Introduction to Outer space and automorphisms of free groups.

Register at: graduate.studies@maths.ox.ac.uk

Prerequisites:

- **Groups**: homomorphisms, normal subgroups, automorphisms, free groups, presentations

- **Topology**: homotopy fundamental group, covering spaces, simplicial complexes, homology, cohomology

- **Homological algebra**: chain complexes, exact sequences, Hom and $\otimes$

Course contents:

Outer space was introduced as a tool for studying (outer) automorphisms of free groups (but has since found other applications)

I hope to cover the following topics

- Automorphisms of F_n, models for F_n
- Definitions of outer space, contractibility
- Bordification and virtual duality
- Assembly maps and homology
- Euler characteristic and number theory
- Lipschitz metric and train tracks

Each topic is substantial so we may not get to all of them.

The first two topics were also covered in my 2015 TCC course. Notes are available on my Warwick web page.

The groups

$F_n = F\langle a_1, \ldots, a_n \rangle$ = free group on n letters

$\mathrm{Aut}(F_n) =$ Group of automorphisms of F_n

Examples of automorphisms:
(suffices to give images of the a_i)

① $\iota_i : a_i \longmapsto a_i^{-1}$ (invert a generator)
$a_k \longmapsto a_k$ if $k \neq i$

② $\sigma : a_i \longmapsto a_{\sigma(i)}$ (permute the generators)

③ $\rho_{ij} : a_i \longmapsto a_i a_j$ (right multiply a_i by a_j)
$a_k \longmapsto a_k$ if $k \neq i$

④ $\lambda_{ij} : a_i \longmapsto a_j a_i$ (left multiply a_i by a_j)
$a_k \longmapsto a_k$ if $k \neq i$

⑤ $\gamma_w : a_i \longmapsto w^{-1} a_i w$ (conjugate all a_i by w)

⑤ is an <u>inner automorphism</u>

$\mathrm{Inn}(F_n)$ = inner automorphisms $\cong F_n$.
$\lhd \mathrm{Aut}(F_n)$

$\mathrm{Out}(F_n) = \mathrm{Aut}\,F_n / \mathrm{Inn}\,F_n$ = outer automorphisms

Express this using a short exact sequence:

$$1 \to F_n \to \mathrm{Aut}\,F_n \to \mathrm{Out}\,F_n \to 1 \, .$$

"$\mathrm{Aut}\,F_n$ is an extension of F_n by $\mathrm{Out}\,F_n$"

There is also an extension of F_n by $\mathrm{Aut}\,F_n$:

$$\begin{tikzcd} 1 \arrow[r] & F_n \arrow[r] & A_{n,2} \arrow[r] & \mathrm{Aut}\,F_n \arrow[r] \arrow[l, dashed, bend left, "s"] & 1 \end{tikzcd}$$

This extension is <u>split</u> so $A_{n,2}$ is a semidirect product

ie $A_{n,2} = \{ (w, \varphi) \mid w \in F_n, \ \varphi \in \mathrm{Aut}\,F_n \}$

with multiplication

$$(w, \varphi) \cdot (u, \psi) = (w\,\varphi(u), \varphi\psi)$$

In fact, for any $k \geq 1$ there is an extension

$$1 \longrightarrow F_n^{\,k} \longrightarrow A_{n,k+1} \longrightarrow Aut(F_n) \longrightarrow 1$$

$$((w_1, \ldots, w_k), \varphi)((u_1, \ldots, u_k), \psi)$$
$$= ((w_1 \varphi u_1, \ldots, w_k \varphi u_k), \varphi\psi)$$

In order to study $Aut(F_n)$ and $Out(F_n)$ we will eventually use all of these groups.

History

The study of $Aut(F_n)$ was begun in early 1900's by Nielsen, Magnus, JHC Whitehead

Nielsen (1924) produced a finite set of generators for Aut(F_n).

namely: τ_i, σ, ρ_{ij}, λ_{ij}

He also produced s finite set of relations.

Notice: abelianization $F_n \longrightarrow \mathbb{Z}^n$ induces

$$\mathrm{Aut}(F_n) \longrightarrow \mathrm{Aut}(\mathbb{Z}^n) = GL(n, \mathbb{Z})$$

[since an automorphism $F_n \to F_n$ preserves $[F_n, F_n]$, so induces

$$\begin{array}{ccc} F_n/[F_n,F_n] & \longrightarrow & F_n/[F_n,F_n] \\ \| & & \| \\ \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^n \end{array}$$
]

The image of σ is a permutation matrix. The image of ρ_{ij} (or λ_{ij}) is an elementary matrix.

These are easily seen to generate $GL_n\mathbb{Z}$,

so $\mathrm{Aut}\, F_n \twoheadrightarrow GL_n\mathbb{Z}$

The kernel contains $\mathrm{Inn}(F_n)$

Nielsen (1917): for $n=2$, the kernel $= \mathrm{Inn}(F_n)$

B. Neumann (1932) used Nielsen's presentation for $\mathrm{Aut}\, F_n$ to get a presentation for $GL(n, \mathbb{Z})$

Magnus (1934) found a finite set
of generators for the kernel, namely

$$x_i \mapsto x_j^{-1} x_i x_j \quad \text{and} \quad x_i \mapsto x_i [x_j, x_k]$$

He asked whether the kernel is finitely
presented.
The answer is "no" for n =3 (proved in 1997
by Krstic and McCool) but still unknown
for n> 3.

