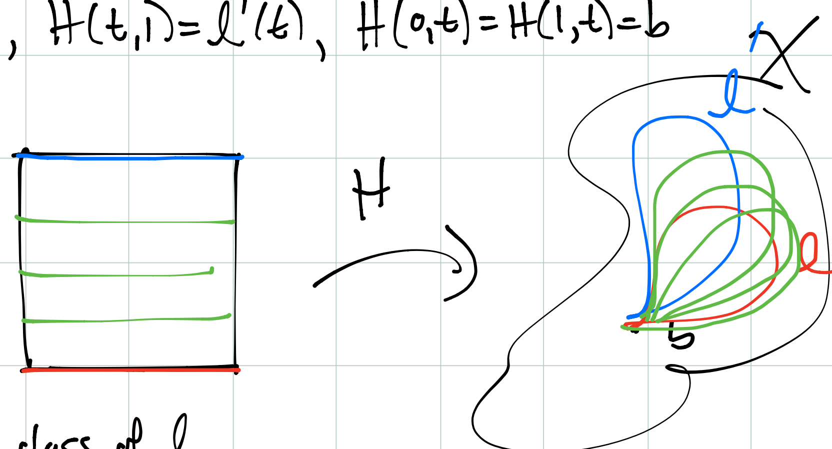
X = top space, $b \in X$ a basept.

A *loop* at b is a (continuous) map $\ell : [0,1] \to X$ w/ $\ell(0) = \ell(1) = b$

Two loops ℓ, ℓ' are *homotopic* if you can *deform* ℓ to ℓ' continuously, ie if $\exists\, H : [0,1] \times [0,1] \to X$

$H(t,0) = \ell(0)$, $H(t,1) = \ell'(t)$, $H(0,t) = H(1,t) = b$

Picture



$[\ell]$ = homotopy class of ℓ

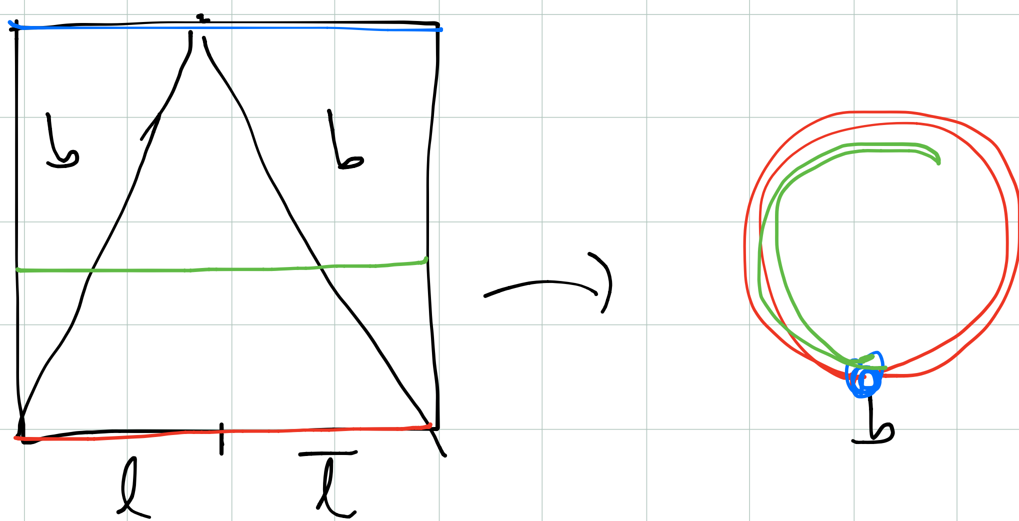
Set of homotopy classes forms a group $\pi_1(X,b)$ called the *fundamental group* of X

$$[\ell][\ell'] = [\ell\ell'], \text{ where } \ell\ell'(t) = \begin{cases} \ell(2t) & 0 \le t \le \frac{1}{2} \\ \ell'(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

id = [constant loop]

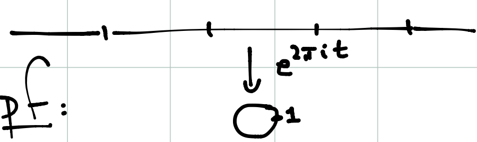
$[\ell]^{-1} = [\bar{\ell}]$, where $\bar{\ell}(t) = \ell(1-t)$

