Lecture 3

Last time we defined the space $\mathcal{O}_{n,s}$

① As a subspace of the sphere complex
$\mathcal{S}(M_{n,s})$ $\quad\left(M_{n,s} = \#_n S^1\times S^2 \setminus \coprod_s B^3\right)$

② As a space of marked metric graphs (g,G), where
G has k leaves $(=\partial G)$ and $g:(R_{n,s},\partial_s) \xrightarrow{\simeq} (G,\partial G)$

The group $A_{n,s} = \pi_0 \operatorname{Diff}(M_{n,s},\partial M_{n,s})/\text{Dehn Twists}$
$= \pi_0 HE(R_{n,s},\partial_s)$
acts on $\mathcal{O}_{n,s}$

$s=0$ $\quad A_{n,0} = \operatorname{Out}(F_n)$ $\quad \mathcal{O}_{n,0} = $ Outer space
$s=1$ $\quad A_{n,1} = \operatorname{Aut}(F_n)$ $\quad \mathcal{O}_{n,1} = $ Autre espace.

Theorem: $\mathcal{O}_{n,s}$ is contractible, the action of $A_{n,s}$ is proper

Will prove this for $\mathcal{S}(M_{n,s})$, using an argument of Hatcher

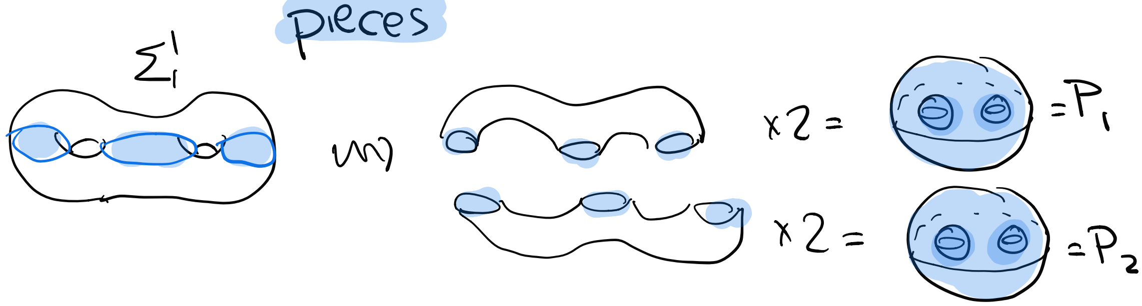
Idea: Fix a maximal sphere system $\Sigma = \{\sigma_1,\dots,\sigma_m\}$
This gives a simplex (also called Σ) of $\mathcal{S}(M_{n,s})$

- Find a canonical path in $\mathcal{S}_{n,s}$ from an arbitrary point $w_1 \Delta_1 + \dots + w_k \Delta_k$ to a point in Σ
- Contract $\mathcal{S}_{n,s}$ to Σ along these paths.

More details: What are these paths?

They are called surgery paths - and the concept is useful in related contexts (eg arc complexes on surfaces)

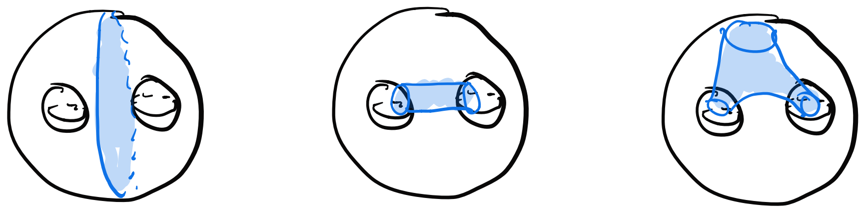
Σ a maximal system $\Rightarrow$ cuts $M_{n,s}$ into 3-punctured spheres $S^3 - \coprod_3 B^3$, called pieces



S any other sphere system. Hatcher's normal form theorem says:

S is isotopic to a system that is
① Transverse to Σ.

