Outer space and Automorphisms of free groups

LECTURE 6

We're studying K_n = spine of outer space

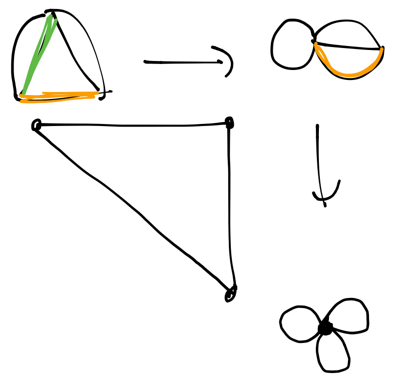
= contractible, cocompact simplicial complex

<u>Today</u>: Cube complex structure, homology computations and connection with Kontsevich's graph homology.

In graphical description of K_n, simplex

:= chain of forest collapses

$$\begin{tikzcd}
G \arrow[r] & G_{\varphi_1} \arrow[r] & (G_{\varphi_1})_{\varphi_2} \arrow[r] & \cdots \arrow[r] & (\cdot\cdot(G_{\varphi_1})_{\varphi_2}\cdots)_{\varphi_k} \\
R_n \arrow[u] \arrow[ur] \arrow[urr] \arrow[urrrr] & & & &
\end{tikzcd}$$

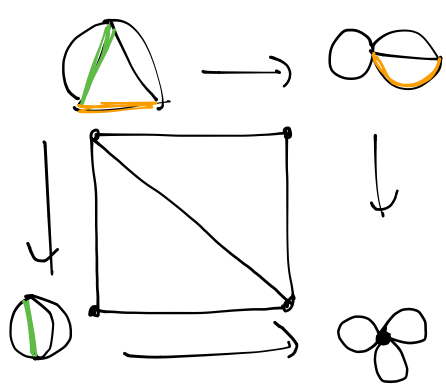


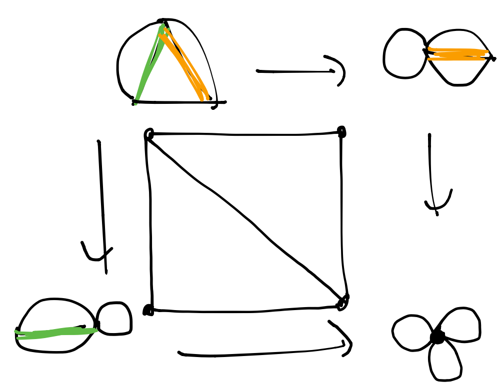
Can regard this as a chain of forests in the first graph

$(G, \Phi_1 \subset \Phi_2, g)$

where Φ_i = inverse image of φ_i

Given $\Phi \subset G$, can collapse the edges one at a time in any order to get a cube in K_n:





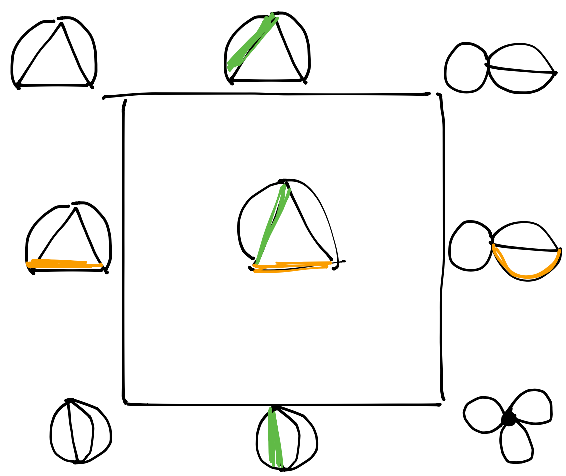
This cube can be oriented by ordering the edges of Φ.