Remark on notation:

For any G, the subgroup of automorphisms that induce the identity on the abelianization is called $IA(G)$

People often use IA_n to mean $IA(F_n)$ but sometimes they mean the image of $IA(F_n)$ in $Out(F_n)$, $= IA(F_n)/Inn(F_n)$

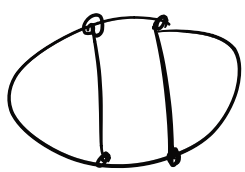
McCool (1974) gave, a more understandable proof that Aut(F_n) is finitely presented using "Whitehead automorphisms" and "peak reduction".

Stallings-(1970's) gave an elegant proof of finite generation using a "folding operation on graphs.

Both Whitehead and Stallings used topological models for the free group F_n and its automorphisms.

Stallings used finite connected graphs with fundamental group F_n, i.e. v-e=I-n

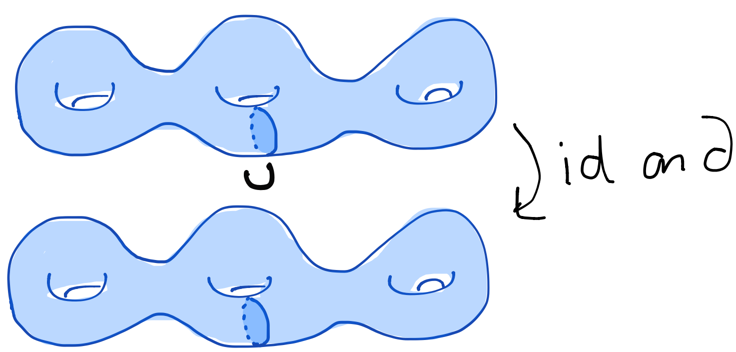
eg



$$v-e = 4-6 = -2$$
$$= 1-3$$

Whitehead used doubled handlebodies

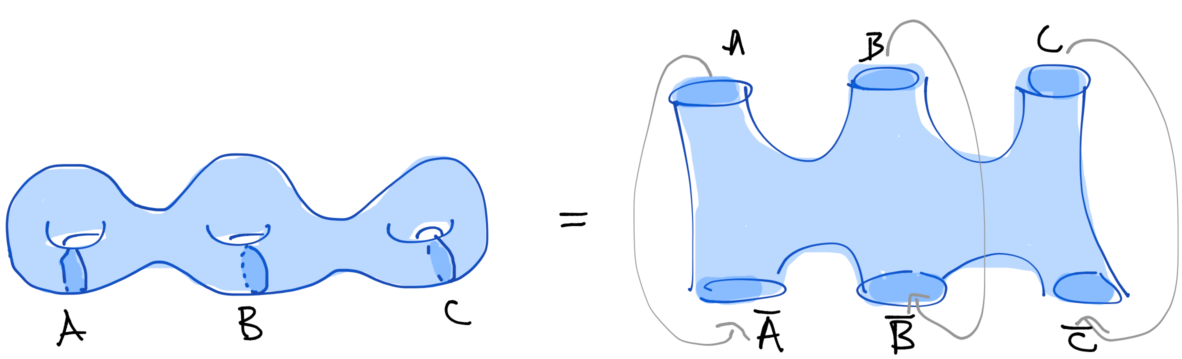
$$M_n = \# S^1 \times S^2, \quad \pi_1(M_n) \cong F_n$$



Graphs are certainly easier to visualize, but doubled handle bodies have advantages, as we will see.

