Outer space and Automorphisms of free groups

LECTURE 1

The study of free groups and their automorphisms goes back more than a century, to the 1880's when the concept of "free group" was introduced by Dyck.

Fundamental results were established by J. Nielsen, JHC Whitehead and W. Magnus in the 1920's and 1930's

From the beginning connections with topology and geometry were important, though much of the work was purely algebraic and combinatorial

More recent work on automorphism groups of free groups starts with Stallings, who introduced new topological techniques in the 70's based on fact π_1(graph) is free.
("The topology of finite graphs" – Inventiones 1983).

The more recent work is also heavily influenced by ideas of Thurston and Gromov who used methods from geometry and dynamics to study groups.

Much of this recent work focuses on analogies between $Out(F_n)$ and

mapping class groups of surfaces

and

lattices in semisimple Lie groups, especially $SL_n\mathbb{Z}$

One reason to suspect such analogies is that there are natural maps

$$① \quad Out(F_n) \twoheadrightarrow GL(n,\mathbb{Z})$$
$$② \quad Mod(S_{g,s}) \hookrightarrow Out(F_n)$$

① Abelianization $F_n \to \mathbb{Z}^n$ induces
$$Aut(F_n) \to Aut(\mathbb{Z}^n) = GL(n,\mathbb{Z})$$
since the commutator subgroup $[F_n, F_n]$ is characteristic

Inner automorphisms map to I, so the map factors through $\rightsquigarrow$ $Out(F_n)$

<u>Nielsen</u> showed $Aut\,F_n \to GL_n\mathbb{Z}$ is surjective. (1924). (We will see a modern proof)

The kernel of the map $\mathrm{Out}(F_n) \longrightarrow GL(n,\mathbb{Z})$ is called IA_n "Identity on the Abelianization"

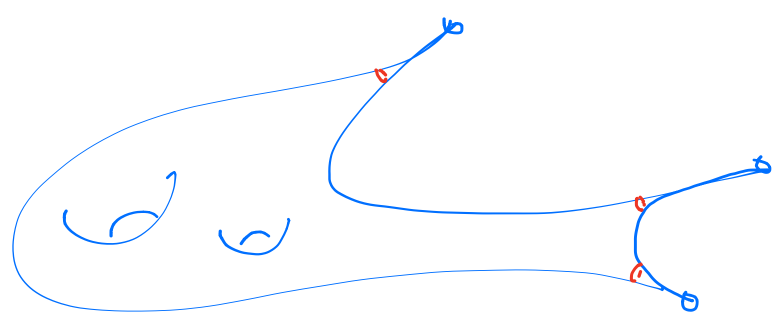
It measures the difference between $\mathrm{Out}(F_n)$ and $GL(n,\mathbb{Z})$ and is still very poorly understood in general.

Nielsen showed $IA_2 = \langle e \rangle$ in 1924

Magnus showed IA_n is finitely generated and asked whether it is finitely presented in 1934.

In 1997 Krstić & McCool showed IA_3 is <u>not</u> finitely presented, but the question remains open for $n>3$.

② $S_{g,s}$ = surface with genus g, s punctures



$$\pi_1(S_{g,s}) \cong F_n, \quad n = 2g+s-1.$$

Any homeomorphism of $S_{g,s}$ induces an automorphism of F_n.

The homeomorphism permutes the punctures, and the (homotopy classes of) simple loops around the punctures, which are cyclic words $u_i \in F_n$.

<u>Zieschang</u> : $Mod(S_{g,s}) \hookrightarrow Out(\pi_1 S_{g,s})$.

The image is $Stab(u_1, \dots, u_s)$, ie outer automorphisms which permute the cyclic words u_i.

<u>Remark</u>

There are often many different punctured surfaces w/ $\pi_1 \cong F_n$, many $\cong$'s.