K_n is a cube complex with cubes $c(g, G, \Phi)$,

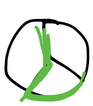
Φ a forest in G

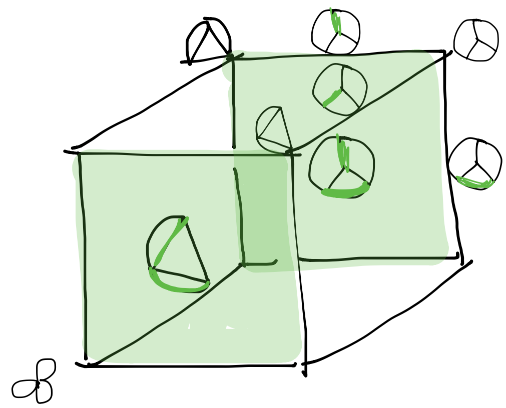
$\underline{\dim}\ c(g, G, \Phi) = \#\text{ edges of } \Phi.$

$\underline{\text{Faces}}$ of $c(g, G, \Phi)$:



$$\partial(G, \Phi, g) = \sum_{e \in \Phi} (G, \Phi - e) - (G_e, \Phi_e)$$

 = 3-cube:

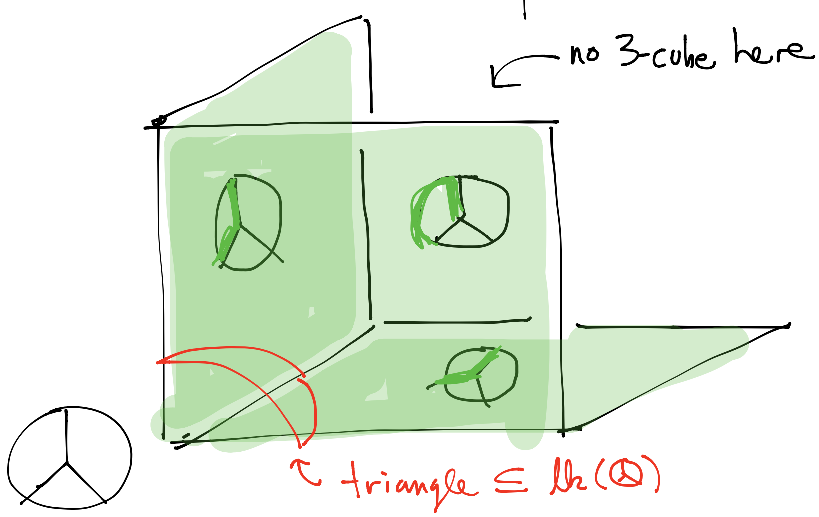


Aside:

Recall a cube complex is NPC (locally CAT(0))

if links of vertices are flag complexes.

K_4 is <u>not</u> a CAT(0) cube complex:



Exercise: Lk(⊗) is homotopic to a wedge of 2-spheres. How many?

Can't even fudge the angles to make a CAT(0) cube complex (Bridson's thesis), and in fact

Thm (Bridson) $\mathrm{Out}(F_n)$ cannot act properly and cocompactly on a CAT(0) metric space.

<u>Recall</u>: $\Gamma < \mathrm{Out}(F_n)$ torsion-free, finite index

$\Rightarrow H^*(\Gamma) = H^*(K_n/\Gamma)$

This is with <u>any</u> coefficients

If we take trivial coeffs in $\mathbb{Q}$ (or any field of char 0), turns out

$$H^*(K_n/\mathrm{Out}F_n; \mathbb{Q}) \cong H^*(\mathrm{Out}F_n; \mathbb{Q})$$

Heuristically: $H^*(H;\mathbb{Q}) = 0$ if H is finite,

so $H^*(\ ;\mathbb{Q})$ doesn't "see" the finite stabilizers, thinks the action is free.

eg $H = \mathbb{Z}/n\mathbb{Z}$ has chain complex

$$\cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$\to C_5 \to C_4 \to C_3 \to C_2 \to C_1 \to C_0$$

tensor w/ $\mathbb{Q}$:

$$\cdots \to \mathbb{Q} \underset{\cong}{\xrightarrow{\times n}} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \underset{\cong}{\xrightarrow{\times n}} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \underset{\cong}{\xrightarrow{\times n}} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \to 0$$

Formally: There is a spectral sequence corresponding to the action of $\mathrm{Out}\,F_n$ on K_n, converging to $H^*(\mathrm{Out}\,F_n)$; E^2-terms correspond to vertex stabilizers if $q>0$, so vanish with trivial $\mathbb{Q}$-coefficients, spectral sequence collapses to a chain complex for $H^*(K_n/\mathrm{Out}\,F_n)$

(Details: K. Brown's book, equivariant homology spectral sequence, (Chap. VII, Sec. 7))

So we can compute $H^*(\mathrm{Out}\,F_n;\mathbb{Q})$ by computing $H^*(K_n/\mathrm{Out}\,F_n;\mathbb{Q})$.