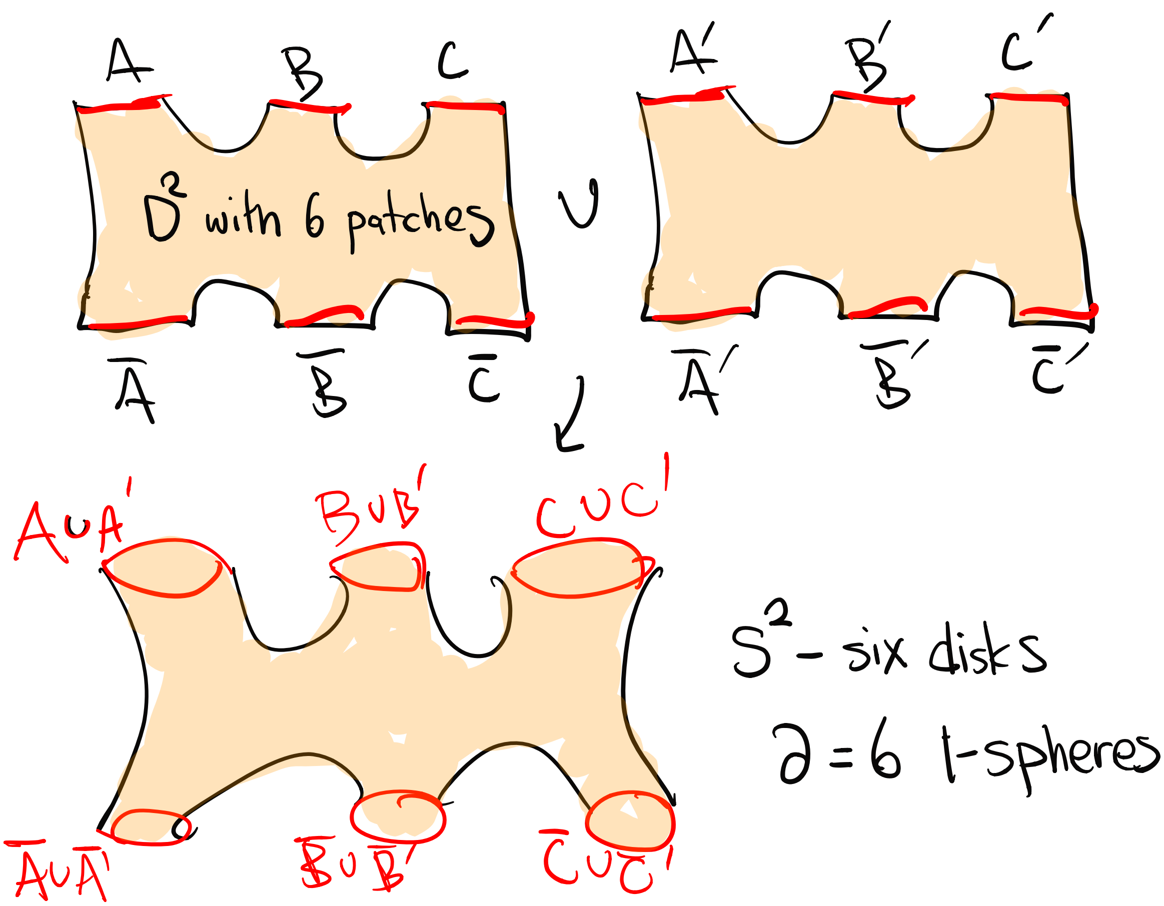
Here's another way to visualize the doubled handlebody:

Cut the handlebody open along n disks before you double it to get a 3-ball with six disk patches:

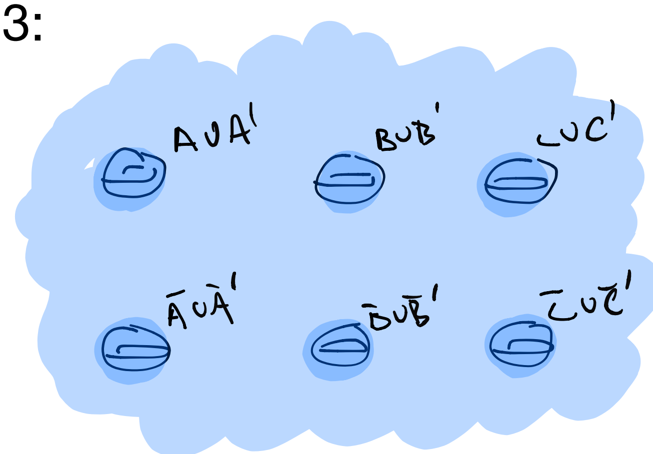


Double this, but don't glue the patches. the result is a 3-sphere with six 3-balls removed, i.e. the six disk patches become six 2-spheres.

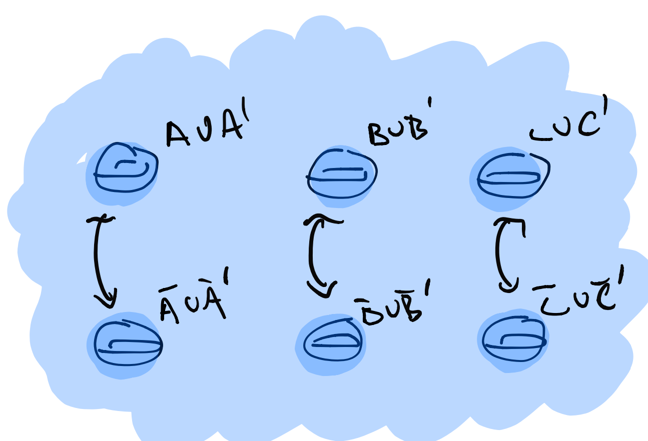
Picture down one dimension:



Back in dimension 3:



To get M_n we now need to glue the spheres back together in pairs, to repair our original cuts.



[Break for 5 minutes]

Let X be any topological space.

If we fix a basepoint $b \in X$ then a homotopy equivalence $X \to X$ sending $b \mapsto b$ induces an automorphism

$$\pi_1(X,b) \to \pi_1(X,b)$$

If we want to allow homotopy equivalences f that do not preserve a basepoint then to get an automorphism of $\pi_1(X,b)$ you need to choose a path γ from b to $f(b)$:

$$\begin{tikzcd} \pi_1(X,b) \arrow[r] \arrow[rr, bend left] & \pi_1(X,f(b)) \arrow[r, "\cong", "\gamma_*"'] & \pi_1(X,b) \end{tikzcd}$$

A different γ changes the automorphism by an inner auto.

So f only induces an outer automorphism of $\pi_1(X,b)$;

Let $HE(X)$ = space of all homotopy equivalences of X, with the compact-open topology

and $HE(X,b)$ those preserving b.

