Thurs, Feb 6

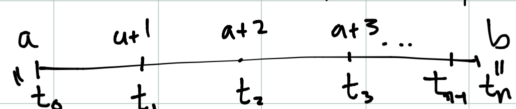
Cleaning up (Tues. we used statement 3 below, will use 1 today)

Lemma: Let X be a geodesic metric space, and $\alpha : [a,b] \longrightarrow X$ a (λ, C)-quasigeodesic. Then there is a continuous $(\lambda, 2(\lambda+C))$-quasigeodesic $\beta : [a,b] \longrightarrow X$ st.

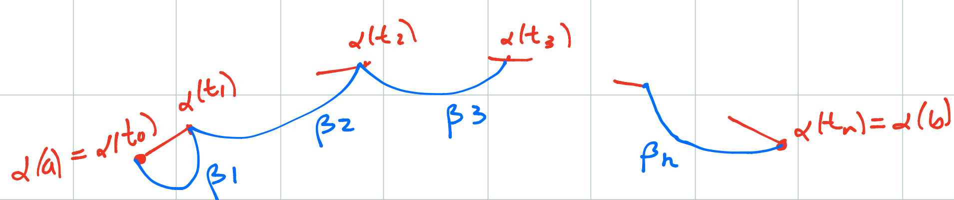
1. $\alpha \subseteq N_{\lambda+C}(\beta)$ and $\beta \subseteq N_{\lambda+C}(\alpha)$
2. $d(\alpha(t), \beta(t)) \leq \lambda + C$
3. $l_\beta(\beta(s), \beta(t)) \leq \lambda' d(\beta(s), \beta(t)) + C'$

where λ' and C' depend only on λ and C

Proof: We divided $[a,b]$ into n pieces of size 1: (except at end)



and formed β from reparamatrized geodesic segments:

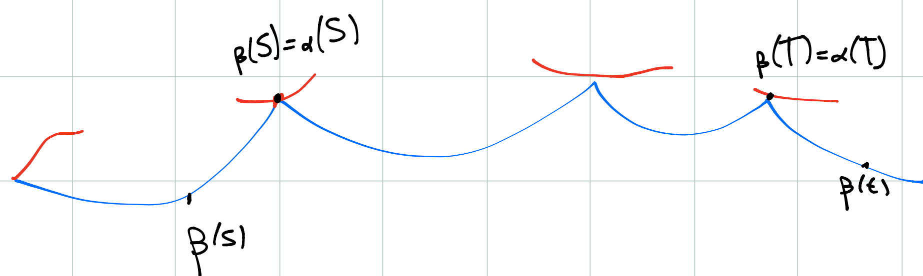


so $\alpha(t_i) = \beta(t_i)$ for all t_i.

Since each segment β_i has length $\leq \lambda + C$,
we get $\beta \subseteq N_{\lambda+C}(\alpha)$ and since α is a (λ, C)-quasigeodesic
we get $\alpha \subseteq N_{\lambda+C}(\beta)$.

<u>Claim</u> β is a $(\lambda, 2(\lambda+C))$-quasigeodesic

Pf: Let $s,t \in [a,b]$ and S,T the closest t_i to s and t:



Then
$$\begin{aligned} d(\beta(s),\beta(t)) &\leq d(\beta(S),\beta(T)) + 2\cdot\tfrac{1}{2}(\lambda+C) \\ &= d(\alpha(S),\alpha(T)) + \lambda + C \\ &\leq \lambda|S-T| + C + (\lambda+C) \\ &\leq \lambda(|s-t|+1) + C + (\lambda+C) \\ &\leq \lambda|s-t| + 2(\lambda+C) \checkmark \end{aligned}$$

The other direction is similar (do it as an exercise).

Since the length of each β_i is $< \lambda + C$, Statement 2 also follows.

Statement 3: Since $l_\beta(\beta(t_i), \beta(t_{i+1})) = d(\alpha(t_i), \alpha(t_{i+1}))$

$= d(\alpha(t_i), \alpha(t_i + 1)) \leq \lambda \cdot 1 + C$,

we get $l_\beta(\beta(t_i), \beta(t_j)) \leq (\lambda + C)|j - i|$

so $l_\beta(\beta(s), \beta(t)) \leq (\lambda + C)(|s-t| + 2) = (\lambda + c)|s-t| + 2(\lambda + c)$

Also $\frac{1}{\lambda}|s-t| - 2(\lambda + C) \leq d(\beta(s), \beta(t))$

(since β is a $(\lambda, 2(\lambda + C))$-quasi-geodesic)

So $(\lambda + c)|s-t| - \lambda 2(\lambda + c)^2 \leq \lambda(\lambda + c) d(\beta(s), \beta(t))$

$(\lambda + c)|s-t| \leq \lambda(\lambda + c) d(\beta(s), \beta(t)) + \lambda 2(\lambda + c)^2$

$(\lambda + c)|s-t| + 2(\lambda + c) \leq \lambda(\lambda + c) d(\beta(s), \beta(t)) + \lambda 2(\lambda + c)^2 + 2(\lambda + c)$

