Lecture 6

We are trying to understand the local structure of $K_{n,s}$, in particular the links of vertices.

We had $lk(\sigma) = lk_-\sigma * lk_+\sigma$

We identified $lk_-(\sigma)$ with the geometric realization of the poset $\Phi(G)$ of forests in G

where $\sigma = (g, G)$ and a forest is a union of internal edges with no cycles

Prop (1) If G has a separating edge then $|\Phi(G)|$ is contractible

(2) If G has a loop e then

$$|\Phi G| = |\Phi(G-e)|$$

(3) If G is connected with no loops or separating edges then

$$|\Phi(G)| \simeq \bigvee S^{v(G)-2}$$

We use the Poset Lemma from last time and induct on $v(G)+e(G)$

Fix e an edge of G. Last time showed

$$\begin{cases} e \text{ a loop} \Rightarrow \Phi(G) = \Phi(G-e) \\ e \text{ separating} \Rightarrow |\Phi(G)| \text{ contractible} \end{cases}$$

Left to consider G with no loops or sep. edges.

Let $\Phi_e = \Phi - \{e\}$ = all forests <u>except</u> $\{e\}$.

Define $f : \Phi_e(G) \longrightarrow \Phi_e(G)$

$$\varphi \overset{\supseteq}{\longmapsto} \begin{cases} \varphi - e & \text{if } e \in \varphi \\ \varphi & \text{if } e \notin \varphi \end{cases}$$

poset map: $\varphi_1 \le \varphi_2 \Rightarrow f(\varphi_1) \le f(\varphi_2)$ ✓

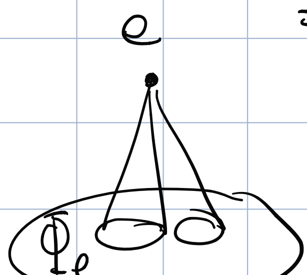
$f(\varphi) \geq \varphi$ ✓

image = forests <u>not</u> containing e

= forests in $G-e \simeq \bigvee S^{v(G)-2}$ by induction

link(e) = forests <u>containing</u> e

$\cong$ forests in $G/e \simeq \bigvee S^{v(G)-3}$ by induction



$|\Phi| = |\Phi_e| \cup |c(lk(e))|$

$|\Phi_e| \cap |c(lk(e))| = |lk(e)|$

Mayer-Vietoris + Van Kampen $\Rightarrow$

$|\Phi|$ is $(v(G)-2)$-dimensional and $(v(G)-3)$-connected

$\Rightarrow \simeq \bigvee S^{v(G)-2}$ ✓

That was the <u>lower link</u>. What about the <u>upper link</u>?

Here it's useful to use the sphere-complex description: a vertex v is a complete sphere system S,

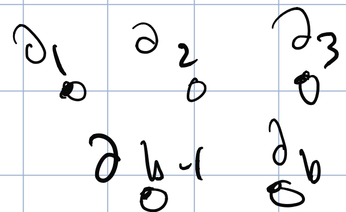
<u>ie</u> $M-S$ = union of punctured 3-balls ("pieces" $P_1, \ldots, P_k$, $k \geq 1$, $P_i = S^3 - \coprod_{b_i} B^3$)

Then $lk_+(S)$ = sphere systems <u>containing</u> S

You can add spheres independently in each piece.

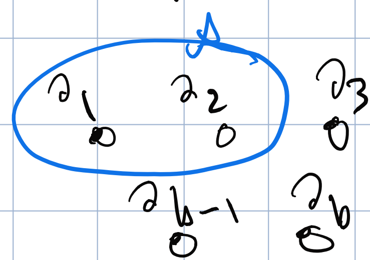
so $lk_+(S) = |\mathcal{S}(P_1)| * \cdots * |\mathcal{S}(P_k)|$

Where $\mathcal{S}(P_i)$ = sphere systems in P_i. So just have to understand $\mathcal{S}(P)$ for P a punctured 3-ball with b boundary spheres:

$P =$ 

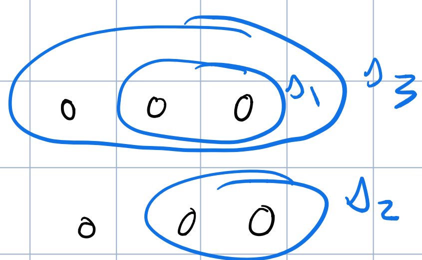
$\partial P = \{\partial_1, \dots, \partial_b\}$

If a sphere $s \subset P$ is not isotopic to a boundary sphere ∂_i, then s is determined up to isotopy by a partition of $\{\partial_1, \dots, \partial_b\}$ into 2 pieces:



