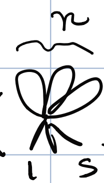
Lecture 5

Last time, used $K_{n,s}$ = spine of $\mathcal{O}_{n,s}$ to find information about $A_{n,s} = \pi_0 HE($  $,\partial)$

$$= \pi_0 \operatorname{Diff}(M_{n,s},\partial)/DT$$

Continuing:

We showed $A_{n,s}$ has tffi subgroups Γ

so $\quad H^*(A_{n,s}) = H^*(K_{n,s}/\Gamma)$

In particular, $\chi(A_{n,s}) = \chi(K_{n,s}/\Gamma)$

For any G with tffi $\Gamma < G$

Defined $\overline{\chi}(G) = \dfrac{\chi(\Gamma)}{[G:\Gamma]}$ the <u>rational</u> Euler characteristic

- Independent of choice of Γ

<u>pf</u> Γ, Γ' tffi $\Rightarrow \Gamma \cap \Gamma'$ tffi

$\Rightarrow \quad X/\Gamma\cap\Gamma'$ covers X/Γ, $\deg = [\Gamma : \Gamma\cap\Gamma']$

$$\Rightarrow \quad \chi(\Gamma\cap\Gamma') = \chi(\Gamma)\cdot[\Gamma:\Gamma\cap\Gamma']$$
$$= \chi(\Gamma')\cdot[\Gamma:\Gamma\cap\Gamma']$$

$$\text{so } \overline{\chi}(G) = \frac{\chi(\Gamma\cap\Gamma')}{[G:\Gamma\cap\Gamma]} = \frac{\chi(\Gamma)\cancel{[\Gamma:\Gamma\cap\Gamma']}}{[G:\Gamma]\cancel{[\Gamma:\Gamma\cap\Gamma']}} = \frac{\chi(\Gamma')}{[G:\Gamma']}$$

If $\Gamma \triangleleft G$, G acts on X freely, then G/Γ acts on freely on X/Γ

If the action of G is not free G/Γ still acts on X/Γ. Suppose X is a simplicial complex, the action of G is simplicial, and the stabilizer of a simplex σ fixes σ.

$$X \longrightarrow X/\Gamma \longrightarrow X/G$$

Then

$$\#(\text{orbit } \sigma) = |G/\Gamma| / \text{stab}\,\sigma = [G:\Gamma] / \text{stab}_{G/\Gamma}\,\sigma = [G:\Gamma] / |\text{stab}_G\,\sigma|$$

In our case:

To calculate $\overline{\chi}(A_{n,s})$ from $A_{n,s} \circlearrowright K_{n,s}$ use simplicial decomposition $\sigma = S_0 \subset \cdots \subset S_k$
$= (g, G, \Phi_1 \subset \cdots \subset \Phi_k)$

Then $\text{stab}(\sigma) = \text{fix}(\sigma)$, ie simplices in $K_{n,s}/A_{n,s}$ lift to simplices in $K_{n,s}/\Gamma$ (and $K_{n,s}$)

so

$$K_{n,s} \longrightarrow K_{n,s}/\Gamma \longrightarrow K_{n,s}/A_{n,s} = Q_{n,s}$$

$$\sigma \cdot A_{n,s} \longrightarrow \tilde{\sigma}_1 \cdots \tilde{\sigma}_\ell \longmapsto \bar{\sigma}$$

$$\uparrow$$

$$\ell = [A_{n,s} : \Gamma] / |\mathrm{stab}(\sigma)|$$

so $$\chi(\Gamma) = \sum_{\bar{\sigma} \in Q_{n,s}} \frac{(-1)^{\dim \sigma}}{|\mathrm{stab}\,\sigma|} [A_{n,s} : \Gamma]$$

$$\bar{\chi}(A_{n,s}) = \sum_{\bar{\sigma} \in Q_{n,s}} \frac{(-1)^{\dim \sigma}}{|\mathrm{stab}\,\sigma|}$$

Recall that a simplex σ of $K_{n,s}$ is

- a chain $S_0 \subset \cdots \subset S_k$ of complete sphere systems

or
- a marked graph in a chain of forests

$$(g, G, \phi_1 \subset \cdots \subset \phi_k) = (g, G, \Phi)$$

$\mathrm{stab}(\sigma)$ fixes σ: $\mathrm{stab}\,\sigma$ sends S_i to some S_j, but then $i=j$ since S_i and S_j have different numbers of spheres.