Action of $\mathrm{Out}(F_n)$ on K_n changes the marking $(G,g)\cdot\varphi = (G, g\circ f)$, in fact acts <u>transitively</u> on markings of G.

So $K_n/\mathrm{Out}(F_n)$ has one vertex for each isomorphism class of graphs G

And one k-simplex for each isomorphism class of chains of k forests $(G, \Phi_1 \subset \dots \subset \Phi_k)$

e.g. If T, T' are (non-equivalent) maximal trees in G, then

$$G \longrightarrow G_T \approx R$$

and

$$G \longrightarrow G_{T'} \approx R$$

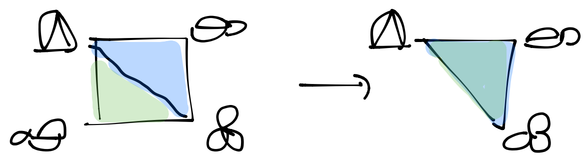
are two *different* edges from G to the rose R.

So $K_n/\mathrm{Out}F_n$ is a union of simplices, but *not* a simplicial complex.

The ∂ is a mess to compute. Easier if you use the cube complex structure:

BUT: $K_n / \mathrm{Out}(F_n)$ is not a union of cubes:

An auto of (G, Φ) induces an auto of F_n which folds the cube:

e.g. 

The stabilizer of the cube $c(G, \Phi, g)$ is isomorphic to $\mathrm{Aut}(G, \Phi)$ (graph isomorphisms preserving Φ)

Prop The quotient $c(G, \Phi, g) / \mathrm{stab}(G, \Phi, g)$ is a cone on a rational homology sphere or disk, depending on whether $\mathrm{Aut}(G, \Phi)$ contains an orientation-reversing element.

Pf: Look at equivariant homology spectral sequence: G acts on X. Two filtrations of

$$C_*X \otimes_{\mathbb{Z}G} C_*EG$$

give two spectral sequences converging to the same thing

$$E^2_{p,q} = H_p(G; H_q X)$$

and

$$E^2_{pq} = H_p(X/G, \mathcal{H}_q) = \bigoplus_{\sigma_q\text{-orbits}} H_q(G_{\sigma_q})$$

For G finite, rat'l coeffs, $X = S^k$, get

$$\begin{array}{c|cccc} k & H_0(G; H_k(S^k)) & 0 & 0 & 0 \\ & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \\ & H_0(G) & 0 & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{|cccccc} & & & & & \\ & 0 & & 0 & & 0 \\ & \bigoplus_{\sigma_0} \mathbb{Q} & \leftarrow & \bigoplus_{\sigma_1} \mathbb{Q} & \leftarrow & \bigoplus_{\sigma_2} \mathbb{Q} \leftarrow \cdots \\ \hline \end{array}$$

This is E^1_{pq}!

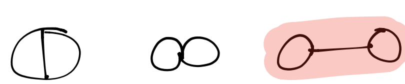
$\uparrow$ $= 0$ except in dims $0, k$.
$= 0$ in dim k if G reverses orientation.