check $l\bar{l} \simeq \text{const}$



First example: $\pi_1 S^1 \cong \mathbb{Z}$

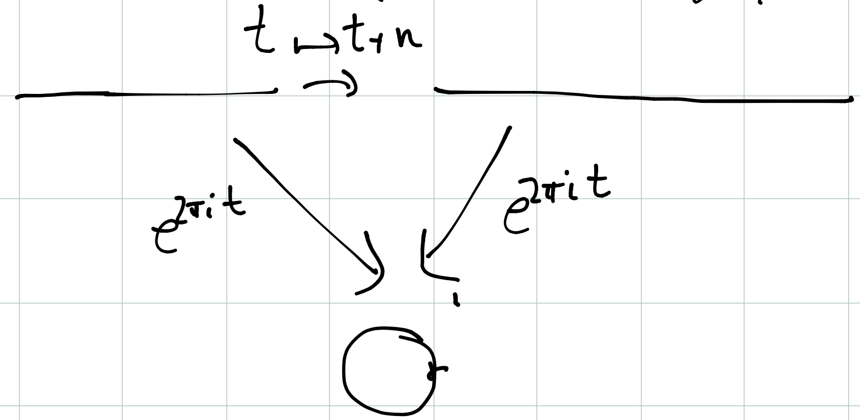
pf:



loops based at 1 lift (locally) to paths in $\mathbb{R}$, ending at integers

Loops are homotopic iff they end at the same integer, so $\pi_1(S^1, 1) \cong \mathbb{Z}$.

$\mathbb{Z} \cong$ translations of $\mathbb{R}$ commuting w/ $e^{2\pi i t}$



$$e^{2\pi i(t+n)} = e^{2\pi i t}\, e^{2\pi i n} = e^{2\pi i t}$$

General phenomenon: $\exists$ covering $\tilde{X} \longrightarrow X$

$\pi_1(\tilde{X}, \tilde{b}) = 1$, $\pi_1(X, b)$ acts freely on $\tilde{X}$

$\tilde{X} \to X$ is the quotient map $\tilde{X} \to \tilde{X}/\pi_1(X, b)$

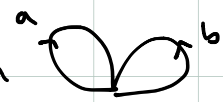
subgroups $G < \pi_1(X, b) \leftrightarrow$ covers $(X_G \longrightarrow X)$

"Galois correspondence"

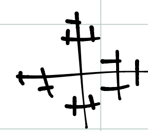
$$\left[\begin{array}{l} X_G = \tilde{X}/G, \quad \pi_1(X_G) \cong G \\ \tilde{X} \longrightarrow \tilde{X}/G \longrightarrow \tilde{X}/\pi_1(X, b) \end{array}\right]$$

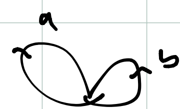
covering maps.

2nd example: $\pi_1(S^1 \vee S^1) \cong F_2$ $\quad (\pi_1 \bigvee_n S^1 \cong F_n)$

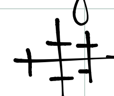
loop in 

$\leftrightarrow$ reduced word in a, b, a^{-1}, b^{-1}

In this case $\tilde{X}$ = tree  w/ $F\langle a,b\rangle$-action

$\tilde{X} / F\langle a,b\rangle$ = 

$G < F\langle a,b\rangle$ $\leftrightarrow$ covering space of 

$\leftrightarrow$ quotient of 

= graph w/ $\pi_1 \cong G$

3rd example: $\pi_1(\text{Graph})$ is free

pf: 1. Every graph X has a maximal tree T

2. $\pi_1 X \cong \pi_1(X/T) \cong \pi_1(\vee S^1)$

So we've sketched a proof of:

<u>Thm</u> Every subgroup of a free group is free

[<u>pf</u> Take a graph X with $\pi_1 X = F$.

$G < F$ corresponds to a <u>cover</u> $X_G \longrightarrow X$.

Covering space of a graph is a graph

$\pi_1(X_G) = G \Rightarrow G$ is free ✓]

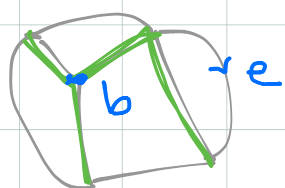
(let's assume F is fin. gen)

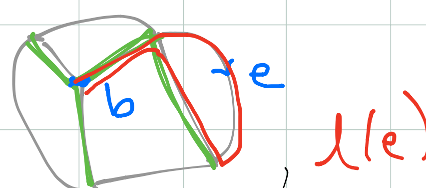
Pf: $T \subset X$ a maximal tree, $b \in T$, S = edges of $X \setminus T$

<u>Claim</u> $\pi_1(X,b) \cong F(S)$

<u>pf</u> Map $F(S) \longrightarrow \pi_1(X,b)$: $e \in S$

$e \longmapsto$ loop $b - i(e) - e - t(e) - b$.





extends to a homomorphism $F(S) \longrightarrow \pi_1(X,b)$

<u>Claim.</u> injective, surjective