② Intersects each piece in disks cylinders and pairs of pants



Furthermore, if the number of intersection circles is minimal, then the intersection pattern on each sphere of Σ is unique

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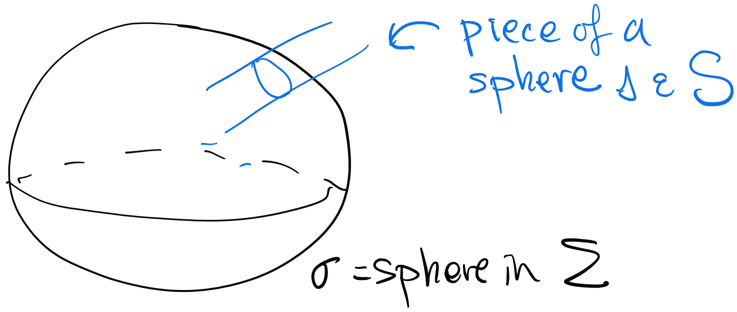
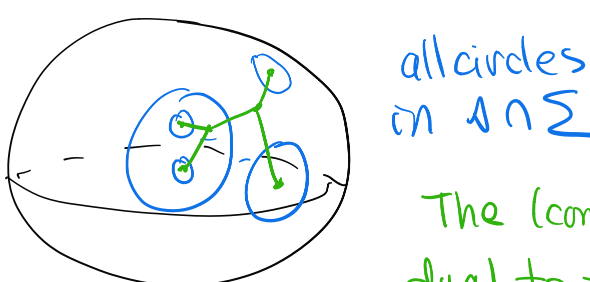
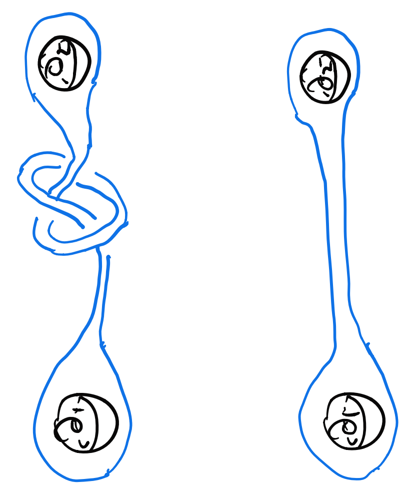




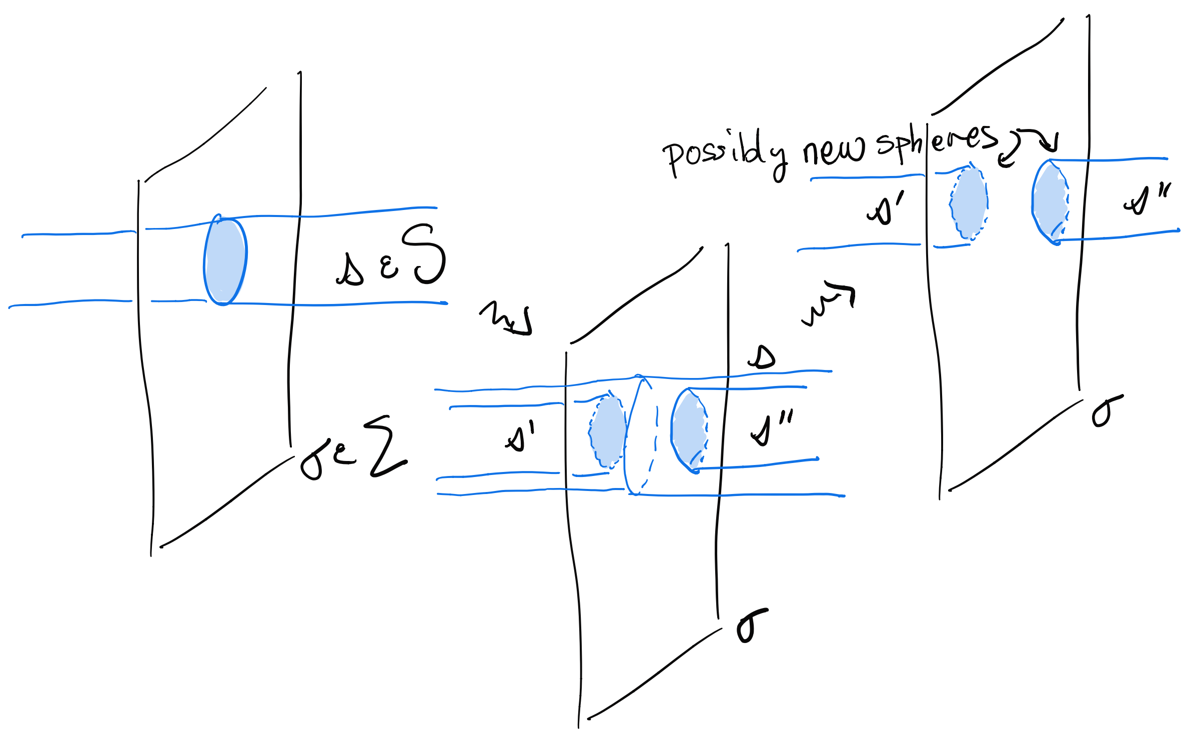
The (combinatorial) tree dual to these circles is unique