Homotopic paths give the same map on π_1

A homotopy (preserving b) is a path in $HE(X)$ $(HE(X,b))$

so actually get maps

$$\pi_0 HE(X) \longrightarrow Out(\pi_1 X)$$

$$\pi_0(HE(X,b)) \longrightarrow Aut(\pi_1(X,b))$$

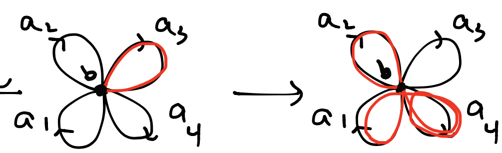
We have X with $\pi_1(X) \cong F_n$

<u>Proposition</u> Let X be a finite connected graph with $\pi_1 \cong F_n$. Then

$$\pi_0 HE(X,b) \longrightarrow \mathrm{Aut}(F_n)$$

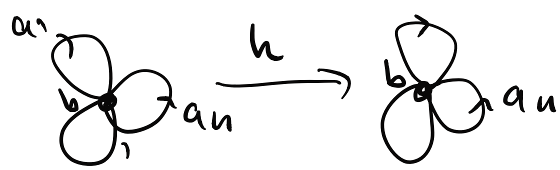
is an isomorphism.

(The proof was left as an exercise, but here's a sketch):

<u>Surjective</u>: $(X,b) \simeq$  $\simeq (X,b)$

Identify $\pi_1(X,b) \cong F\langle a_1, \ldots, a_n \rangle = F_n$

Given $\varphi \in \mathrm{Aut}(F_n)$, construct a h.equiv.
say $\varphi(a_i) = w_i$ $h: a_i \mapsto w_i$.

<u>Injective</u>:  induces id

on $\pi_1(R_n, b) = F_n\langle a_1, \ldots, a_n \rangle$

$\Rightarrow$

each loop a_i is sent to a loop htpic to to a_i, fixing b. You can perform these homotopies simultaneously to get $h \simeq id$.

(same pf for general graph G choose a maximal tree to identify $\pi_1(G, b)$ with F_n)

Now consider $X = M_n = \# S^1 \times S^2$

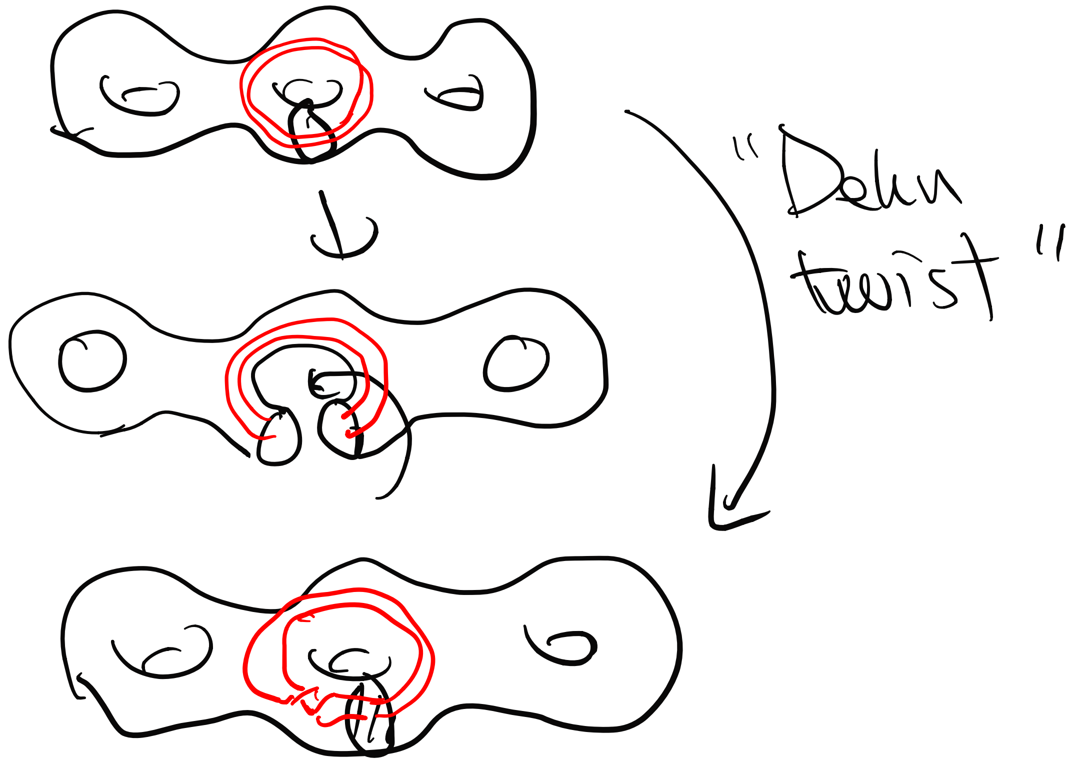
Actually, start with the (undoubled) handlebody

$$X = H_n = \# S^1 \times D^2$$

Every homotopy equivalence is homotopic to a homeomorphism (unlike for graphs!) so look at

$$\pi_0(\mathrm{Homeo}\ H_n) \to \mathrm{Out}(F_n)$$

Turns out this map has a large kernel:



The "Dehn twist" in this disk is not homotopic to id, but induces the identity on π_1,

ie it's in the kernel.
It has infinite order.