The two green lines give

$$l_\beta(\beta(s), \beta(t)) \leq \lambda' d(\beta(s), \beta(t)) + C'$$

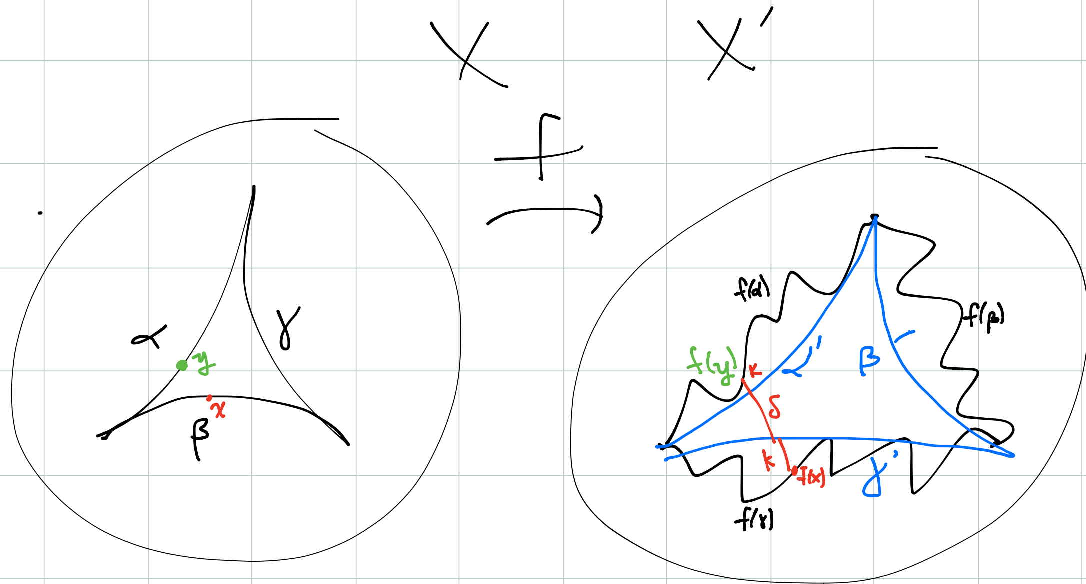
with $\lambda' = \lambda(\lambda + c)$ and $C' = 2(\lambda + c)(\lambda(\lambda + c) + 1)$ ✓

Now let's remember what we wanted this for:

Thm: X, X' quasi-isometric, X' δ-hyperbolic

Then X is hyperbolic.

Pf: Let f be a (λ, c)-quasi-isometry, Δ a geodesic triangle in X and x a point of Δ



Let α, β, γ be the sides of Δ, and let $x \in \beta$

Let α' be a geodesic in X' joining the endpoints of $f(\alpha)$

Similarly let β', γ' be geodesics corresponding to β, γ

$f(\alpha)$, $f(\beta)$ and $f(\gamma)$ are quasi-geodesics, so are within Hausdorff distance $K = K(\lambda, c)$ of α', β' and γ' respectively.

[$f(\alpha)$ is within Hausdorff distance K_1 of a <u>continuous</u> quasi-geodesic by Statement 1 of the Lemma, which is within Hausdorff distance K_2 of α'. So take $K = K_1 + K_2$]

So let $b \in \beta'$ be a point within K of $f(x)$. There is a point on either α' or γ' within δ of b, say $a \in \alpha'$

Then there is $f(y) \in f(\alpha)$ within K of a.

Now $d(f(x), f(y)) \leq 2K + \delta$

But f is a (λ, C)-quasi-isometry, so

$$2K + \delta \geq d(f(x), f(y)) \geq \frac{1}{\lambda} d(x, y) - c$$

$$\Rightarrow \quad d(x,y) \leq (2K + \delta + c)\lambda$$

so X is $\lambda(2K + \delta + C)$-hyperbolic.

Q: Suppose I can show a group is hyperbolic. What does that tell me about the group?

A: G is finitely presented, centralizers of elements are (virtually) cyclic, G has solvable word, conjugacy and isomorphism problems, G is automatic: there are normal forms for elements which make computer computations feasible...

Next goal:

Thm If G is hyperbolic then G is finitely presented.

(note we are assuming here G is fin. gen, need to show it's determined by a finite no of relations.)

Idea of proof: S = (finite) generating set for G,

Notation: $F(S) \twoheadrightarrow G$.
$$w \longmapsto \overline{w}$$

$$\mathcal{C} = \mathcal{C}(G,S) = \text{Cayley graph}$$

Recall a word $w \in F(S)$ gives a <u>loop</u> in $\mathcal{G}$ if and only if $\overline{w} = 1$ in G.

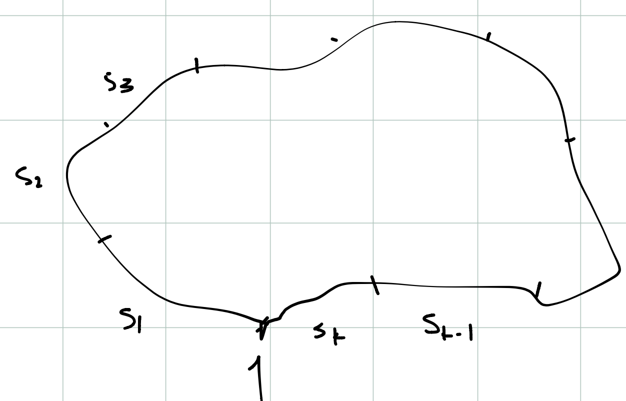
We are assuming $\mathcal{G}$ is δ-hyperbolic

Let $R = \{ w \in F(S) \mid \overline{w} = 1 \text{ and } \text{length}(w) \leq 8\delta \}$

<u>Claim</u> $\langle S \mid R \rangle$ is a presentation of G, ie

R normally generates $\ker(F(S) \longrightarrow G)$, ie

every w with $\overline{w} = 1$ is a product of conjugates of elements of length $\leq 8\delta$. //



$w = s_1 \cdots s_k$

Induct on $\text{length}(w) = |w|$. $|w| \leq 8\delta \Rightarrow$ true!