$\sigma = (g, G, \Phi)$

$\Rightarrow \mathrm{stab}(\sigma) \cong \mathrm{Aut}(G, \Phi)$

so $$\overline{\chi} = \sum_{[(G, \phi_1 \subset \cdots \subset \phi_k)]} \frac{(-1)^k}{|\mathrm{Aut}(G, \phi_1 \subset \cdots \subset \phi_k)|}$$
(isomorphism classes)

Want to simplify this expression in order to calculate it.

$$= \sum_{[G]}' \sum_{[\phi_1 \subset \cdots \subset \phi_k \subset G]} \frac{(-1)^k}{|\mathrm{Aut}(G, \Phi)|}$$

$\mathrm{Aut}\, G$ acts on chains $\phi_1 \subset \cdots \subset \phi_k$ in G

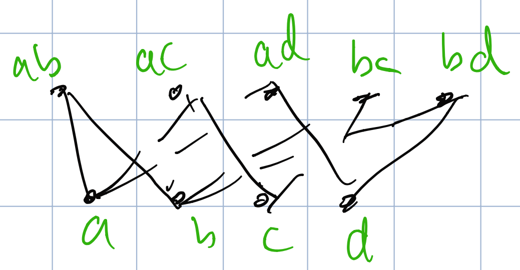
$$|\mathrm{Aut}\, G| = \#(\text{orbit } \sigma) \cdot |\mathrm{Aut}(G, \Phi)|$$

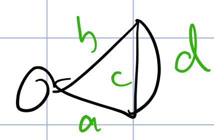
so $$\overline{\chi} = \sum_{[G]} \sum_{\phi_1 \subset \cdots \subset \phi_k \subset G} \frac{(-1)^k}{|\mathrm{Aut}\, G|}$$

claim

$$\sum_{\substack{\phi_1 \subset \cdots \subset \phi_k \\ \text{chains of forests in } G}} (-1)^k = \sum_{\substack{\phi \\ \text{forests in } G}} (-1)^{e(\phi)}$$

pf





filter the poset of forests by # of edges in the forest:

$$F_1 \subset F_2 \subset \cdots \subset F_{v(G)-1} = \text{maximal trees in } G$$

$|F_i|$ is obtained from $|F_{i-1}|$ by coning off $|\text{proper subforests of } F_i|$, one for each $\phi \in F_i$.

every subset of Φ is a subforest.

so $|$proper subforests of $\Phi|$ is the ∂ of an $(e(\Phi)-2)$-sphere, ie it's an $(e(\Phi)-1)$-sphere

so the core is an $e(\Phi)$-cell.

$$\therefore \chi(\text{poset of forests}) = \sum_{\Phi \in G} (-1)^{e(\Phi)}$$

$$\sum_{\substack{\Phi_1 \subset \cdots \subset \Phi_k \\ \text{chains of forests} \\ \text{in } G}} (-1)^k \qquad \square$$

Now define $\displaystyle \tau(G) = \sum_{\substack{\varphi \subset G \\ \text{a forest}}} (-1)^{e(\varphi)}$

(including the empty forest)

