Thurs., March 6

We're proving: g of infinite order in a hyperbolic group $G \Rightarrow \{g^i\}_{i\geq 0}$ is a quasi-geodesic

Still need λ, C with $d(1, g^s) \geq \frac{1}{\lambda}s - C$.

We've got: For any $R>0$, $\exists$ $e(R) \leq 2KR+K$

s.t. $d(1, g^{e(R)}) > R$ (here $K = \#$ vertices in $B_{12\delta}(1)$)

Note: $2KR+K \leq 2KR + RK = 3KR$.

Simplifies things to use $e(R) \leq 3KR$

Also note $R \leq d(1, g^{e(R)}) \leq e(R) \cdot d(1,g)$, so

$$R/d(1,g) \leq e(R) \leq 3KR$$

Claim $d(1, g^{3KR}) \geq R$ for all R.

Pf: Suppose $d(1, g^{3KR_0}) < R_0$ for some R_0

say $d(1, g^{3KR_0}) = R_0 - \varepsilon$

For $s > 3KR_0$, write $s = n(3KR_0) + R_1$ with $R_1 < 3KR_0$

$$\left(\text{so } \frac{s}{3K} = nR_0 + \frac{R_1}{3K} \geq nR_0\right)$$

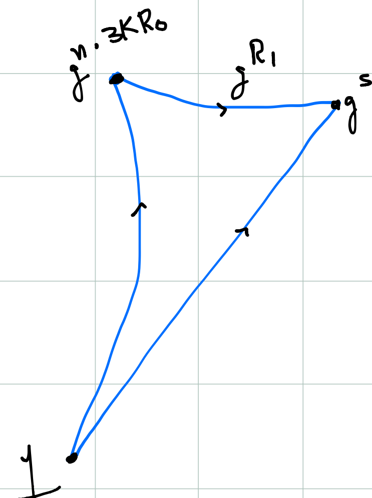
Then $d(1, g^s) \leq d(1, g^{n \cdot 3KR_0}) + d(g^{3KR_0}, g^{3KR_0 + R_1})$

$$\leq n \cdot d(1, g^{3KR_0}) + d(1, g^{R_1})$$

$$= n(R_0 - \varepsilon) + \text{Const}$$

$$< nR_0 \text{ for } s \text{ sufficiently large.}$$

$$\leq \frac{s}{3K}$$



Now take $s = e(R)$ with $e(R)$ sufficiently large.

(recall $R/d(1,g) \leq e(R) \leq 3KR$, so we can make $e(R)$ large)

Then the above says $d(1, g^{e(R)}) < \frac{e(R)}{3K} \leq R$,

contradicting $d(1, g^{e(R)}) \geq R$.

Now we can finish the proof that $\{g^i\}_{i=0}^{\infty}$ is a quasi-geodesic:

Write $s = 3Ki + j$, $0 \le j < 3K$

Then $d(1, g^s) \geq d(1, g^{3Ki}) - d(1, g^j)$

$$\geq i - \max_{j<3K} d(1, g^j)$$

$$= \left[\frac{s}{3}\right] - \max$$

$$\geq \underset{\substack{\uparrow \\ \lambda}}{\frac{1}{3}} s - \underset{\substack{\uparrow \\ c}}{(\max + 1)} \quad \checkmark$$

So $1, g, g^2, g^3 \cdots\cdots \longrightarrow g_\infty \in \partial X$

$\{g^i\}$ a quasi-geodesic ray stays a bounded distance from a geodesic ray $\gamma_+ \longrightarrow g_\infty$.

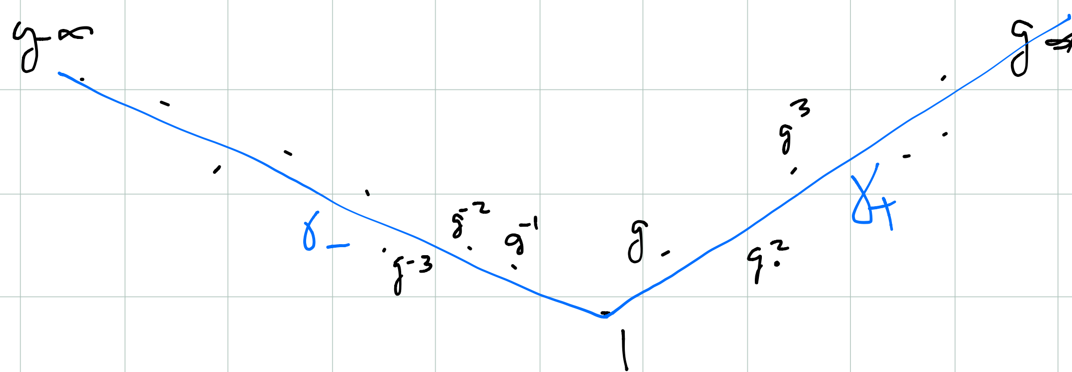
Similarly $1, g^{-1}, g^{-2}, \ldots \longrightarrow g_{-\infty}$, close to a geodesic ray $\gamma_- \longrightarrow g_{-\infty}$

