Lecture 4

We have $K_{n,s}$ = spine of $\mathcal{O}_{n,s}$, w/ action of $A_{n,s}$

- contractible cube complex
- cocompact
- stabilizers are finite (proper action)
- $\dim = 2n-3+s$

vertices of $K_{n,s}$ = complete sphere systems in $M_{n,s}$

Thm (Culler, Khramtsov) Every finite subgroup of $A_{n,s}$ stabilizes some vertex v of $K_{n,s}$

Cor $A_{n,s}$ has only finitely many conjugacy classes of finite subgroups

pf: Since $K_{n,s}/A_{n,s}$ is compact, there is a finite subcomplex $D \subset K_{n,s}$ whose images cover $K_{n,s}$ (a fundamental domain).

So $v = v_0 \cdot h$ for some $h \in A_{n,s}$, $v_0 \in D$

and $\operatorname{stab} v = h^{-1}(\operatorname{stab} v_0)\, h$ ✓

Next I'll give some applications to homology:

Thm (Hurewicz) If G acts freely on a contractible CW-complex X

Then $H^*(X/G)$ ($H_*(X/G)$) is an invariant of G

Write $H^*(G)$ ($H_*(G)$)

X/G is called a $K(\pi,1)$ for G (or a $K(G,1)$)

Every group has a $K(\pi,1)$.

$A_{n,s}$ does not act freely on $K_{n,s}$, but any torsion-free subgroup does. Claim: $A_{n,s}$ has tffi subgroups:

Thm (Baumslag-Taylor) The kernel of the map $\mathrm{Out}(F_n) \longrightarrow GL_n\mathbb{Z}$ is torsion-free.

Thm $GL_n(\mathbb{Z})$ has torsion-free subgroups of finite index (tffi subgroups)

Pf Let $K_n = \mathrm{Ker}\ GL_n(\mathbb{Z}) \longrightarrow GL_n(\mathbb{Z}/p\mathbb{Z})$ for some $p \geq 3$ is prime.

If K_n is not torsion-free, then there is some $A \in K_n$ of prime order l

ie $A \equiv I \bmod p$, $A^l = I$

so $A = p^\alpha B + I$, $(I + p^\alpha B)^l = I$

$B \not\equiv 0 \bmod p$

$$(I+p^{\alpha}B)^{\ell} = I + \ell p^{\alpha}B + \binom{\ell}{2} p^{2\alpha}B^{2} + \cdots + \binom{\ell}{\ell} p^{\ell\alpha}B^{\ell}$$

so $$\ell \cdot p^{\alpha} B = -\sum_{i=2}^{\ell} \binom{\ell}{i} p^{i\alpha} B^{i} \qquad (*)$$

If $\ell \neq p$, then α is the exact power of p dividing $\ell p^{\alpha} B$. But RHS is divisible by $p^{2\alpha}$ $*$.

If $\ell = p$, $p^{\alpha+1}$ is the exact power of p dividing $\ell p^{\alpha} B$, but $p^{2\alpha+1}$ divides the RHS:

p divides $\binom{p}{i}$ if $i \geq 2, i < p$

$p \geq 3 \Rightarrow p^{2\alpha+1}$ divides $p^{p\alpha}$

Cor: $A_{n,s}$ has tffi subgps

If ker $\operatorname{Out} F_n \longrightarrow GL(n,\mathbb{Z})$ torsion free

ker $A_{n,s} \longrightarrow \operatorname{Out}(F_n)$ torsion-free

So take Γ = any tffi subgroup of $GL(n,\mathbb{Z})$. The inverse image in $A_{n,s}$ is tffi in $A_{n,s}$.

Since $A_{n,s}$ acts properly on $X = \mathcal{O}_{n,s}, K_{n,s}$ any tffi $\Gamma < A_{n,s}$ acts freely, so by Hurewicz, $H^*(X/\Gamma) = H^*(\Gamma)$

Cor: • $H^i(\Gamma) = 0$ for $i > 2n-3+s$
• $H^i(\Gamma)$ is finitely generated for all i

Def: G virtually has a given property if it has a finite-index subgroup with that property.