The proof of Hatcher's theorem relies heavily on Laudenbach's theorem that homotopic spheres in $M_{n,k}$ are isotopic, which uses the "light bulb trick":

These two spheres are isotopic:

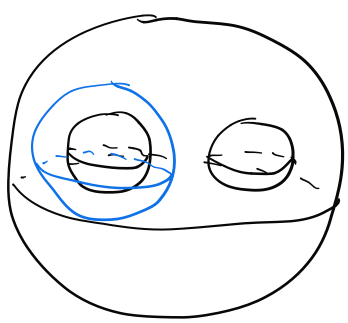


Surgery:



Surgery path:

- surger all innermost circles of intersection
- transfer weight uniformly onto the new non-trivial spheres until the surgered spheres have weight 0
- continue until all intersections are gone. The remaining (weighted) spheres are parallel to the spheres in Σ, so give a point in the simplex Σ



every embedded non-trivial sphere is isotopic to one of the ∂ components

It remains to prove surgery paths vary continuously. – Hatcher does this.
An easier proof is in Hatcher–V., using the spine of Outer space (coming up).

This shows $\mathcal{S}(M_{n,s})$ is contractible. What about $\mathcal{O}_{n,s}$?

Lemma: If S is a complete system, then the result of surgery is still complete

Proof Exercise (use Van Kampen)

Corollary: The deformation retraction
$\mathcal{S}_{n,s} \to \Sigma'$ along surgery paths
restricts to $\Theta_{n,s}$

There are other proofs that $\Theta_{n,0}$ is contractible
- using marked graphs
- using actions on trees

One more description of $\Theta_n = \Theta_{n,0}$

G a finite connected graph
$\pi_1 G$ acts on $\tilde{G}$ by deck transformations
This is a free action: every elt moves every point.

A marking $g: R_n \longrightarrow G$
induces $g_* \pi_1(R_n, b) = F_n \longrightarrow \pi_1(G, g(b))$
so get an action of F_n on the tree $\tilde{G}$

If all vertices of G have valence ≥ 3 this action is <u>minimal</u>: there is no invariant subtree.

If G has a metric, get an invariant metric on $\tilde{G}$

If $(g, G) \sim (g', G')$, the isometry

$G \longrightarrow G'$ lifts to an isometry $\tilde{G} \to \tilde{G}'$

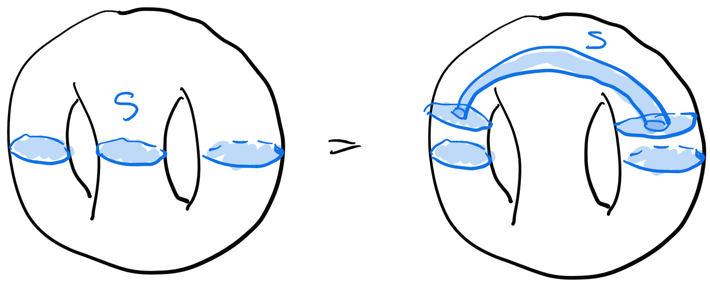
commuting with the action of F_n

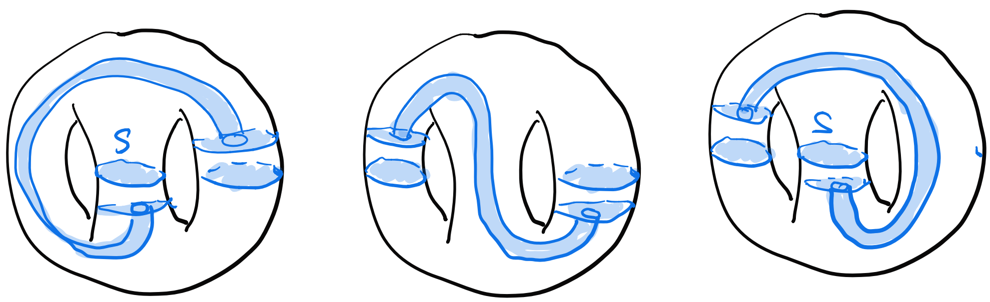
Def: $\mathcal{O}_n$ is the space of free minimal isometric actions of F_n on metric simplicial trees.