What happens when you double H_n?

- $X = M_n = \# S^1 \times S^2$

The Dehn twist magically becomes finite order!

$(\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z})$

Laudenbach (1973):

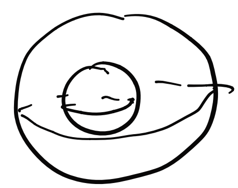
The map

$$\pi_0(HE(M_n)) \twoheadrightarrow Out(F_n)$$

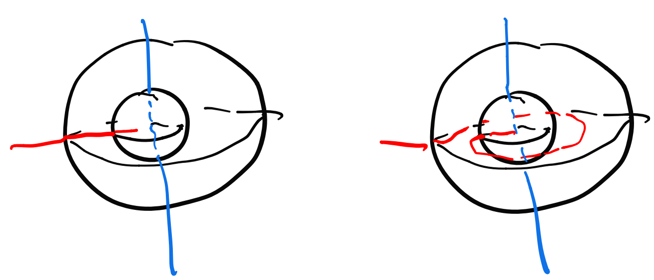
has a finite kernel $(\mathbb{Z}/2\mathbb{Z})^n$, generated by Dehn twists in 2-spheres.

The Dehn twist in H_n occurs in a nbd $D^2 \times I$ of the disk D^2

The Dehn twist in M_n occurs in a nbd $S^2 \times I$ of the 2-sphere S^2



outside sphere is fixed
inside sphere is rotated 360° —
intermediate spheres are rotated
between 0 and 36



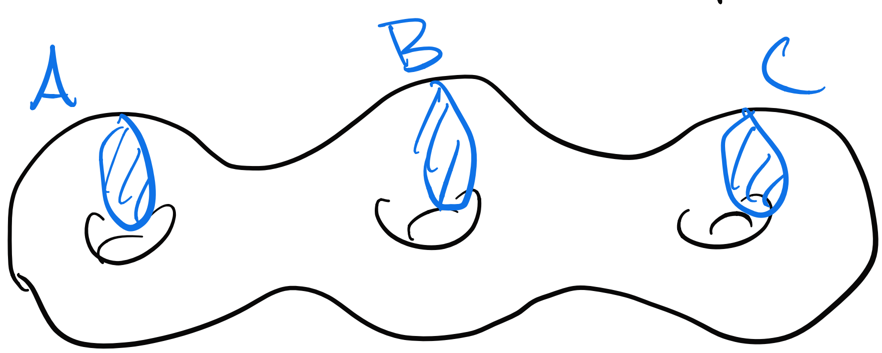
This is a loop in $SO(3)$.
$\pi_1(SO(2)) = \mathbb{Z}/2 \Rightarrow D^2 = id.$

Whitehead used M_n to answer:
when is a set of words $w_1, \dots, w_n$ a basis for F_n?
ie When is the homomorphism sending

$$a_i \longmapsto w_i \text{ an automorphism?}$$

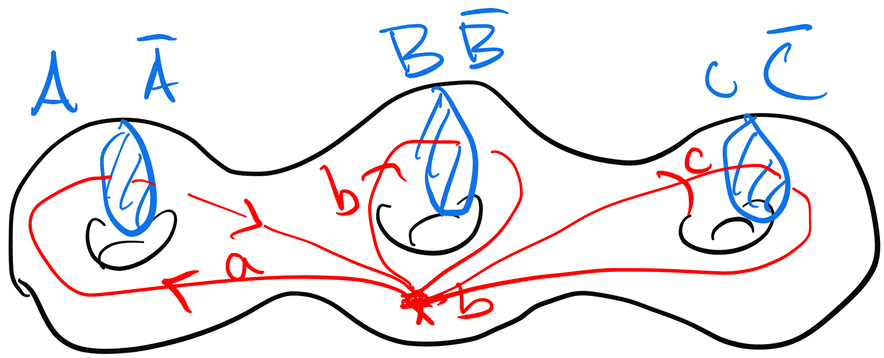
His key tool was the star graph of an automorphism.