Claim There is a bi-infinite geodesic γ

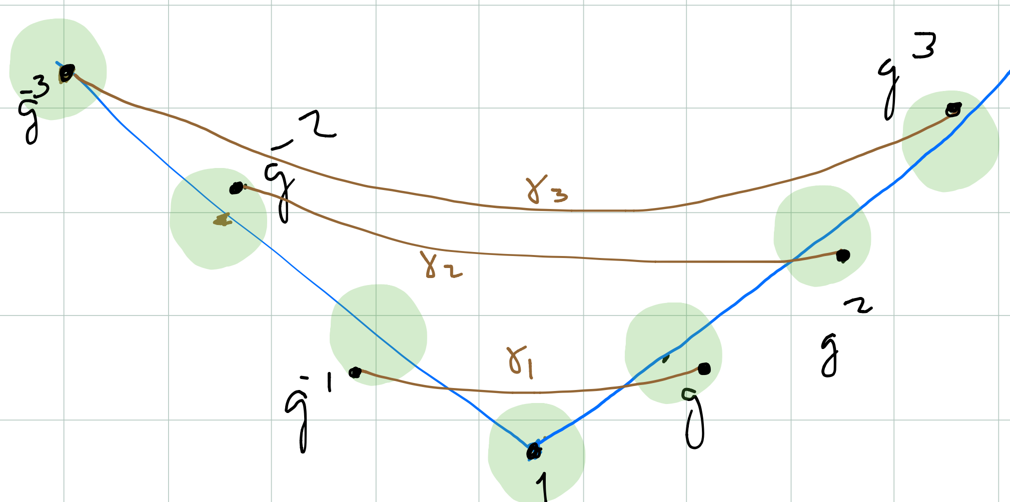
one direction $\longrightarrow g_\infty$, other $\longrightarrow g_{-\infty}$, and

$\{g^i\}$ stays within bounded distance K of γ

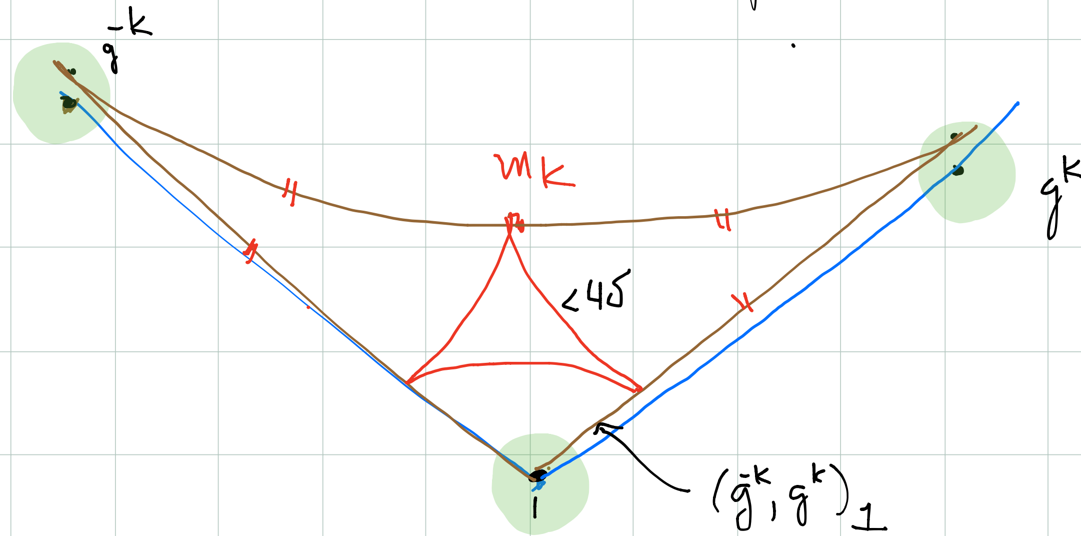
for all $i \in \mathbb{Z}$.