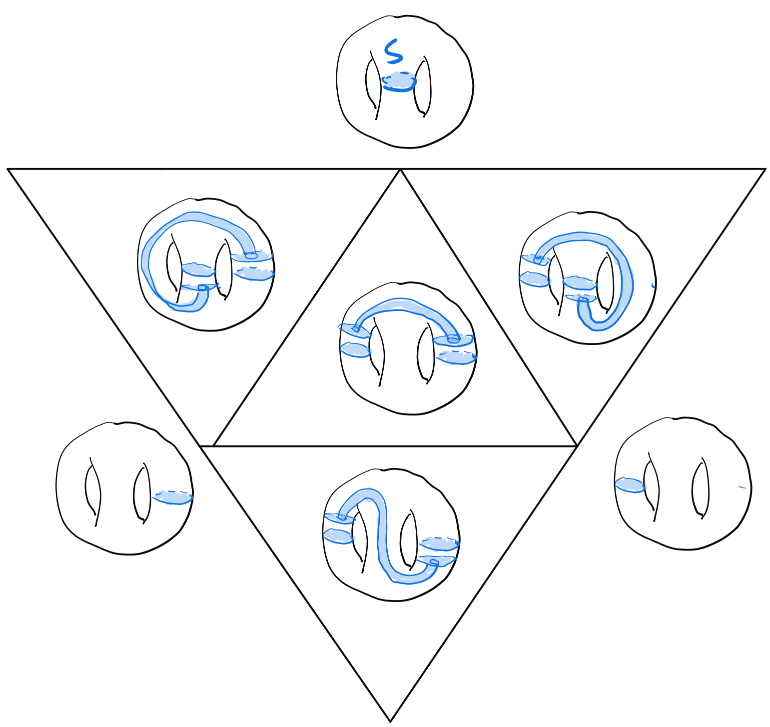
Nice concise definition, easy to generalize to groups other than F_n (eg free products - see Guirardel-Levitt - deformation spaces of actions on trees).

but tricky to define $\mathcal{O}_{n,s}$ for $s>0$, hard to see structure of space. - eg decomposition into (open) simplices