Idea: Choose n 2-spheres

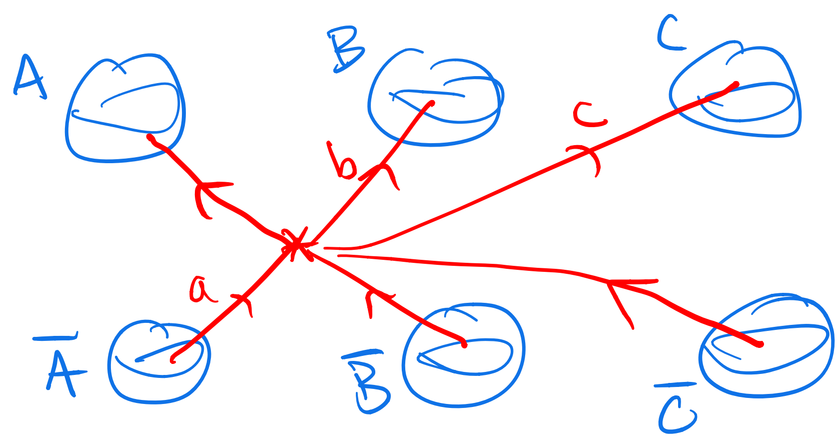


that cut M_n into a $2n$-punctured 3-sphere

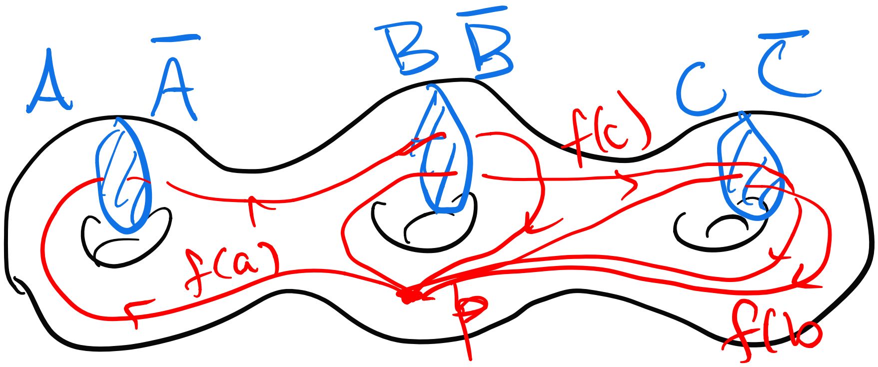
and a dual basis for $\pi_1 M_n$

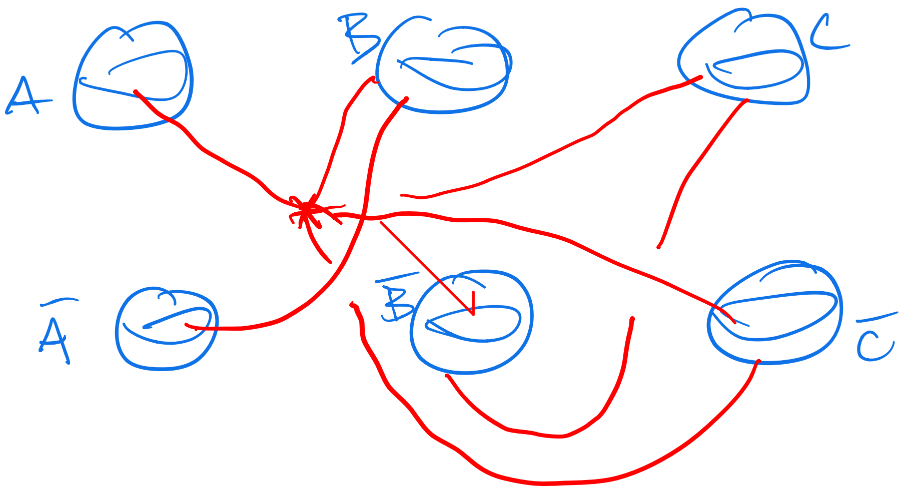


in our other picture:

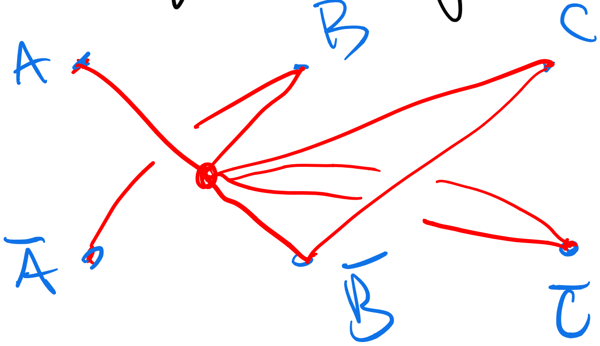


Now apply a diffeomorphism
$$M_n \longrightarrow M_n$$
and watch what happens to the red graph:





If you squint you see a graph, called the <u>star graph</u> of the automorphism



It has an edge x to y whenever $xy^{\pm 1}$ occurs in some $f(a_i)$.

Whitehead observed:

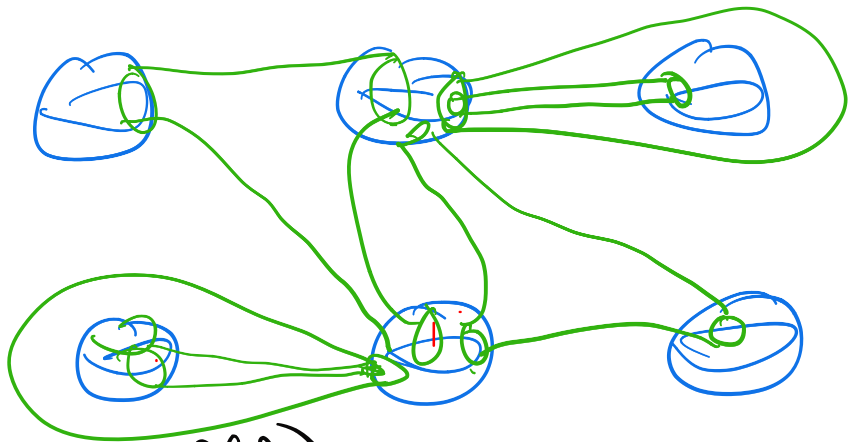
This graph has a <u>cut vertex</u> (other than the basepoint)

(B in this example)

To see why, think what happens to the spheres A, B, C under the diffeomorphism.