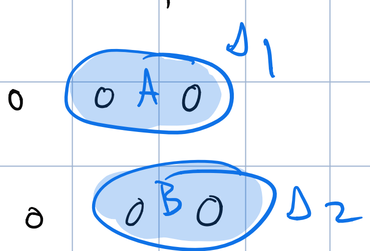
$S = \{s_1, \dots, s_e\}$ is a sphere system in $\mathcal{S}(P)$ iff the partitions are pairwise <u>compatible</u> i.e. they have sides A, B s.t. $A \cap B = \emptyset$

eg

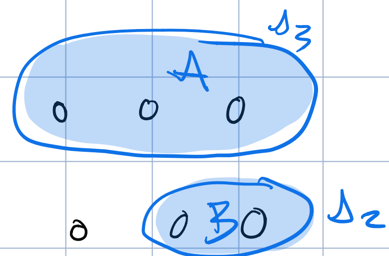


$S = \{s_1, s_2, s_3\}$

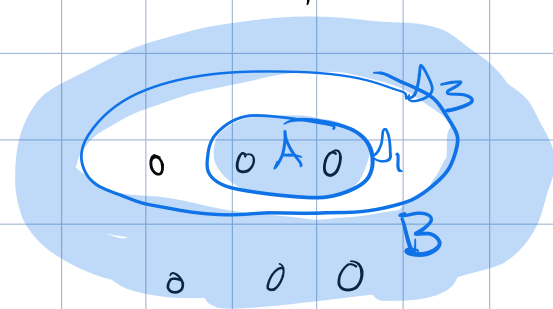
s_1, s_2 compatible:



s_2, s_3 compatible:

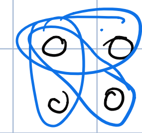


s_1, s_3 compatible:

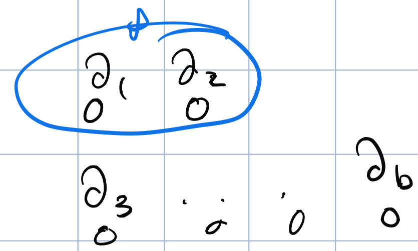


Let $\mathcal{S}(P)$ = sphere systems in P, ordered by $\subseteq$

Prop $P = S^3 - \coprod_b B^3 \Rightarrow |\mathcal{S}(P)| \simeq \bigvee S^{b-4}$

Pf Induct on b. $\underline{b=4} \Rightarrow$  $|\Sigma| = S^0 \cup S^0$

$\underline{b>4}$: Let $s = (\partial_1, \partial_2 \mid \partial_3 \dots \partial_b)$:



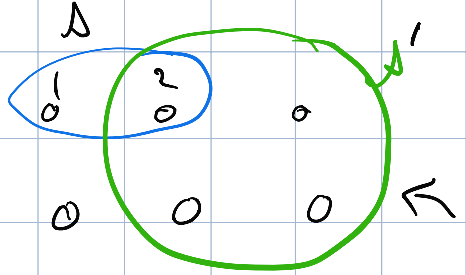
Let $\Sigma_s \subset \mathcal{S}(P)$ = sphere systems compatible w/ s

Then $S \overset{\subseteq}{\longmapsto} S \cup s \overset{\supseteq}{\longmapsto} s$ are poset maps

$\Sigma_s \to \Sigma_s$ satisfying the Poset lemma, showing

$\Sigma_s \simeq pt$.

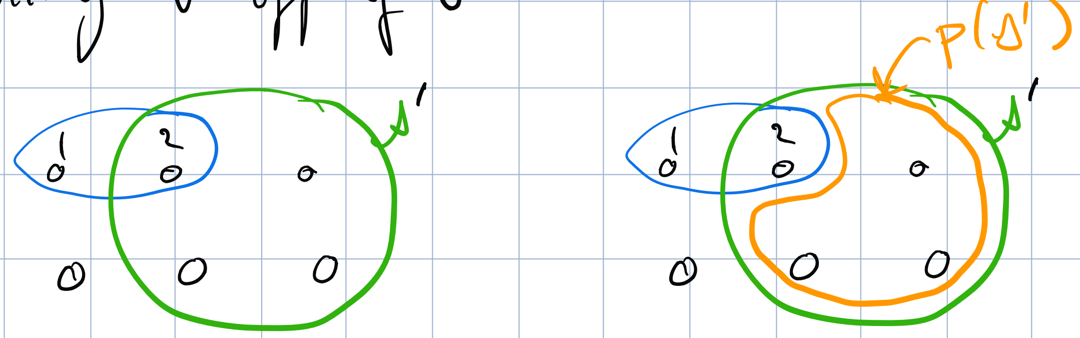
$\mathcal{S}(P) - \Sigma_s$ = systems containing some sphere s' that separates ∂_1 from ∂_2 ie $\underline{\text{crosses}}$ s