$A_{n,s}$ has virtual cohomological dimension (VCD) $\leq 2n-3+s$

Prop if $\Gamma \geq \Gamma'$ then $\mathrm{cD}(\Gamma) \geq \mathrm{cD}(\Gamma')$
(pf: Any chain complex computing $H^*(\Gamma)$ is also a chain complex for Γ')
or: X/Γ' is a covering space of X/Γ, so has same dimension.

$A_{n,s}$ contains a free abelian subgroup of rank $2n-3+s$

eg: The subgroup of $A_{n,1}$ generated by $\rho_{i1}: x_i \mapsto x_i x_1$ and $\lambda_{i1}: x_i \mapsto x_1 x_i$ is free abelian of rank $2n-2$.

Exercise: Find a free abelian subgroup of rank $2n-3+s$ for any s.

Since the r-torus T^r is a $K(\mathbb{Z}^r, 1)$, $cd(T^r) = r$.

Cor $\mathrm{vcd}(A_{n,s}) = 2n-3+s$

If you use (co)homology with trivial rational coefficients the cohomology of any finite group is trivial. The proof uses the transfer map:

G acts freely on contractible X,

$H < G$ finite index $\hookrightarrow$ $X/H \xrightarrow{p} X/G$ finite cover

p induces $H^*G \longrightarrow H^*H$, induced by $C_*H \hookrightarrow C_*G$

But there's also a map $H^*H \longrightarrow H^*G$ $\quad C_*G \longrightarrow C_*H$

$$\sigma \longmapsto \sum_{p\tilde{\sigma}=\sigma} \tilde{\sigma}$$

composition is $\sigma \longmapsto [G:H]\sigma$, which is an isomorphism if $H^*(G)$, $H^*(H)$ are rational vector spaces

So if G is finite, can take $H = \langle 1 \rangle$, get

$$\begin{tikzcd} H^*(G;\mathbb{Q}) \arrow[r] \arrow[rr, bend left, "\times |G| \cong"] & H^*(\langle 1 \rangle, \mathbb{Q}) \arrow[r] & H^*(G;\mathbb{Q}) \end{tikzcd}$$

$$H^*(\langle 1 \rangle, \mathbb{Q}) = 0 \text{ if } * > 0$$

Heuristics: If G acts on contractible X 'properly' (with finite stabilizers) then rational cohomology thinks it's a free action, so $H^*(X/G;\mathbb{Q}) = H^*(G;\mathbb{Q})$

Formally: There is a spectral sequence

$$E_1^{pq} = \bigoplus_{[\sigma_p]} H^q(\operatorname{stab} \sigma_p)$$

$$\Rightarrow H^{p+q}(G)$$

For $\mathbb{Q}$-coeffs this collapses to the line $q=0$, where it is just the cellular cochains of X/G.

So we can compute the rational (co)homology of $\operatorname{Out}(F_n)$ by computing the quotient $K_n/\operatorname{Out}(F_n)$

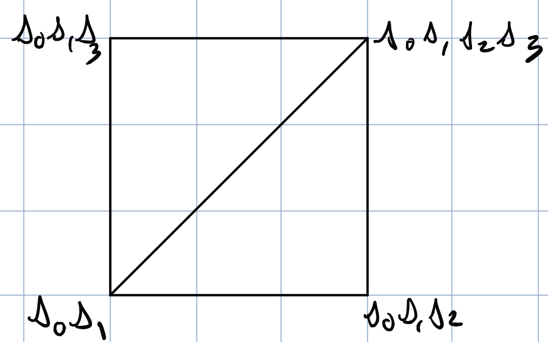
This is feasible for (very) small n.
Easier with the graph picture

Cube of K_n : $S \subset S'$ complete sphere systems

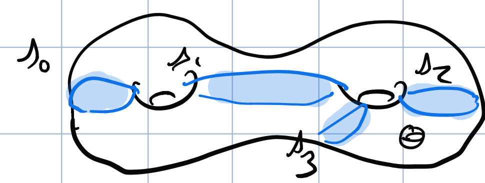
$c(S \subset S')$ = all chains you can make from S to S'

$$S' = S \cup s_1 \cup \dots \cup s_k$$

$$S \subset S, s_1 \subset S, s_1 s_2 \subset \dots \subset S s_1 \dots s_k = S'$$