If $f(A)$ intersects The spheres A B and C, they cut $f(A)$ into pieces, which are planar surfaces.

But $f(A)$ is still a sphere,



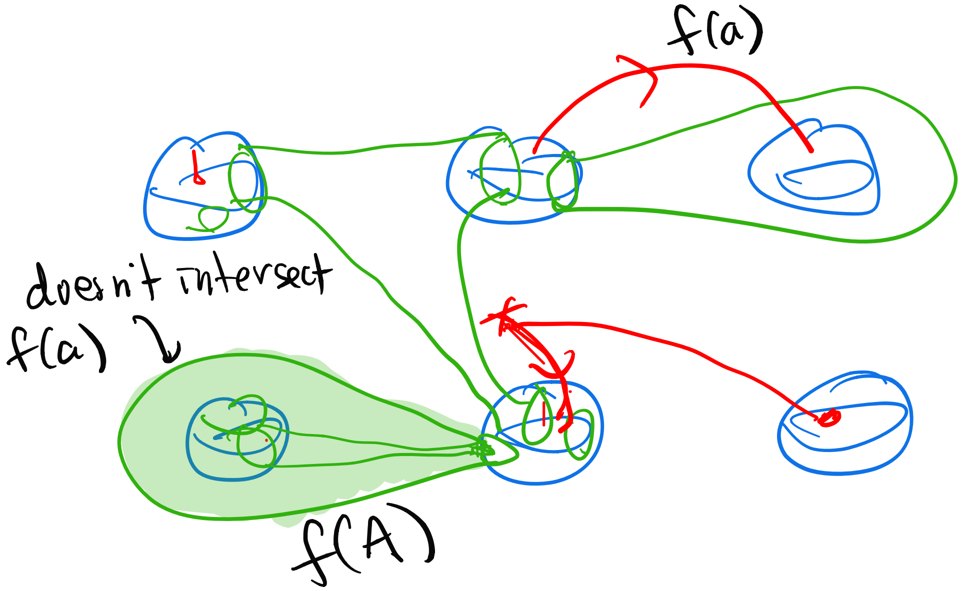
f(A)

so at least two of these pieces are disks.

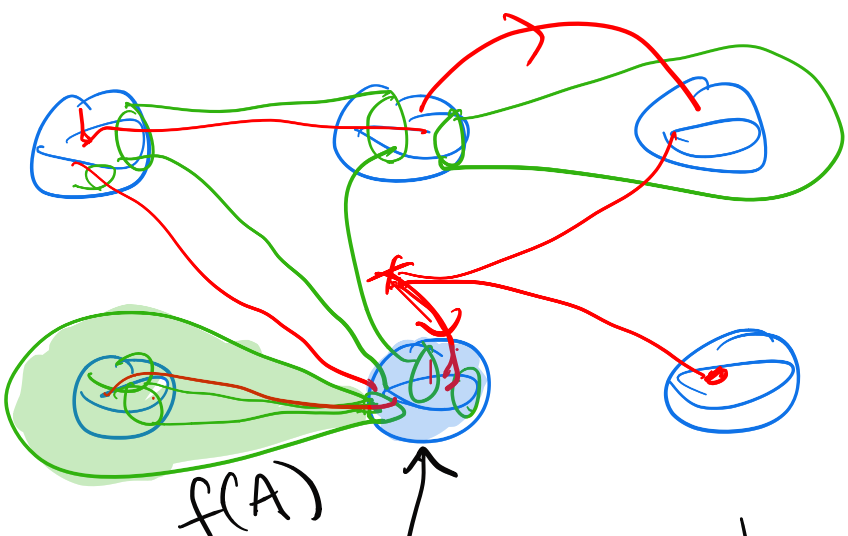
Now a intersected A in one point

So f(a) intersects f(A) in one point.

In particular, f(a) misses one of the disks



f(A) also misses f(b) and f(c) altogether

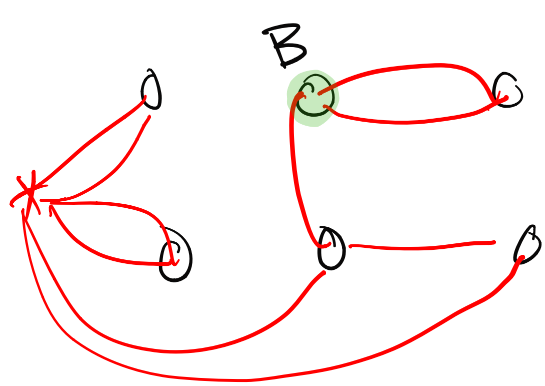


so is a cut vertex!

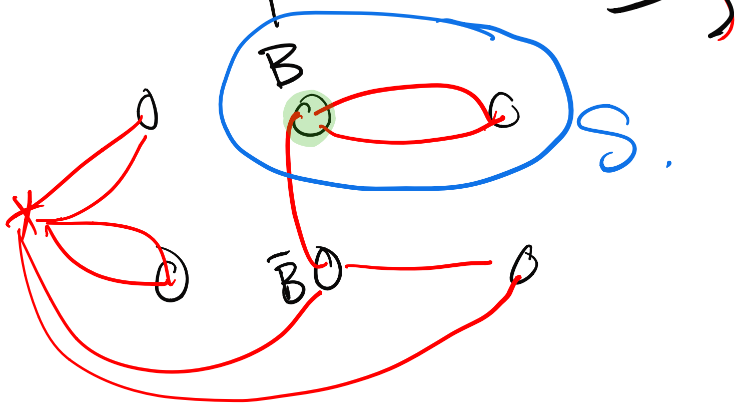
Part of the star graph is inside, part outside.