If s' crosses s, Define the $\underline{\text{size}}$ of s' to be the # of elts in the side containing ∂_2

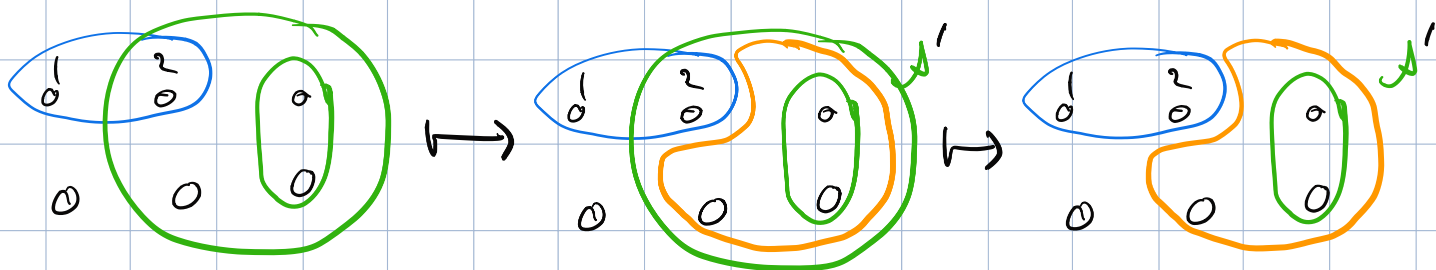
Note a system can contain at most one sphere of each size, and size $\leq b-2$

If s' has size >2, define $p(s')$ by "pushing s' off of s'":



let $\Sigma_s^{b-2} = \Sigma_s \cup$ (systems containing some s' of maximal size $b-2$)

Now $$\Sigma_s^{b-2} \longrightarrow \Sigma_s^{b-2} \longrightarrow \Sigma_s^{b-2}$$
$$S \overset{\leq}{\longmapsto} S \cup p(s') \overset{\geq}{\longmapsto} S \cup p(s') \setminus s'$$



are poset maps satisfying the Poset lemma, with image in Σ_s

so Σ_s^{b-2} is contractible, too.

Now for each $k \geq 3$

$$\Sigma^k_\delta = \Sigma^{k+1}_\delta \cup (\text{systems containing } \delta' \text{ of size } k)$$

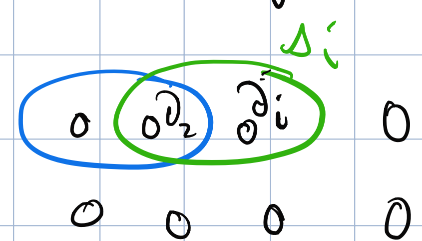
Then
$$\begin{array}{ccccc} \Sigma^k_\delta & \longrightarrow & \Sigma^k_\delta & \longrightarrow & \Sigma^k_\delta \\ \delta & \longmapsto & \delta \cup p(\delta') & \longrightarrow & \delta \cup p(\delta') - \delta' \end{array}$$

has image in $\Sigma^{k+1}_\delta \simeq pt$

What's left?

$\delta(P) - \Sigma^3_\delta$ = Systems containing δ' of size 2 crossing δ

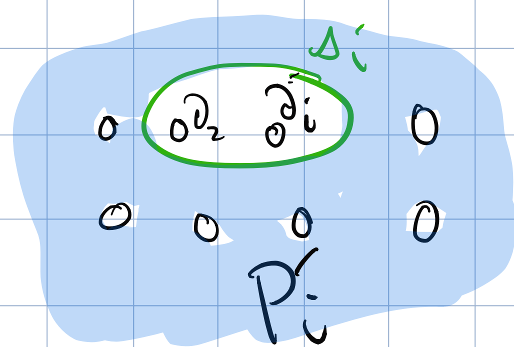
There are $n-2$ such δ':



$\delta_i = (\partial_2, \partial_k \mid \partial_1, \ldots)$ $\quad i = 3, \ldots, n$

