Thurs, Feb 13

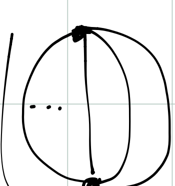
Last time: stated Švarc-Milnor lemma, got half through proof. Entire proof is in Feb 11 notes.

Consequences:

1. S = closed orientable surface. Then $\pi_1(S)$ acts properly and cocompactly on $\mathbb{H}^2$, which is $\frac{\log(3)}{2}$-hyperbolic. So surface groups are hyperbolic.

(Exercise: what if S is non-orientable?)

2. F_n = free group of rank $n \geq 2$. Take any trivalent graph Γ with $\pi_1(\Gamma) \cong F_n$. Then $\tilde{\Gamma}$ is the infinite trivalent tree T_3. Since F_n acts freely and cocompactly on T_3, F_n is quasi-isometric to T_3; in particular, all $F_n \sim F_m$ if $n, m \geq 2$. Also $F_n \sim T_{n+1} \Rightarrow T_n \sim T_3$ for all $n \geq 4$.

$$F_n = \pi_1\Bigg($$  $$\Bigg)$$

3 An n-dimensional *hyperbolic manifold* is a complete Riemannian mfld of constant sectional curvature -1. The only such manifold which is simply-connected is $\mathbb{H}^n$, so $\pi_1 M$ acts (freely) on $\tilde{M} = \mathbb{H}^n$ by isometries. Since any geodesic triangle in $\mathbb{H}^n$ is contained in a geodesic subspace isometric to $\mathbb{H}^2$, $\mathbb{H}^n$ is hyperbolic (whew!). If M is compact, then by Švarc-Milnor $\pi_1 M$ is hyperbolic.

($\pi_1 M$ is a discrete torsion-free subgroup of $SO^+(n,1) = \mathrm{Isom}(\mathbb{H}^n)$ – this shows any such subgroup is hyperbolic.

4. A finite-index subgroup of a hyperbolic group is hyperbolic. More generally, a finite-index subgroup H of a f.g. group G is quasi-isometric to G, since H also acts cocompactly on $\mathcal{C}(G,S)$

<u>Def</u>: G, G' are <u>commensurable</u> if they contain isomorphic finite-index subgroups.

5. Let $K < G$ be a finite normal subgroup. Then $G \sim G/K$: If $S = \{s_1, \dots s_n\}$ generates G, then $SK = \{s_1K, \dots, s_nK\}$ generates G/K, and G acts on $\mathcal{C}(G/K, SK)$ cocompactly with finite stabilizers $(\cong K)$, so $G \sim \mathcal{C}(G/K, SK) \sim G/K$.

<u>Def</u>: G, G' are <u>virtually isomorphic</u> (VI) if they are commensurable or have isomorphic quotients by finite normal subgroups.

In general quasi-isometric groups need not be VI. If quasi-isometry $\Rightarrow$ VI for G, G is said to be <u>rigid</u>.

A <u>class</u> of groups is said to be rigid if any group quasi-isometric to a group in the class is virtually isomorphic to a (possibly ifferent) group in the class.

<u>Examples of rigidity</u>

1. $\mathbb{Z}^n$

2. F_n $(n \geq 2)$

4. The class of hyperbolic groups

3. The class of finitely presented groups

$\vdots$

n. The class of lattices in simple Lie groups (like $SL_n\mathbb{Z}$)

Recall from exercises that F_n is <u>not</u> quasi-isometric to $\mathbb{Z}$ (="F_1"), because $\mathbb{R} \not\sim T_3$ (exercises)

(idea: Suppose $f: \mathbb{R} \to T_3$ was a quasi-isometry

Since $d(f(n), f(n+1))$ is bounded, and $d(f(0), f(n)) \to \infty$ as $n \to \infty$, the points $f(n)$ must be marching towards ∞ near some ray. Similarly, $f(-n) \to \infty$ along some ray

Thus f is not quasi-surjective)