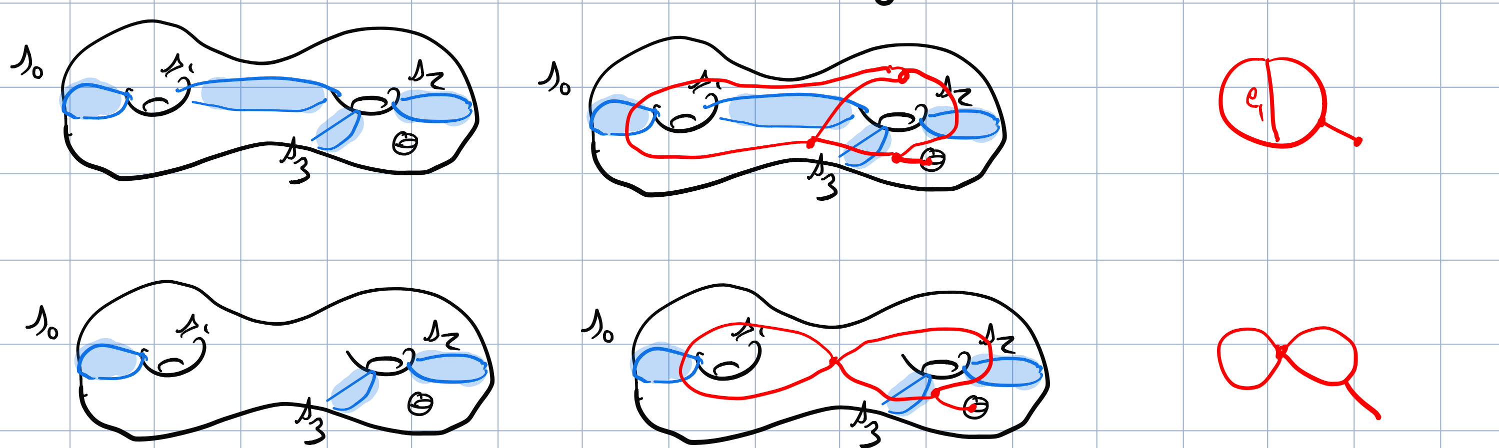
$\longleftrightarrow$



In terms of graphs

$M_{n,k} \ni S = s_1 \dots s_r \longleftrightarrow$ graph G with r internal edges, k leaves, $\pi_1(G) \cong F_n$

$S \setminus s_i \longleftrightarrow$ graph obtained by collapsing e_i



You can only remove s_i if all components of $M \setminus S \setminus s_i$ are still 1-connected

This corresponds to:

You can only collapse e_i if it is not a loop.

You can only collapse $e_1, \dots, e_i$ if they form a forest in the graph G

ie in the graph picture, a cube in $K_{n,s}$ is a marked graph together with a forest $\Phi \subset G$.

To compute the quotient, need to understand the stabilizer of a cube

$$K_{n,s} \longrightarrow K_{n,s}/A_{n,s}$$

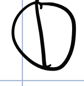
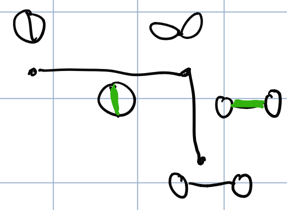
$$\text{marked cube} \longmapsto \text{cube}/\text{stab(cube)}$$

In sphere picture, we have stab(S) = diffeos mapping $S \circlearrowleft S$ (may permute the spheres & comp. components)

In graph picture this is combinatorial autos of G (may permute the edges and vertices of G)

stab$(S \subset S')$ permutes the spheres in S', but must send $S \circlearrowleft$

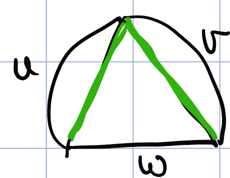
stab$(g, G, \Phi) \cong \mathrm{Aut}(G, \Phi)$ = autos sending $\Phi \circlearrowleft$

eg $\operatorname{stab}(g, $  $) \cong \mathbb{Z}/2 \times \Sigma_3$ $\quad K_{2,0}/A_{2,0} =$ 