The fact that the star graph of an automorphism always contains a cut vertex leads to an algorithm for deciding when a given set of words $w_1, \ldots, w_n$ is a basis.

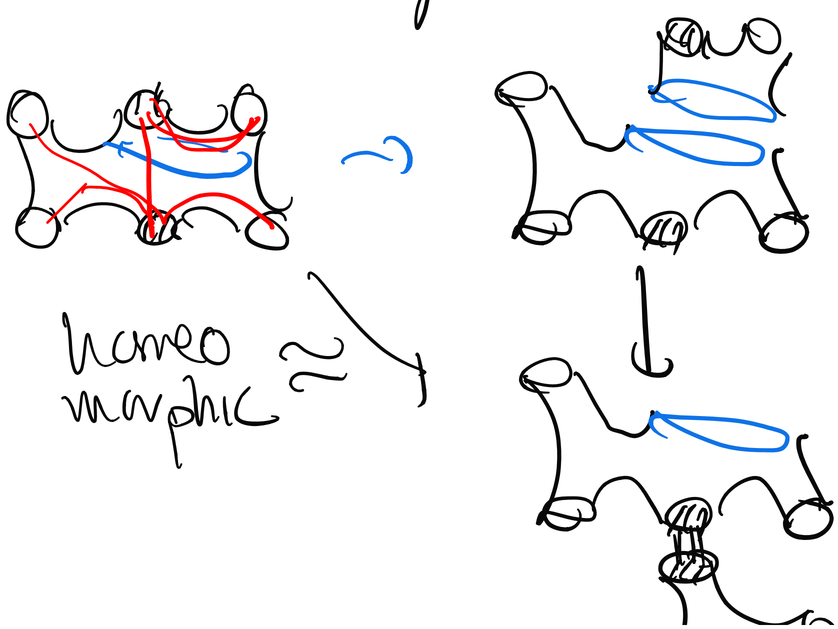
If it is, then $\varphi: a_i \longmapsto w_i$ is an automorphism, so the stargraph of φ has a cut vertex

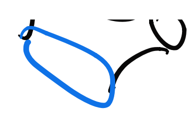


so you can draw a sphere
that intersects the star graph
in fewer points than B,

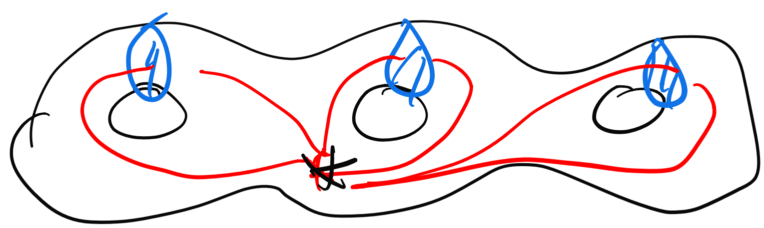


and separates B from $\bar{B}$.
now apply a homeomorphism
that interchanges B and S.

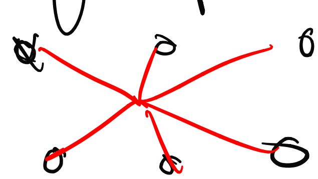




If $\{w_i\}$ is a basis you can continue until # of n pts of star graph with new sphere system is minimal, i.e $= n$.



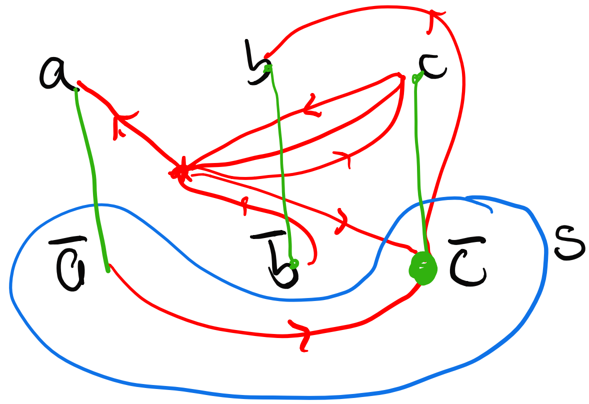
there are n intersections and stargraph is



If at any point you don't have a cut vertex, then $w_1, \dots, w_n$ wasn't a basis!

Example

$$a \longmapsto a\bar{c}$$
$$b \longmapsto \bar{c}$$
$$c \longmapsto cb$$



$\bar{C}$ is a cut vertex. S is a new sphere separating C from $\bar{C}$

replacing $\bar{C}$ by S has the effect of replacing a by ac, which gives

$$a\bar{C} \longrightarrow ac\bar{C} = a$$
$$\bar{C} \longmapsto \bar{C} = \bar{c}$$
$$cb \longmapsto cb = cb$$

with shorter total length. (=4)