$\uparrow$ computes $H_*(S^k/G)$

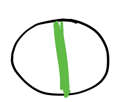
Cor $H_*(K_n/\mathrm{Out}\,F_n)$ is the homology of the chain complex C_* with $C_k = \bigoplus_{(G,\Phi)} \mathbb{Q}$

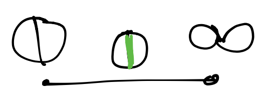
where
- G has rank n
- Φ is a forest in G with k edges
- (G,Φ) has no orientation-reversing automorphisms

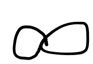
and
- $$\partial(G,\Phi) = \sum_{e\in\Phi} (G,\Phi - e) - (G_e,\Phi_e) = \partial_R(G,\Phi) - \partial_C(G,\Phi)$$

eg: $n=2$. vertices 

↑ remove separating edges

edges 

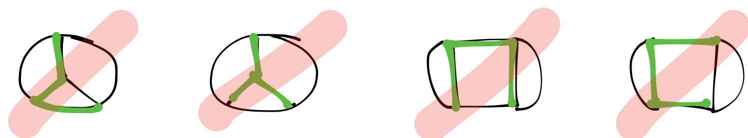
(no or-reversing autos!) quotient 

$\partial($  $) =$  $-$ 

So $H^*(\mathrm{Out}\,F_2;\mathbb{Q}) = 0$ for $* > 0$.

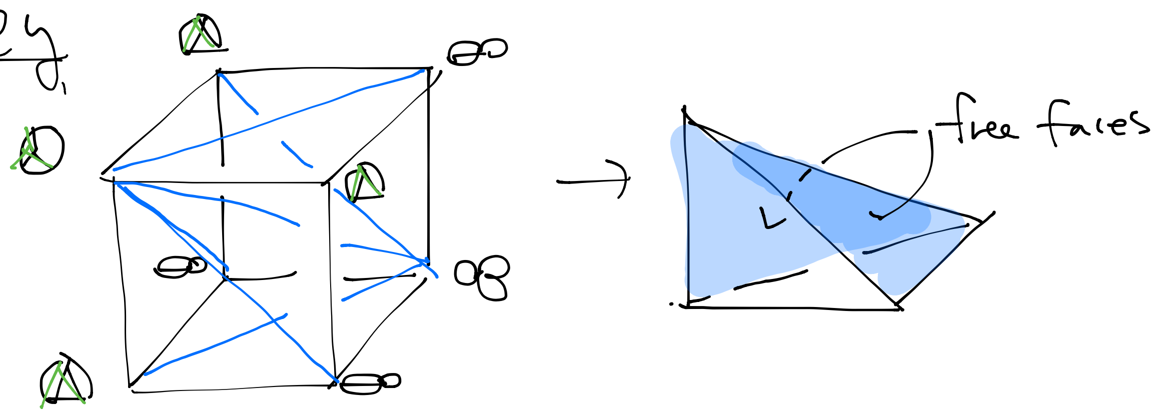
$H^*(\mathrm{Out}\,F_3)$: K_n is 3-dimensional

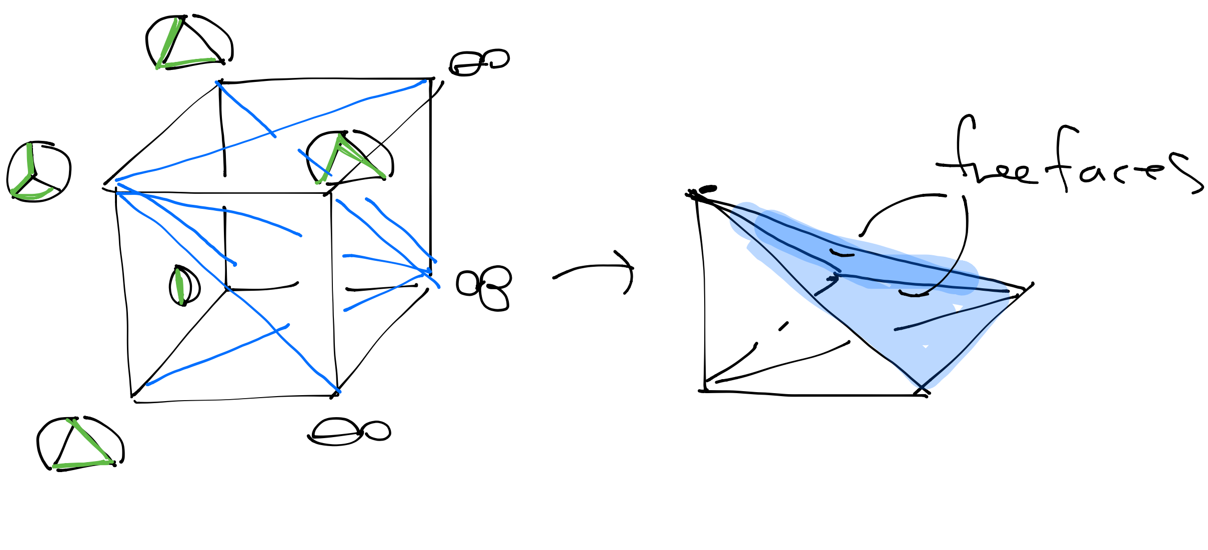
3-cubes:



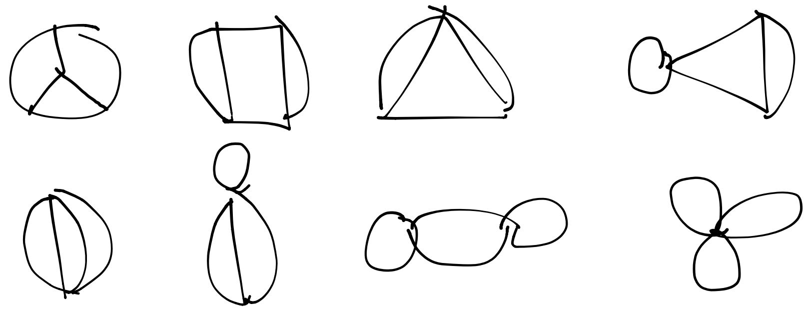
Each has an orientation-reversing automorphism, so a free face in $K_3/\mathrm{Out}\,F_3$, so $H^3 = 0$

eg.

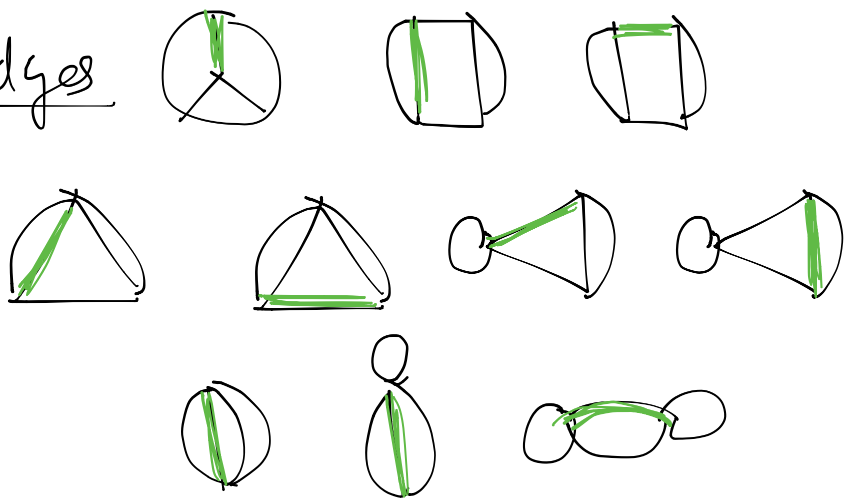




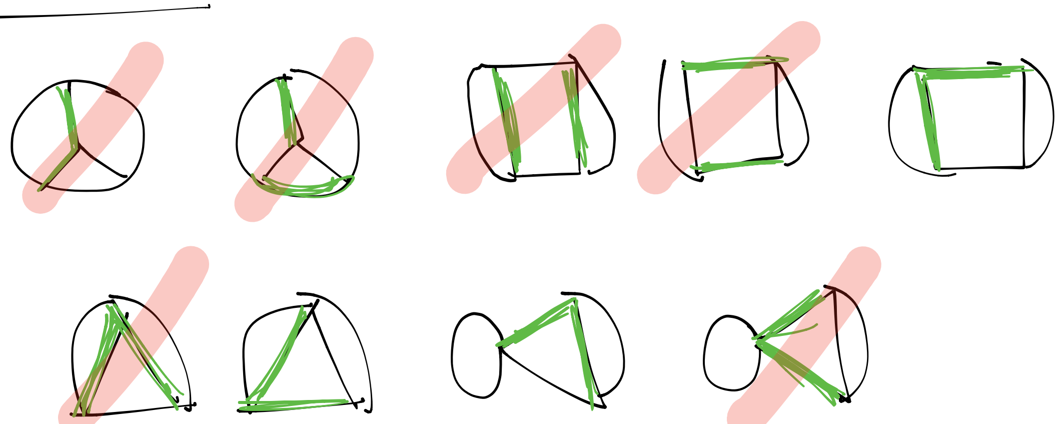
8 vertices:



10 edges



2-cubes



chain complex

$$C_3 \to C_2 \to C_1 \to C_0 \to 0$$

$$0 \to \mathbb{Q}^3 \to \mathbb{Q}^{10} \to \mathbb{Q}^8 \to 0$$

Exercise: Compute ∂ maps, show

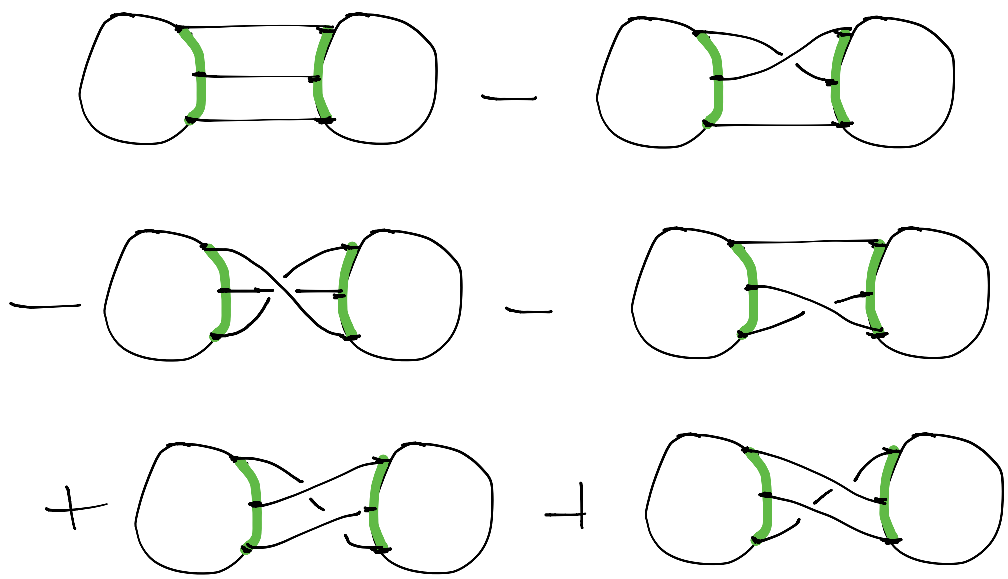
$$H_* = 0 \ , \ * > 0$$

$n=4$ is about the limit of reasonable computability

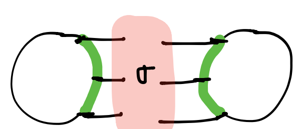
Get $H_1 = H_2 = H_3 = H_5 = 0$ but

$$H_4(\mathrm{Out}\, F_4) \cong \mathbb{Q} \, !$$

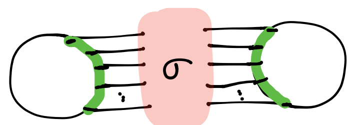
Here is the 4-cycle which generates $H_4(\mathrm{Out}\, F_4)$: it is made out of (quotients of) 4-dimensional cubes



$$= \sum_{\sigma \in \Sigma_3} \mathrm{sign}(\sigma) \cdot$$



Easy to generalize:

$$\mu_k = \sum_{\sigma \in \Sigma_{2k+1}} \mathrm{sign}(\sigma) \cdot$$ 

This is a cycle, called the k-th Morita cycle

μ_1, μ_2, μ_3 are not boundaries

<u>Conjecture</u> (Morita) μ_k represents a non-zero homology class for all k.