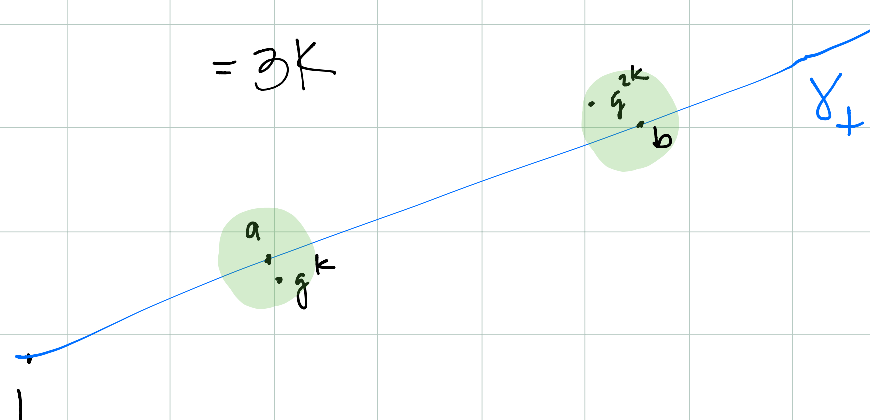
Let γ_k = geodesic from g^{-k} to g^k



m_k = midpoint of γ_k



$$
\begin{aligned}
(g^{-k}, g^{k})_1 &= d(g^k, 1) + d(g^{-k}, 1) - d(g^k, g^{-k}) \\
&= d(1, g^k) + d(g^k, g^{2k}) - d(1, g^{2k}) \\
&\le d(1,a) + K + d(a,b) + K - d(1,b) + K \\
&= 3K
\end{aligned}
$$



So $d(1, m_k) < 3K + 4\delta$ for all k

Thus for ∞ many k, γ_k passes through the same m.

Use these to construct γ as we always do, inductively on larger and larger balls around m, by passing to subsequences. We get a geodesic which is geodesic in <u>both</u> directions and connects $g_{-\infty}$ to g_∞. We call it an <u>axis</u> for g

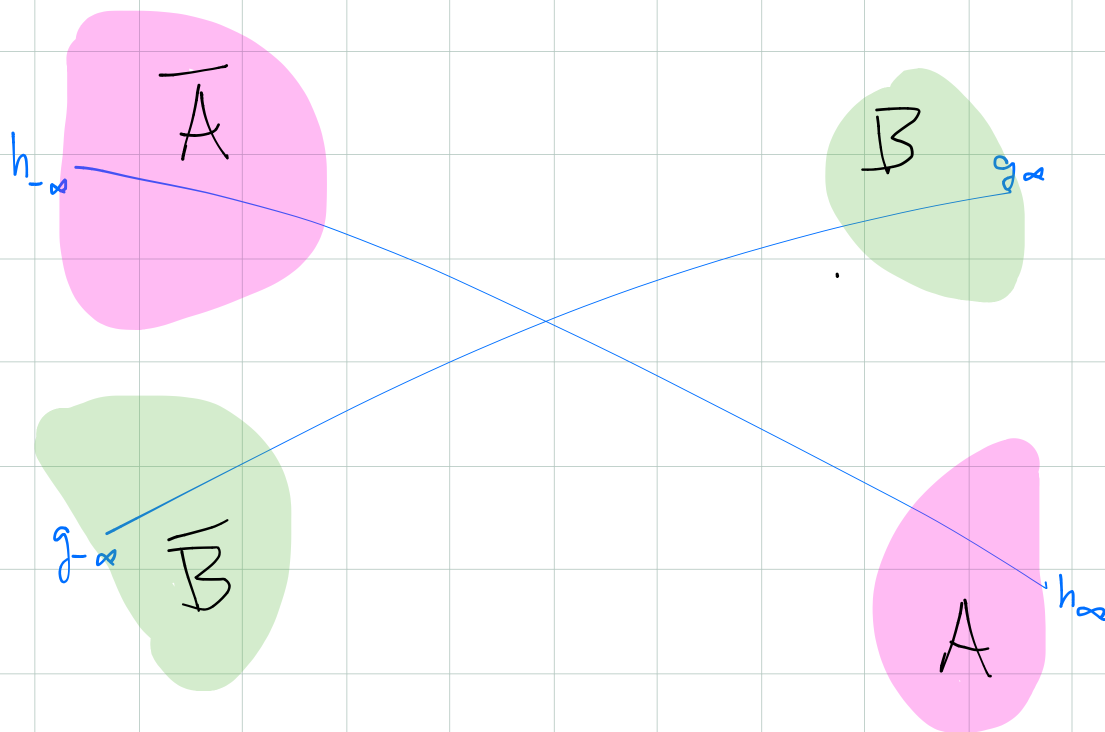
Using this ray to compute gives $(g_{-\infty}, g_\infty)_m = 0$; in particular $g_\infty \neq g_{-\infty}$.

Now take two infinite order elements g and h with $h_\infty \neq g_\infty, g_{-\infty}$

Choose R w/ $N_R(h_\infty) \cap N_R(g_\infty) = \emptyset$,

ie for i,j sufficiently large, $(g^i, h^j)_1 < R$

Then $(g^{-i}, h^{-j})_1 < R$ too, so $g_{-\infty}$ and $h_{-\infty}$ are also distinct



Claim For k sufficiently large,

$$g^k(A \cup \bar{A} \cup B) \subset B$$
$$g^{-k}(A \cup \bar{A} \cup \bar{B}) \subseteq \bar{B}$$
$$h^k(A \cup B \cup \bar{B}) \subset A$$
$$h^{-k}(\bar{A} \cup B \cup \bar{B}) \subset \bar{A}$$

So by a ping-pong argument, no word in $a = g^k$, $b = h^k$ is the identity, ie $\langle g^k, h^k \rangle$ is a free subgroup