Then $\boxed{\displaystyle \overline{\chi}(A_{n,s}) = \sum_{[G]}' \frac{\tau(G)}{|\mathrm{Aut}\, G|}}$

where $\mathrm{Aut}\, G$ = autos of G fixing ∂G

Exercise

Some properties of $\tau(G) := \sum_{\phi \subset G} (-1)^{e(\phi)}$

① $\tau(\cdot) = 1$

② e a loop $\Rightarrow$ $\tau(G) = \tau(G-e)$

③ $\tau(G_1 \amalg G_2) = \tau(G_1)\,\tau(G_2)$

pf $(-1)^{e(\phi_1)+e(\phi_2)} = (-1)^{e(\phi_1)}(-1)^{e(\phi_2)}$

④ $\tau(G) = \tau(G-e) - \tau(G/e)$

pf: forests not containing e ; forests containing e

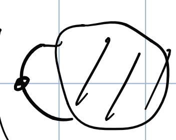
⑤ G has a sep edge $\Rightarrow$ $\tau(G) = 0$

pf: $\tau(G) = \underset{\phi}{\tau(G-e)} - \underset{\phi \cup e/e}{\tau(G/e)}$ (terms cancel)

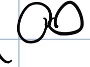
④ $\tau\Big($  $\Big) = 1-k$ pf: induction

k edges

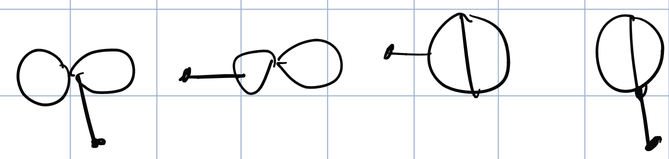
⑤ G has bivalent vertex. Removing it changes sign of τ

$\tau\Big($  $\Big) = -\tau\Big($  $\Big)$

To calculate $\overline{\chi}(\mathrm{Out}\,F_2)$:

$\tau($  $) = 1$ $\quad$ $\tau($  $) = 0$ $\quad$ $\tau($  $) = -2$

$$= \frac{1}{8} - \frac{2}{12} = \frac{3}{24} - \frac{4}{24} = -\frac{1}{24} \checkmark$$

$\overline{\chi}(\mathrm{Aut}\,F_2)$: 

$$\tau: \quad 1 \quad -1 \quad 2 \quad -2$$

$$\overline{\chi} = \frac{1}{8} - \frac{1}{4} + \frac{2}{4} - \frac{2}{6}$$

$$= \frac{3}{24} - \frac{6}{24} + \frac{12}{24} - \frac{8}{24} = \frac{1}{24}$$

Smillie-V, Zagier calculated $\overline{\chi}(\mathrm{Out}\,F_n)$ for $n \leq 100$, found

- $\overline{\chi}(\mathrm{Out}\,F_n) < 0$
- $|\overline{\chi}(\mathrm{Out}\,F_n)|$ grows very fast

Could have also got the answer for $\mathrm{Aut}\, F_2$ using the following property of $\overline{\chi}$:

Prop $1 \to A \to B \to C \to 1$ a short-exact sequence of groups

if $\overline{\chi}(A), \overline{\chi}(B)$ and $\overline{\chi}(C)$ are defined,

then $\overline{\chi}(B) = \overline{\chi}(A)\overline{\chi}(C)$.

Cor
$$\overline{\chi}(A_{n,s}) = \overline{\chi}(\mathrm{Out}\, F_n) \cdot \overline{\chi}(F_n^{\,s})$$
$$= \overline{\chi}(\mathrm{Out}\, F_n) \cdot (1-n)^s$$

ie
$$\overline{\chi}(\mathrm{Aut}\, F_n) = \overline{\chi}(\mathrm{Out}(F_n)) \cdot (1-n)$$
$$\overline{\chi}(\mathrm{Aut}\, F_2) = \overline{\chi}(\mathrm{Out}\, F_2)(-1)$$

So we can turn the tables and use $\overline{\chi}(A_{n,s})$ to study graphs with s leaves.

Borinsky (2018) used this plus some number theory to get

Thm
- $\overline{\chi}(\mathrm{Out}\, F_n) < 0$ for all $n \geq 2$
- $\overline{\chi}(\mathrm{Out}\, F_{n+1}) = \dfrac{-\Gamma(n+\frac{1}{2})}{n \log^2 n} + O\left(\dfrac{\Gamma(n+\frac{1}{2})}{n \log^4 n}\right)$

where Γ is the Γ-function, defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

and by analytic continuation for $\operatorname{Re} z < 1$

$$\Gamma(1) = 0$$

satisfies $\Gamma(z+1) = z\Gamma(z)$

(i.e. $\Gamma(n) = (n-1)!$)

undefined for $z \in \{0, -1, -2, \ldots\}$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \quad , \quad \Gamma\left(n+\tfrac{1}{2}\right) = \frac{2n!}{4^n n!}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

This implies $|\bar{X}(\text{Cont } F_n)|$ grows more than exponentially in n.

