Lecture 2

Last time we introduced two models for automorphisms of free groups, based on two spaces

① X = finite connected graph with $\pi_1(X) \cong F_n$

$\left.\begin{array}{l} \mathrm{Aut}(F_n) = \pi_0 HE(X, b) \\ \mathrm{Out}(F_n) = \pi_0 HE(X) \end{array}\right\}$ Exercise, sketch in notes from Lecture 1

② X = doubled handlebody $M_n = \#_n S^1 \times S^2$

$$1 \to (\mathbb{Z}/2)^n \to \pi_0(\mathrm{Diff}(X, b)) \to \mathrm{Aut}(F_n) \to 1$$

$$1 \to (\mathbb{Z}/2)^n \to \pi_0(\mathrm{Diff}(X)) \to \mathrm{Out}(F_n) \to 1$$

generated by Dehn twists in 2-spheres (Laudenbach's theorem)

Used M_n to prove Whitehead's theorem:
Defined the star graph of a set of words

Thm: $w_1, \ldots, w_n$ is a basis for $F_n \Rightarrow$ star graph has a cut vertex other than b

Didn't have time to show you how to use this. It leads to an algorithm to decide whether $w_1, \ldots, w_n$ is a basis ie whether $a_i \longmapsto w_i$ is an automorphism. This is spelled out in the notes to lecture 1, (which are online), including an example.

Exercise: Read this. Now take three arbitrary sets of words in 3 letters and decide whether they are bases

I can now define the groups $A_{n,s}$:

$$A_{n,0} = \pi_0 HE(M_n) \big/ \text{Dehn twists} = Out(F_n)$$

$$A_{n,1} = \pi_0 HE(M_n, b) \big/ \text{Dehn twists} = Aut(F_n)$$

Forgetting the basepoint gives the map

$$Aut(F_n) \to Out(F_n)$$

Pushing b around a loop in M_n gives an inner automorphism of $\pi_1(M_n, b) = F_n$, Instead of a basepoint b, can think of a tiny hole, ie remove a 3-ball B^3 from M_n

$$A_{n,1} = \mathrm{Aut}(F_n) = \pi_0 HE(M_n \setminus B^3, \partial B^3)/DT$$

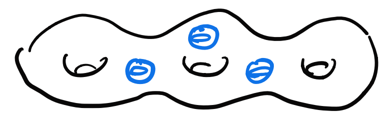
h.equivs fixing ∂B^3.

The map $A_{n,1} \longrightarrow A_{n,0}$ is induced by

$$M_n \setminus B^3 \hookrightarrow M_n$$

To get $A_{n,s}$, remove s balls:

$$M_{n,s} = M_n - \coprod_s B^3$$



$$A_{n,s} := \pi_0 HE(M_{n,s}, \partial M_{n,s})/DT$$

filling in any ball gives an inclusion

$M_{n,s} \hookrightarrow M_{n,s-1}$, inducing a map

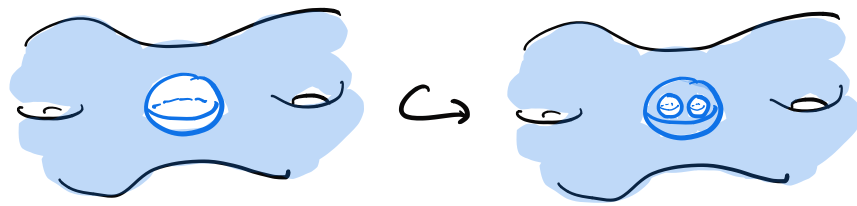
$$A_{n,s} \longrightarrow A_{n,s-1}$$

with kernel $F_n = \pi_1(M_{n,s})$

If $s > 1$, there is a section (actually $s-1$ sections)

$A_{n,s-1} \longrightarrow A_{n,s}$ induced by

$$M_{n,s-1} \hookrightarrow M_{n,s}$$



Iterating, get

$$1 \to (F_n)^{s-1} \to A_{n,s} \to \mathrm{Aut}(F_n) \to 1$$

and

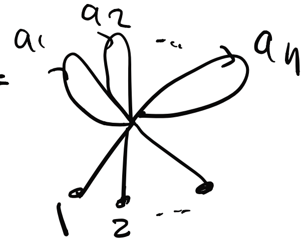
$$1 \to (F_n)^{s} \to A_{n,s} \to \mathrm{Out}(F_n) \to 1$$

<u>But</u>: there is no section at the last step $\quad \mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$

(There are finite subgroups of $\mathrm{Out}(F_n)$ that do not lift to $\mathrm{Aut}(F_n)$)

In the graph model, take a rose with n petals and leaves

$R_{n,s} =$



$\partial_k =$ univalent vertizes $1, \dots, s$

Define

$$A_{n,s} = \pi_0 HE(R_{n,s}, \partial_s)$$

Outer space $\mathcal{O}_n$ (with action of $Out(F_n)$)