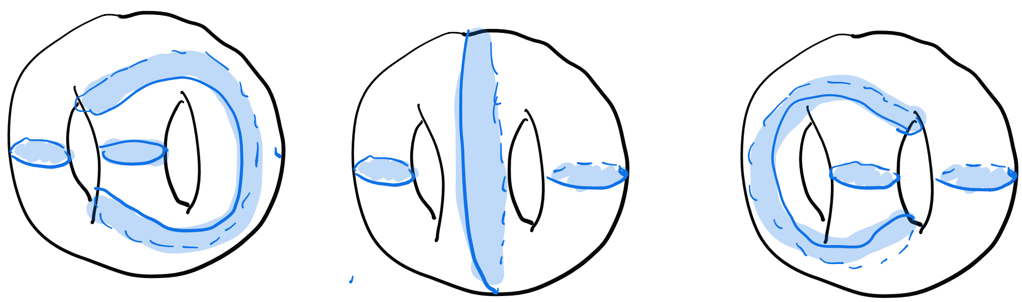
Pictures $n=2$, $s=0$



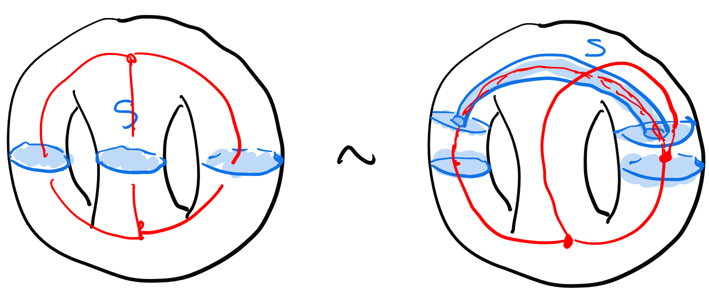


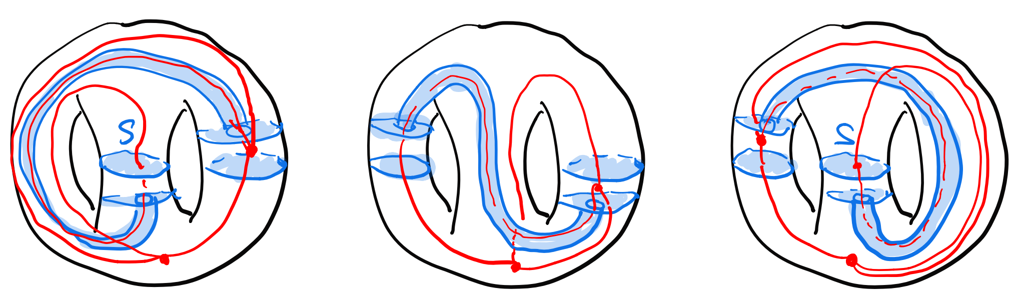


If we allow separating spheres we also get triangles



Marked graphs dual to sphere systems

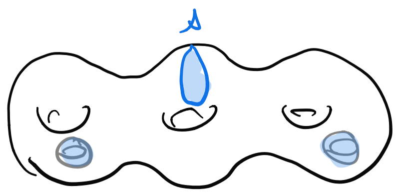


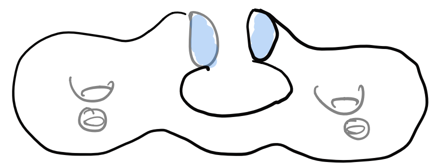


The action of $A_{n,s}$ on $\mathcal{S}_{n,s}$ is cocompact but not proper:

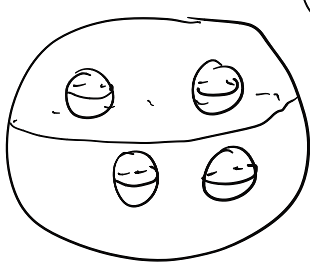
cocompact: There are only finitely many isotopy classes of sphere systems up to diffeomorphism

$$\operatorname{stab}(s) = \pi_0 \operatorname{Diff}(M_{n,s}, s \cup \partial) = \pi_0 \operatorname{Diff}(M_{n-1,s+2}, \partial)$$





But, if $S = \{s_1, \dots, s_k\}$ is complete, each component P of $M - S$ is a punctured ball,



so any diffeo of P fixing ∂P is isotopic to the identity.

A diffeo of $M_{n,s}$ realizing an element of $A_{n,s}$ has to fix $\partial M_{n,s}$ but can permute the s_i

Conclusion: $\operatorname{stab}(S)$ is finite if S is complete

ie $A_{n,s}$ acts properly on $\mathcal{O}_{n,s}$

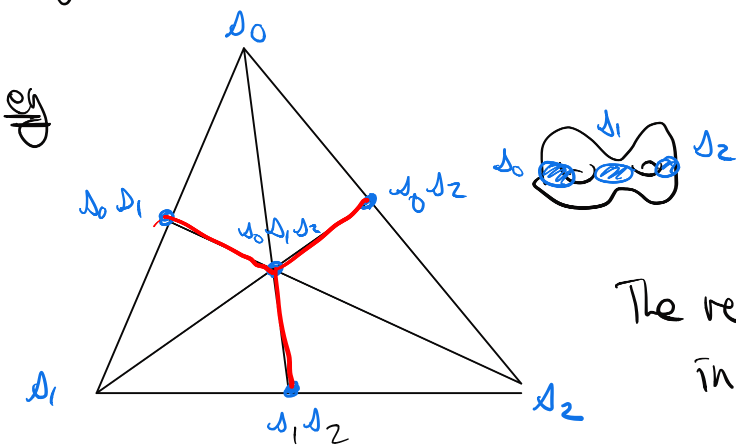
By restricting to $\mathcal{O}_{n,s}$ we gained properness, but lost cocompactness.