<u>Stallings</u>: $\exists\, \varphi \in Out\, F_n$ not in the image of <u>any</u> $Mod(S_{g,s})$

<u>eg</u> $\quad \varphi : \begin{cases} x \mapsto y \\ y \mapsto z \\ z \mapsto xy \end{cases}$

Suppose φ <u>is</u> in the image of some $Mod(S_{g,s})$

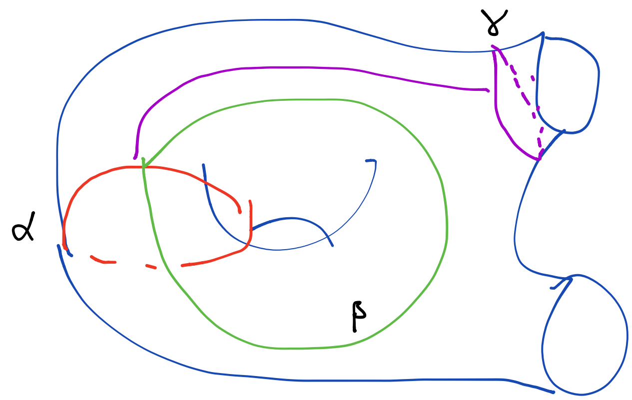
Rank $= 3 \Rightarrow \psi$ could come from a homeomorphism of a 4-punctured sphere or 2-punctured torus.

I'll do $S = S_{1,2}$ = punctured torus:

If $h : S \to S$ is a homeomorphism then some power h^k preserves orientation and fixes ∂S.

<u>Claim</u> $(h^k)_* : H_1 S \to H_1 S$ has eigenvalues $1, \lambda, 1/\lambda$.

<u>Pf</u>:



$\pi_1 S$ generated by α, β, γ, so

$H_1 S$ generated by $[\alpha], [\beta], [\gamma]$

$h^k(\gamma) = \gamma$ so $[\gamma]$ is an evector for h^k_* with eigenvalue 1

So matrix is $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in SL(3,\mathbb{Z})$

$\Rightarrow$ eigenvalues $= \{1, \lambda, 1/\lambda\}$

Matrix in blue corner is map on T^2 given by filling in holes and extending h.

Exercise: What are the possible eigenvalues if $S = S_{0,4}$?

Exercise For any $h: S_{g,s} \longrightarrow S_{g,s}$ preserving orientation and fixing ∂, eigenvalues of $h_*: H_1 S_{g,s} \longrightarrow H_1 S_{g,s}$ include $(s-1)$ 1's, and the rest come in pairs $\{\lambda, 1/\lambda\}$

Now back to our automorphism $\varphi: x \to y \to z \to xy$

The induced map on $H_1 S \cong \mathbb{Z}^3$ is $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

which has eigenvalues: $\lambda_1 > 1$, $\lambda_2, \lambda_3 < 1$.

note $\lambda_1^{-1} \neq \lambda_2, \lambda_3$ since $\lambda_1 \lambda_2 \lambda_3 = \pm 1$

(Powers of φ have these properties, too)

Lattices are often studied via their action on the symmetric space $K\backslash G$

Mapping class groups act on the Teichmüller space $\mathcal{T}_{g,s}$

Outer space was introduced in the mid-80's as an analog for $\mathrm{Out}(F_n)$ of symmetric space or Teichmüller space

Outer space can be thought of as a space of finite graphs.
Many structures in mathematics are paramaterized by finite graphs, and so Outer space has found connections, eg, with

* Moduli spaces of punctured surfaces
* Systems of spheres in a doubled handlebody
* tropical curves
* Feynman diagrams
* Invariants in symplectic modules
* Classical modular forms

and even

* phylogenetic trees

Shortly after Outer space appeared,
Bestvina and Handel introduced traintracks for free group automorphisms, as an analog of Thurston's train tracks for surface automorphisms.

The literature has since exploded, and I can't hope to cover all of it in this course. I hope to produce a guide to the literature by the end of the course, for those who want to learn more.

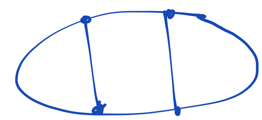
This course will begin at the beginning, and we'll see how far we get.