Do all $\mathcal{O}_{n,s}$ (with action of $A_{n,s}$) at once:

② Sphere systems in $M_{n,s}$

An embedded $S^2 \hookrightarrow M_{n,s}$ is non-trivial if it does not bound a B^3 and is not isotopic to a boundary component

Isotopy classes of embedded 2-spheres are compatible if they have disjoint representatives

The sphere complex $\mathcal{S}_{n,s}$ is the simplicial complex with

vertices = isotopy classes $[s]$ of nontrivial 2-spheres

k-simplices σ_k = compatible collections of $(k+1)$ distinct isotopy classes.

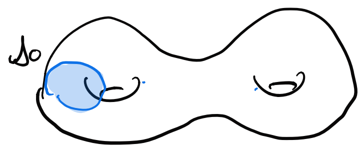
$= ([s_0], \dots, [s_k])$ = sphere system

A point in σ_k can be specified by giving its barycentric coordinates, ie

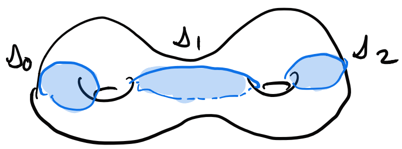
$p = w_0[s_0] + \dots + w_k[s_k]$, $\quad w_i \in \mathbb{R}_{\geq 0}$, $\sum w_i = 1$

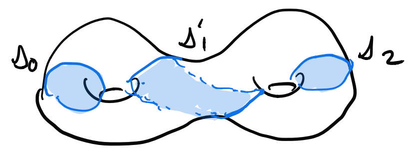
$p \in \text{interior}(\sigma_k)$ iff all w_i are > 0.

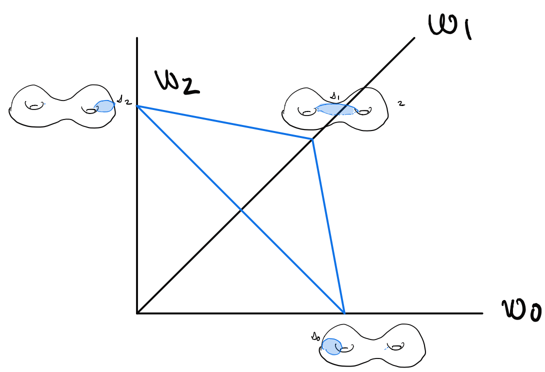
eg $M_{2,0}$ vertex

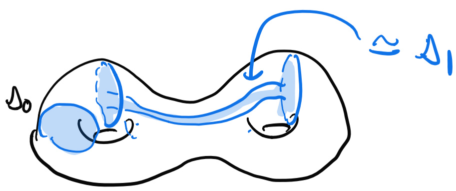


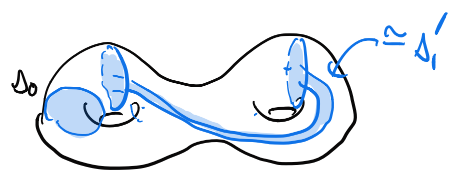
2-simplices





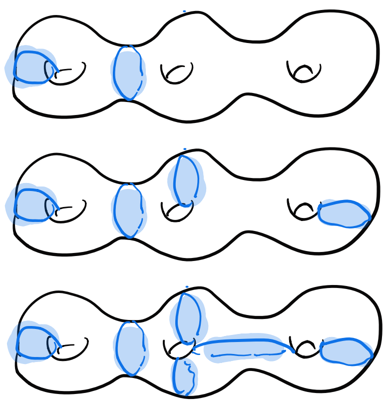






Exercise Find more 1-simplices with s_0 as a vertex

A set of compatible isotopy classes of 2-spheres is complete if there are representatives $s_0, \dots, s_r$ that cut $M_{n,s}$ into punctured balls



incomplete

complete

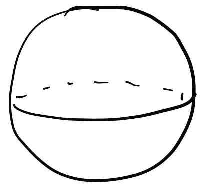
maximal

Exercise: How many spheres are there in a maximal system?