We will return to this next week

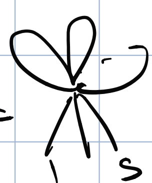
More about the structure of $K_{n,s}$

Local structure: link of a vertex
- (g, G) or • S = complete sphere system

Lower link: Easier to think of the graph description

Upper link: Easier to think of sphere systems.

We will do the lower link first.

$R = R_{n,s} =$  = standard rose,

$\pi_1 \equiv F_n$

vertex (g, G)

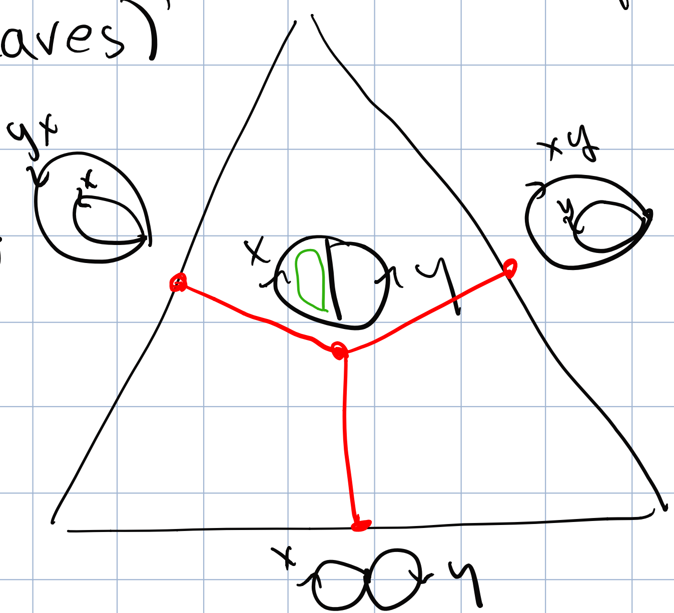
$$\begin{array}{c} G \\ g \big\uparrow \\ R \end{array}$$

connected to (g', G') if there's a forest collapse $G \xrightarrow{c} G'$

making
$$\begin{tikzcd} G \arrow[rr, "c"] & & G' \\ & R \arrow[ul, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$$
commute

(as usual, this is up to homotopy, all maps and homotopies must fix leaves)

eg



different forests give different marked graphs (even if they're isomorphic as graphs)

(removing different spheres gives non-isotopic sphere systems!)

So Lower link (g, G)

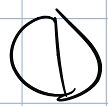
$\cong$ |partially ordered set of forests in G|

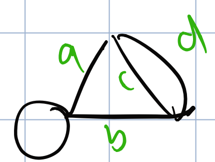
$:= |\Phi(G)|$

Prop: If G has a separating edge then $|\Phi(G)|$ is contractible

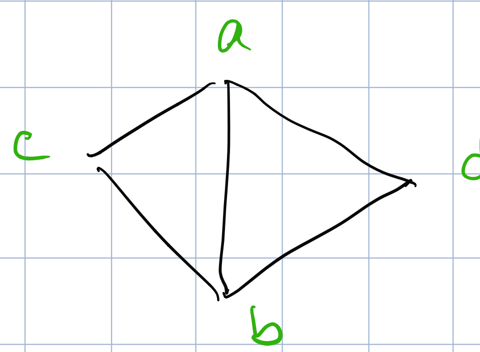
If G has a loop ℓ, then $\Phi(G) = \Phi(G-\ell)$