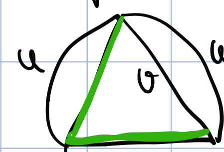
$\operatorname{stab}(g,$  $) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

Exercise: 1 Compute the stabilizers of all cubes in $K_{2,1}$ and the cell structure of $K_{2,1}/\mathrm{Aut}(F_2)$

2 Compute the stabilizers in $\mathrm{Out}(F_3)$ of the cubes



and



where u, v, w is a basis for F_3. Draw the images of these cubes in $K_{3,0}/\mathrm{Out}(F_3)$

3 Compute $H_*(\mathrm{Out}(F_3); \mathbb{Q})$

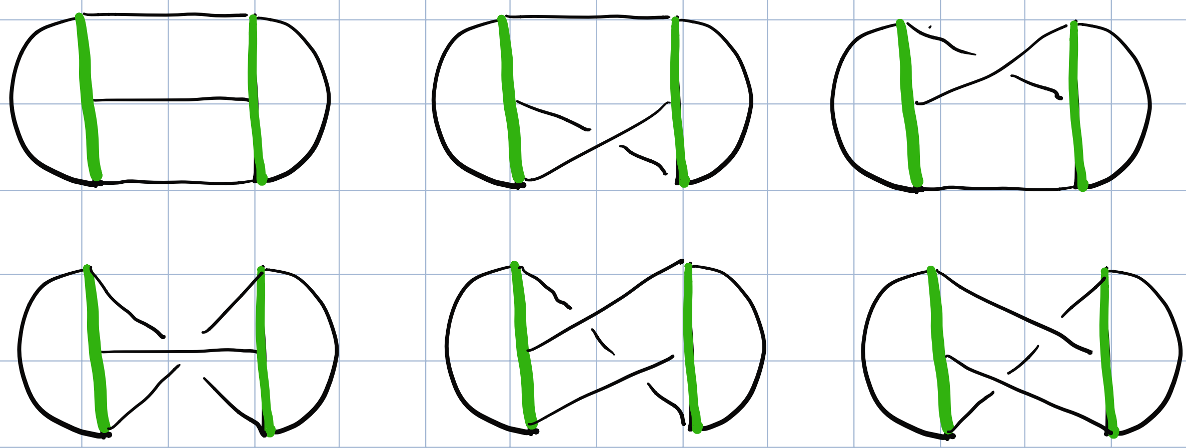
It's more tedious to compute $H_*(\mathrm{Out}(F_4); \mathbb{Q})$ and that's about the limit of hand calculation. The answer turns out to be very interesting:

$H_0 \cong \mathbb{Q}$

$H_4 \cong \mathbb{Q}$

$H_i = 0 \quad i \neq 0, 4$

The non-trivial class in dimension 4 can be represented by a cycle consisting of the images of six cubes:



Homology is hard to compute. Easier is the Euler characteristic $\chi = \sum (-1)^i \operatorname{rank} H_i$ because you can compute it on the chain level

$$\chi = \sum (-1)^i \operatorname{rank} C_i = \sum_\sigma (-1)^{\dim \sigma}$$

The Euler characteristic of a group is the Euler characteristic of a $K(\pi,1)$ for G

(But we don't have a $K(\pi,1)$ for $A_{n,s}$)

χ has nice features: eg if $Y \to X$ is a covering map of finite CW-complexes, of degree d, then $\chi(Y) = d \cdot \chi(X)$

So, if G acts freely and cellularly on X with finite quotient, and $H < G$ has finite index d, then $X/_H \longrightarrow X/_G$ is a covering map and

$$\chi(H) = d \cdot \chi(G)$$

$$\chi(G) = \frac{\chi(H)}{[G:H]}$$, independent of the choice of H.

If G only acts properly on X w/ finite quotient but has a tffi subgroup H,

define $\overline{\chi}(G) = \frac{\chi(H)}{[G:H]}$

Since the intersection of any two tffi subgroups is tffi, this is independent of the choice of H.

It's called the rational Euler characteristic of G

(unfortunate name, because replacing $\mathbb{Z}$ by $\mathbb{Q}$ and rank by dimension gives the usual Euler characteristic)