Definition $\mathcal{O}_{n,s}$ is the union of open simplices $\sigma_k = ([s_0], \dots, [s_k])$ in $\mathcal{S}_{n,s}$ such that $\{[s_0], \dots, [s_k]\}$ is a complete sphere system

Any diffeomorphism of $M_{n,s}$ takes sphere systems to sphere systems, preserving completeness, ie have an action of $\pi_0(\mathrm{Diff}\, M_{n,s}, \partial)$ on $\mathcal{S}_{n,s}, \mathcal{O}_{n,s}$

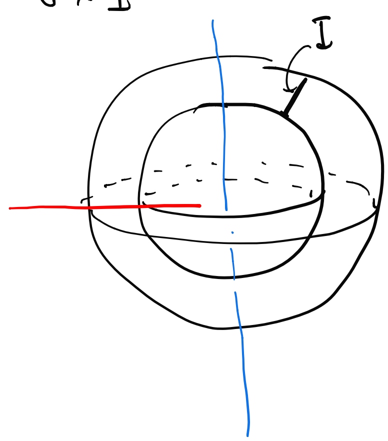
Fact Dehn twists act trivially on 2-spheres idea: σ

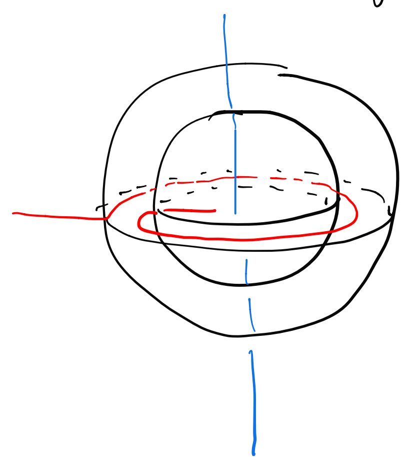


D_σ = Dehn twist in σ

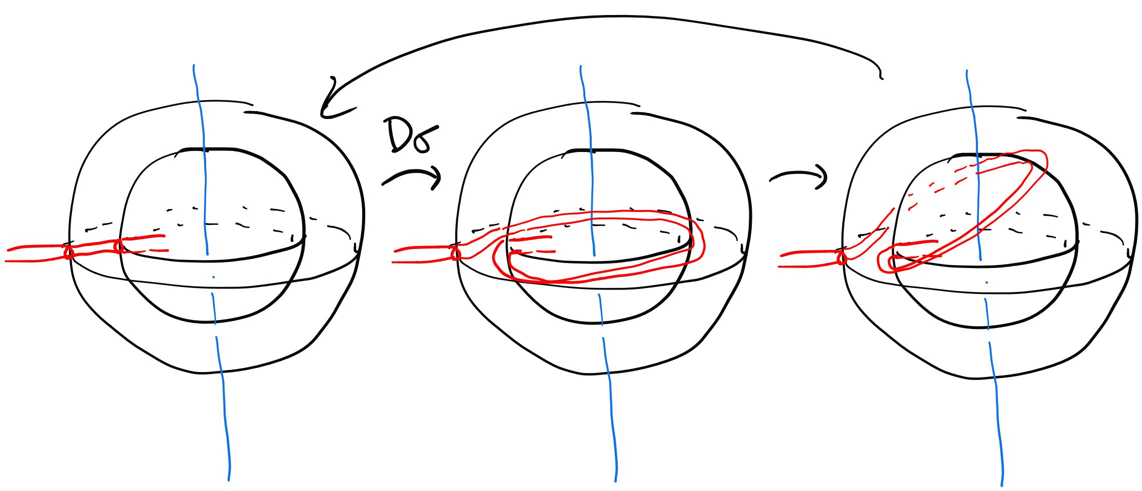
= suspension of Dehn-twist in equator

$\sigma \times I$





s = another 2-sphere. Isotop so s is transverse to $\sigma \times I$. Tube of intersection gets twisted, but can isotope it back to s



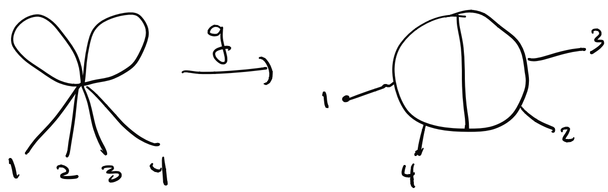
So get an action of $\pi_0 HE(M_{n,s},\partial)/DT = A_{n,s}$

on $\mathcal{S}_{n,s}$, $\mathcal{O}_{n,s}$

① Another description of $\mathcal{O}_{n,s}$

Space of marked metric graphs (g, G)

where



- G is a graph with s leaves labeled $1,\dots,s$, ∂G = leaves
- All non-leaf vertices are at least trivalent.
- Internal edges of G have weights w_i with $\sum w_i = 1$
- $g: (R_{n,s}, \partial_{n,s}) \to (G, \partial G)$ is a label-preserving homotopy equivalence
- $(g, G) \sim (g', G')$ if there is a label-preserving isometry $(G,\partial G) \xrightarrow{f} (G', \partial G')$

s.t. $f \circ g \simeq g'$ by a label-preserving homotopy $H_t: (R_{n,s}, \partial_{n,s}) \to (G', \partial G')$:

$$\begin{tikzcd} (G,\partial G) \arrow[rr, "f"] & & (G',\partial G') \\ & R_{n,s} \arrow[ul, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$$

$f \in A_{n,s}$ acts by: realise f as a map $R_{n,s} \to R_{n,s}$

Then $(g, G) \cdot f = (g \circ f, G)$.