Morita found these cycles via the relation with Kontsevich's graph homology of the Lie operad

To make the connection, need to simplify our chain complex.

We have

$$C_k = \bigoplus_{\substack{(G,\Phi) \\ e(\Phi)=k}} \mathbb{Q} = \bigoplus_{e(\Phi)=k} (G,\Phi)$$

We can decompose this further by # vertices of G

$$C_k = \bigoplus_{p+q=k} C_{p,q}$$

where $C_{p,q} = \bigoplus_{\substack{vG=p \\ e\Phi=p+q}} (G,\Phi)$

Top-dim'l cubes are

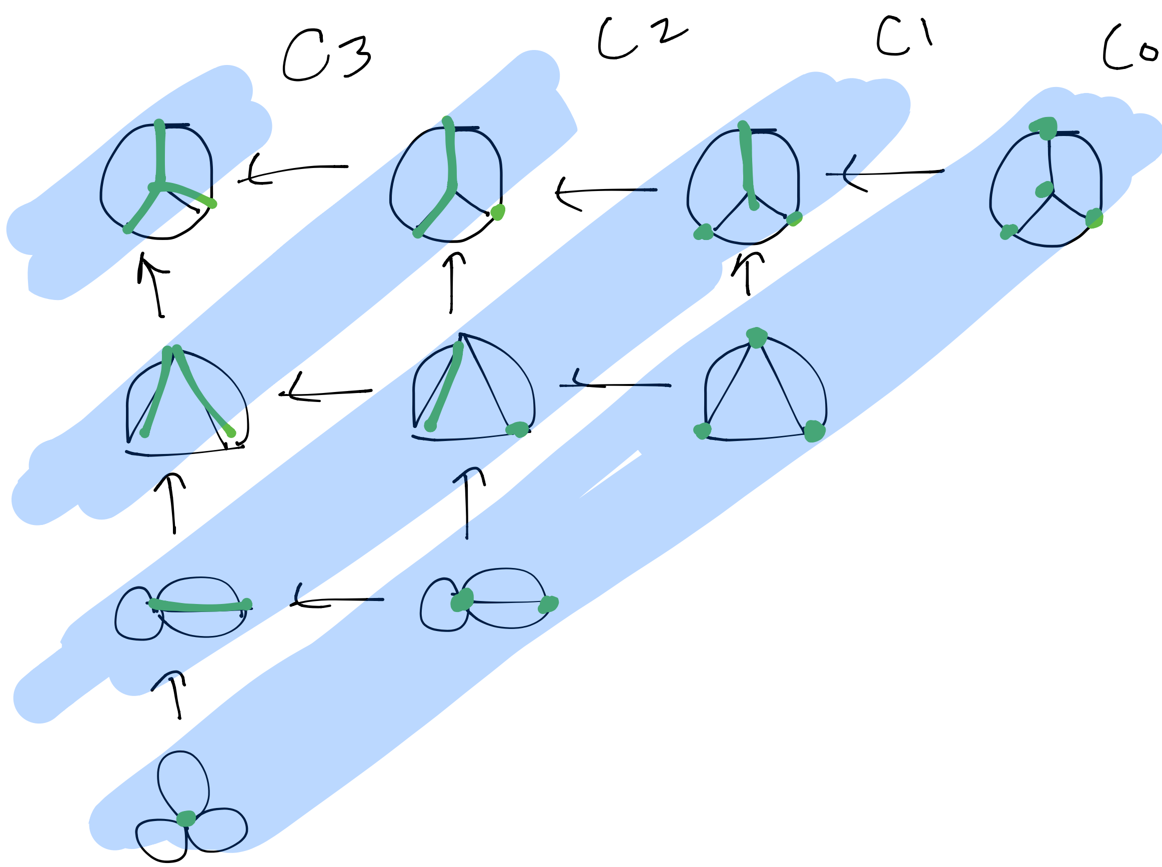
$(G,T) =$ (trivalent graph, maximal tree)