Start pre-Outer space. Want to study autos of free groups using topology. Need a topological model for a free group; then self-maps of this model can represent automorphisms.

Need X = topological space w/ $\pi_1 X \cong F_n$

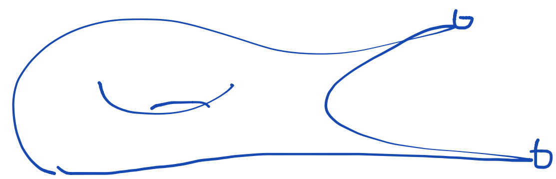
Candidates:

1. Finite connected graph G $(v-e=1-n)$

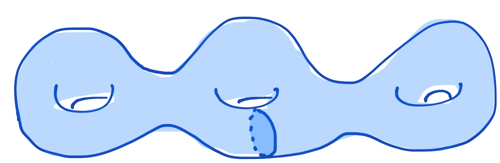


$v-e=4-6=-2$

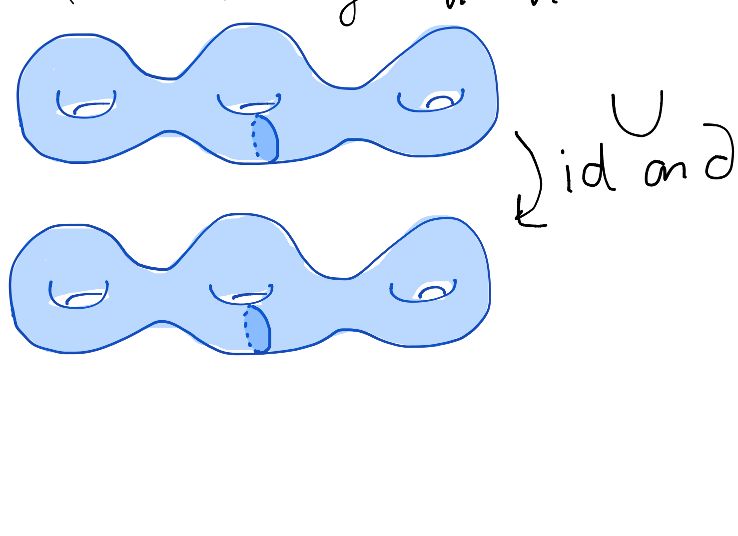
2. Punctured surface $S_{g,s}$ $(2g+s-1=n)$



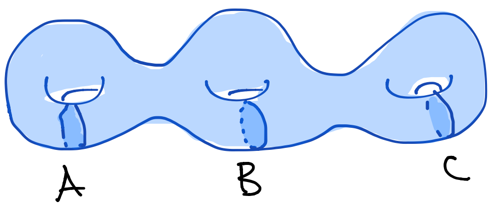
3. Handlebody $H_n = \#_n S^1 \times D^2$



4. Doubled handlebody $M_n = \#_n S^1 \times S^2$



Another way to visualize the doubled handlebody.



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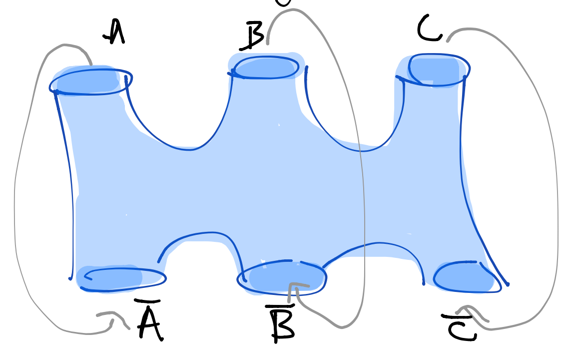
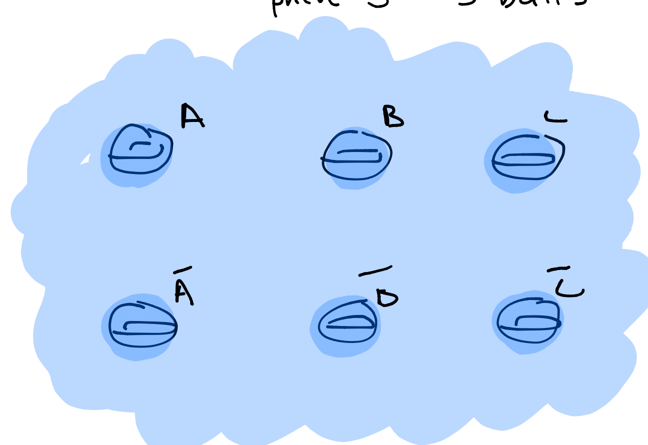