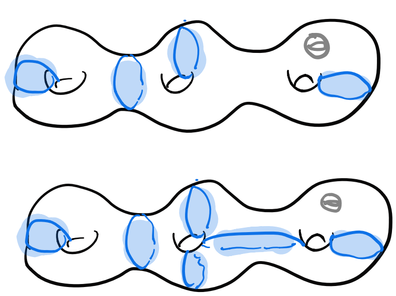
Exercise: Show this gives a well-defined action of $A_{n,s}$.

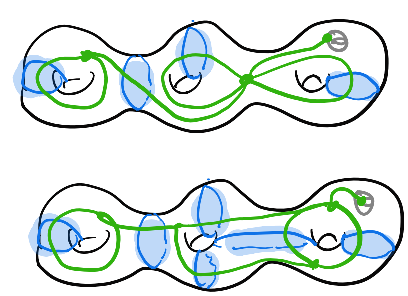
This was the original description of $\mathcal{O}_n$ = Outer space of rank n = $\mathcal{O}_{n,0}$

Relation between the descriptions (Hatcher):

complete sphere systems $\longrightarrow$ marked graphs

one vertex in each component
one edge for each sphere

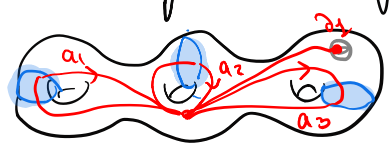


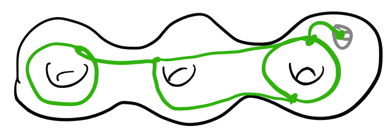


marking:

Fix a minimal complete sphere system

Identify its dual graph with $R_{n,s}$





Each loop in the green graph starting on a ∂ component is homotopic to a path in $R_{n,s}$ (rel ∂). This gives a homotopy equivalence $h : (G, \partial G) \longrightarrow (R_{n,s}, \partial_s)$. We take its homotopy inverse for g.

" The red graph identifies $\pi_1(M_{n,s})$ with F_n, so loops in the green graph with elements of F_n."

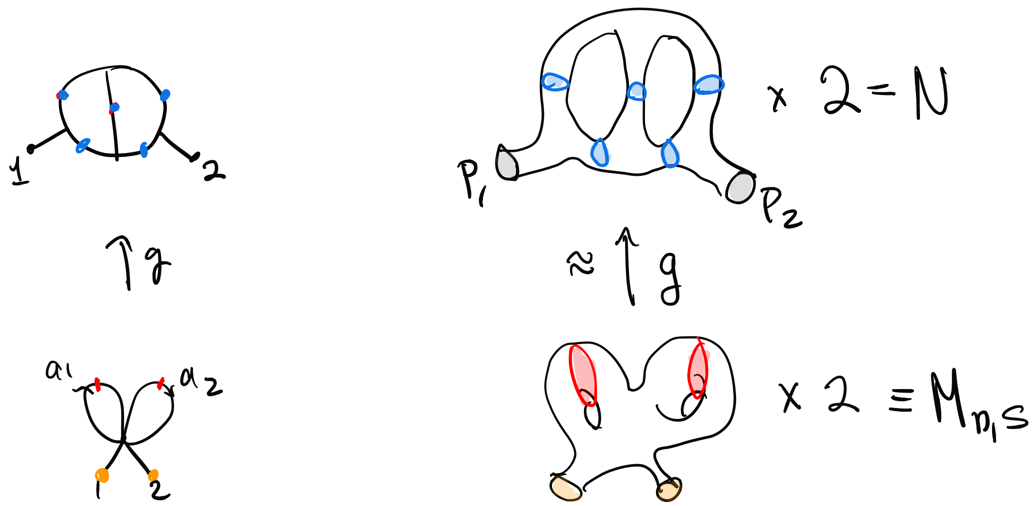
Marked graphs $\longrightarrow$ complete sphere systems

- Choose a point p_e in each internal edge e
- Fatten G to $H_n = \#_n S^1 \times D^2$

$p_e \longmapsto D_e$ = disk

leaf $l_i \longmapsto$ disk P_i on ∂H

- Double H, but not the P_i



The resulting manifolds are diffeomorphic
Choose a diffeomorphism realizing g, fixing ∂
Pull back the spheres to get a sphere system in $M_{n,s}$.

Theorem. $\mathcal{O}_{n,s}$ is contractible. $A_{n,s}$ acts properly (ie with finite stabilisers)

Proof (A. Hatcher for $s > 0$, using sphere systems)