If G is connected with no separating edge, no loop then $|\Phi(G)| \simeq \bigvee S^{v(G)-2}$

eg $G =$  $\Phi(G) = 3$ points $\simeq \bigvee S^0$

$G =$ 

$\Phi(G)$ has 4 1-edge forests, 5 2-edge forests



$\simeq S^1 \vee S^1$

Proof Induct on $v(G) + e(G)$

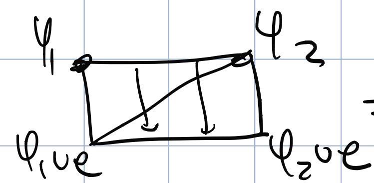
$e \in G$ an edge

e a loop $\Rightarrow \Phi(G) = \Phi(G-e) \Rightarrow$ done by induction.

(e a separating edge $\Rightarrow \varphi \mapsto \varphi \cup e \to e$ is a poset map, contracting Φ to a point.

φ_1 doesn't contain $e \Rightarrow \varphi_1 \cup e$ is in the cone

$\varphi_1 < \varphi_2$



$\Rightarrow$ can fill in square

$\sigma = \varphi_1 \subset \dots \subset \varphi_{k+1} \Rightarrow$ can fill in $\sigma \times I$ by "prism operator"

$\Rightarrow$ can def retract to cone.

This argument is formalized in the

Poset Lemma

P a poset, $f : P \to P$ a poset map

$(x \leq y \Rightarrow f(x) \leq f(y))$

If $f(x) \geq x$ for all x, then $|P| \simeq |f(P)|$

(or if $f(x) \leq x$ for all x)

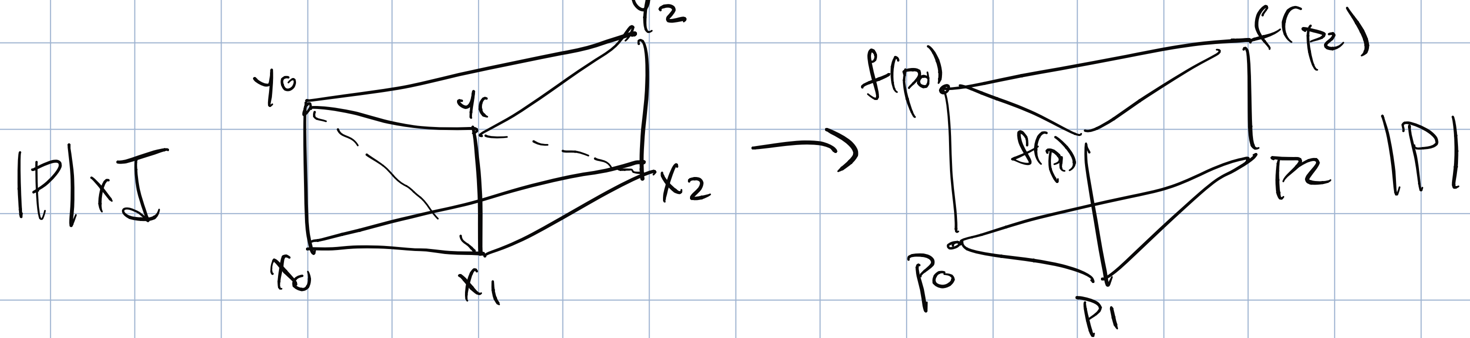
<u>Proof</u>: Suppose $f(x) \geq x \quad \forall x$

Then $H : |P| \times I \longrightarrow |P|$

is defined using the prism operator, which decomposes $\sigma \times I$ into a union of simplices.

A k-simplex x in P is a k-chain $p_0 \leq \cdots \leq p_k$

f a poset map $\Rightarrow f(p_0) \leq \cdots \leq f(p_k)$



simplices of $\sigma \times I = x_0 \cdots x_i y_i \cdots y_k$

define homotopy $P \times I \longrightarrow P$

by sending

works because $f(p_i) \geq p_i$

$$(p_0 \leq \cdots \leq p_k) \times I \longmapsto \sum_i (p_0 \leq \cdots \leq p_i \leq f(p_i) \leq \cdots \leq f(p_k))$$

✓

We'll complete the proof of the proposition next time.