After doubling, this becomes

M_n with 3 spheres A, B, C

Sphere S^3 − 3 balls



Any homotopy equivalence $X \to X$ induces an outer automorphism of $\pi_1 X$
(If you want an automorphism, need to fix a basepoint $b \in X$ and only consider maps $(X,b) \to (X,b)$.)

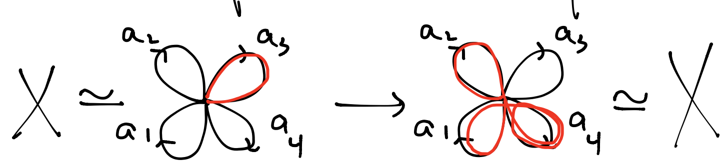
So if you fix an identification $F_n \cong \pi_1 X$ you get an elt of $\mathrm{Out}(F_n)$

Some models are better than others for studying $\mathrm{Out}(F_n)$

Homotopic maps give the same map on $\pi_1 X$ so get a map $\pi_0 HE(X) \longrightarrow \mathrm{Out}(F_n)$ for any X.

① $\underline{X \text{ a graph}}$: Then this map is an isomorphism

Surjective:

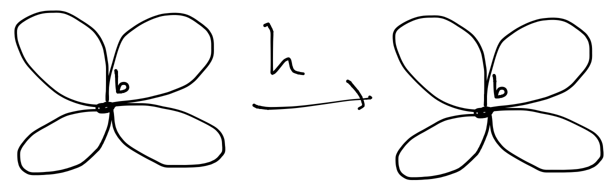


Identify $\pi_1(X) \cong F\langle a_1, \dots, a_n\rangle = F_n$

Given $\varphi \in \mathrm{Aut}(F_n)$, construct a h.equiv.
say $\varphi(a_i) = w_i$ $\qquad h: a_i \mapsto w_i$.

Injective: If $h: X \to X$ induces $id: \pi_1 X \to \pi_1 X$, then h is homotopic to id.

pf for X a rose:



Composing w/ a homotopy, if necessary, we may assume $h(b) = b$

Each loop a on the left goes to a loop homotopic to a. These homotopies can be performed independently on each petal, giving a homotopy of h to id.

② X a punctured surface $S_{g,s}$

Homotopic homeomorphisms of X are isotopic, so

$$\begin{array}{c} \pi_0(\mathrm{Homeo}(X)) = \mathrm{Mod}(X) \hookrightarrow \pi_0(HE(X)) \\ \downarrow \cong \text{ Zieschang} \\ \mathrm{Stab}(u_1, \ldots u_s) \lneq \mathrm{Out}(F_n) \end{array}$$

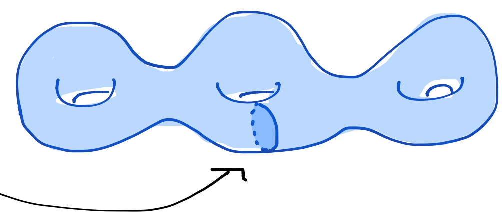
(We saw that not every φ can be realized on a surface (unless $g=s=1$)