To get a cocompact action, we restrict even further, to the <u>spine</u> of $\mathcal{O}_{n,1}$:

Let $\mathcal{S}'(M_{n,s}) :=$ Barycentric subdivision of $\mathcal{S}(M_{n,s})$.
k-simplex = chain $S_0 \subset S_1 \subset \cdots \subset S_k$ of spheresystems

Definition. The spine $K_{n,s}$ is the subcomplex of $\mathcal{S}'(M_{n,s})$ spanned by complete systems.

eg



The red simplices are in $K_{3,0}$

Prop: $K_{n,s}$ is a deformation retract of $\mathcal{O}_{n,s}$

Pf: Each point $x \in \mathcal{O}_{n,s}$ is

$$x \doteq w_0 \Delta_0 + \cdots + w_k \Delta_k$$

where $S = \{\Delta_0, \ldots, \Delta_k\}$ is a complete sphere system $w_i > 0$ and $\sum w_i = 1$.

x is in some simplex of $\mathcal{S}'$, say

$$S_0 \subset \cdots \subset S_r = S$$

$$x = b_0 S_0 + \cdots + b_r S_r \quad , b_i > 0, \sum b_i = 1$$

Let S_i be the first of $S_0, \ldots S_r$ which is complete

Move x to the point

$$\lambda(b_i S_i + \cdots + b_r S_r)$$

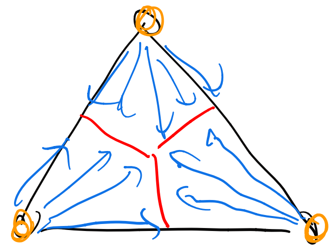
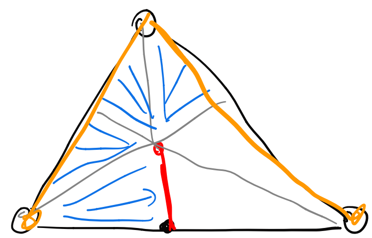
where $\lambda(b_i + \cdots + b_r) = 1$

by linearly shrinking $b_1, \ldots, b_{i-1}$

and expanding $b_i, \ldots, b_r$.

Since S_i is complete, then any $S_j \supset S_i$ is also complete, so

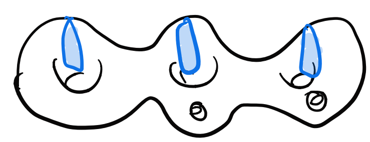
$$\lambda(b_i S_i + \cdots + b_r S_r) \in K_{n,s},$$

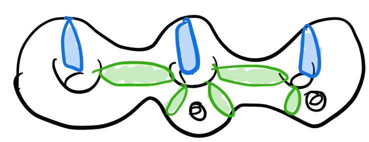
Corollary: $K_{n,s}$ is contractible

Proposition: $\dim K_{n,s} = 2n-3+s$

pf It takes at least n spheres to cut $M_{n,s}$ into punctured balls



And you need $3n-3+s$ to cut it into "pants"



So a simplex in $K_{n,k}$ has length at most $(3n-3+s)-n = 2n-3+s$.

The action of $A_{n,s}$ on $O_{n,s}$ restricts to $K_{n,s}$ so is still proper and is now cocompact (there is one orbit of vertices for each diffeomorphism type of sphere systems, one orbit of r-simplices for each diffeomorphism type of chains $S_0 \subset \dots \subset S_r$)

Cube complex structure of $K_{n,s}$

Let $S = \{s_0, \dots, s_i\} \subset S' = \{s_0, \dots, s_{i+k}\}$ be complete sphere systems

Let $S_j = S \cup \{s_{i+1}, \dots, s_{i+j}\}$ $\quad 1 \le j \le k$

Then $S = S_0 \subset S_1 \subset \dots \subset S_k = S'$ is a k-simplex of $K_{n,s}$

Any ordering of $\{s_{i+1}, \dots, s_{i+k}\}$ gives a different k-simplex.
These all fit together to form a k-dimensional cube

eg $S = \{s_0\, s_1\}$ $\quad S' = \{s_0\, s_1\, s_2\, s_3\}$