$(vG = 2n-2,\ e\Phi = 2n-3)$

next are $\quad v(G) = 2n-2,\ e\Phi = 2n-4$

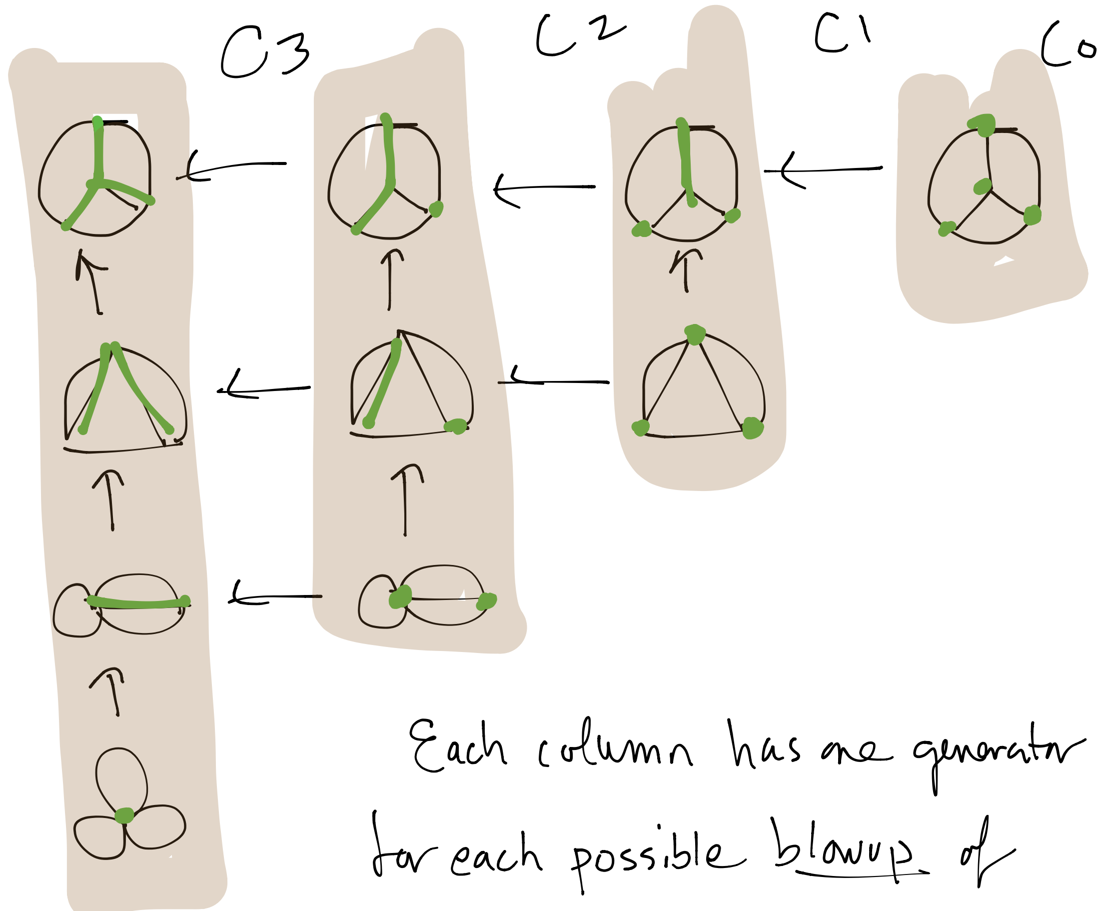
and $vG = 2n-3,\ eG = 2n-4$

etc.



We can compute $H^*(C_*)$ by first computing the vertical homology, then the horizontal H_*

(this is spectral sequences, but don't tell anybody)



Each column has one generator for each possible blowup of the graphs G at the bottom

This is the chain complex for $\mathcal{lk}_{>G}$.

We know $\mathcal{lk}_{>G} \simeq \bigvee S^{v(G)-2}$, so has no homology except in the top dimension

So same is true for $\mathcal{lk}_{>G}/\mathrm{Aut}(G) \subset K_n/\mathrm{Out}(F_n)$

(again by equivariant H_* spectral sequence)