③ X = handlebody H_n

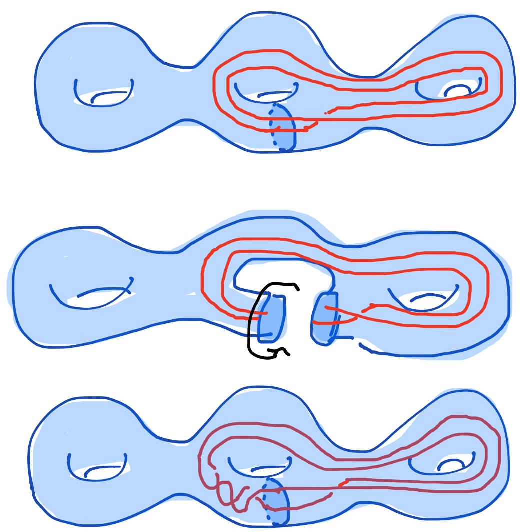
$$\pi_0(\mathrm{Homeo}(H_n)) \longrightarrow \mathrm{Out}\,F_n$$

is surjective but not injective:

"Dehntwist" on disk



is not $\simeq$ id, but does nothing to $\pi_1 H_n$, so is in kernel.



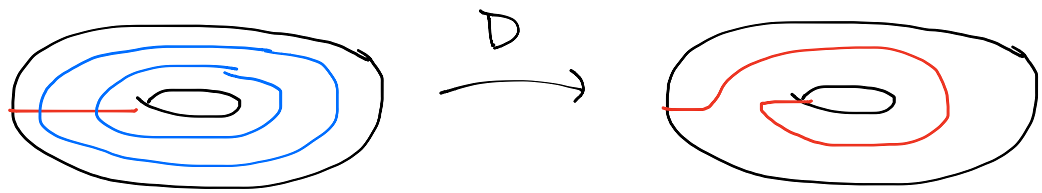
(4) X = doubled handlebody M_n

$$\pi_0(\mathrm{Homeo}\, M_n) \longrightarrow \mathrm{Out}(F_n) \quad \text{is surjective}$$

Also not injective, but closer:

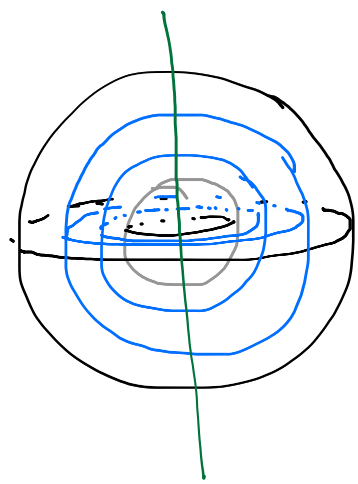
<u>Thm</u> (Laudenbach) kernel is generated by Dehn twists in 2-spheres $S^2 \subset M$

Dehn twist in a surface: supported on an annulus



concentric circles rotate thru angles $0 < t < 2\pi$

In a 3-manifold, Dehn twist supported on $S^2 \times I$



Pick an axis of rotation
Fix outside sphere, rotate more and more as you go inside, rotate 360 on inside sphere (= loop in $SO(3)$).

Get a path in $SO(3)$, infact a loop.

$\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. $\Rightarrow D^2 = id$.

Laudenbach shows kernel is a finite 2-group $(\mathbb{Z}/2)^n$

We will use ① and ④ to study automorphisms.

We're not the first:

④ Was used by Whitehead, eg to answer
when is $a_i \longmapsto w_i$ an automorphism?

eg $\left\{ \begin{array}{l} a \longmapsto ab \\ b \longmapsto b a^{-1} b c^{-1} \\ c \longmapsto cab \end{array} \right.$ Is this invertible?

A: No. But $\left\{ \begin{array}{l} a \mapsto ab \\ b \mapsto bcab \\ c \mapsto b^{-1} a^{-1} c \end{array} \right.$ is

Whitehead invented a combinatorial algorithm to decide, for any set of distinct cyclic words $\{w_i\}$, whether $a_i \longmapsto w_i$ is an automorphism.
We'll see this in the next lecture.