Let P'_i = outside of δ_i

$= S^3 - \coprod_{b-1} B^3$

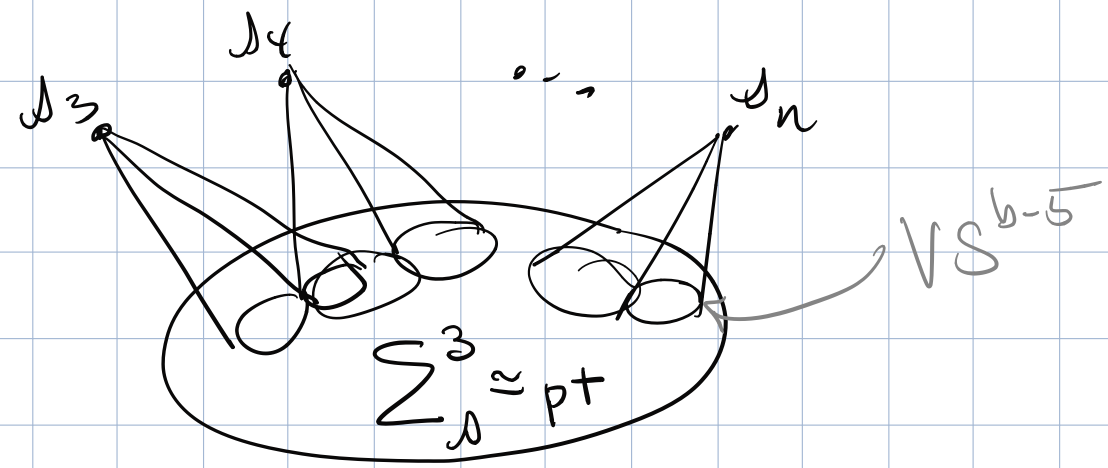


Then

$lk(s_i)$ = sphere systems compatible w/ s_i

= sphere systems in P_i'

$\simeq \bigvee S^{b-5}$ by induction



$\therefore$ Van Kampen + Mayer-Vietoris give $|S(P)| \simeq \bigvee S^{b-4}$ ✓

Now put this all together to determine the link of $S \in K_{n,s}$

Prop $\mathcal{M}(S) \simeq \bigvee S^{n-2+s}$

Pf

$$\mathcal{M}(S) = \mathcal{M}_-(S) * \mathcal{M}_+(S)$$

If S cuts M into k pieces, it corresponds to a graph with k internal vertices and $|S|$ internal edges, with $k - |S| = 1 - n$

$$\simeq \bigvee S^{v(G)-2} * \left(\bigvee S^{b(P_1)-4} * \cdots * \bigvee S^{b(P_k)-4}\right)$$

$$= \bigvee S^{k-2} * \left(\bigvee S^{b_1-4} * \cdots * \bigvee S^{b_k-4}\right)$$

Exercise Finish the proof.

use: $$b_1 + \cdots + b_k = 2|S| + s$$

$$k - |S| = 1 - n$$

and $\bigvee S^a * \bigvee S^b \simeq \bigvee S^{a+b+1}$

Next: The spine encodes $A_{n,s}$ in a very strong way if $n \geq 3$.

We will continue to think of $K_{n,s}$ as a simplicial complex (instead of a cube cplex)

Any elt of $A_{n,s}$ gives a simplicial automorphism $K_{n,s} \longrightarrow K_{n,s}$

so we get a map

$$A_{n,s} \longrightarrow \mathrm{Aut}(K_{n,s})$$

Theorem: This map is an isomorphism.

This is a combinatorial analog of Royden's theorem for Teichmüller space:

$$\text{(Royden)} \quad \mathrm{Mod}^{\pm}(S_{g,s}) \to \mathrm{Isom}(\mathcal{T}_{g,s})$$

is an isomorphism, where the metric on $\mathcal{T}_{g,s}$ is the Teichmüller metric.

For simplicity we will stick to $s=0$

Observe:

- If $\mathrm{lk}_-(s)$ and $\mathrm{lk}_+(s)$ are both non-empty, then $\mathrm{diam}(\mathrm{lk}(s)) = 2$.

  (since $\mathrm{lk}(s) = \mathrm{lk}_-(s) * \mathrm{lk}_+(s)$)

- If S is minimal then $\mathrm{lk}(S) = \mathrm{lk}_-(s)$ has diameter 6.

  proof:

$$\mathrm{lk}(S) = \mathrm{lk}_+(S)$$

Since S is minimal, $M-S$

is connected, ie has one piece

$P \approx S^3 - \coprod_{2n} B^3$ $\quad \partial_1 \circ \ \circ \ \circ \ \cdots \ \circ \quad \overset{\partial_{2n-1}}{\circ}$

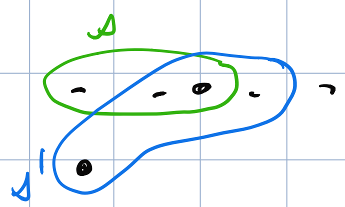
$\partial_{2n} \circ$

Let s_i be the sphere $(\partial_1 \partial_2 .. \partial_i \,|\, \partial_{i+1} \cdots \partial_{2n-1}, \partial_{2n})$ $i=2,..2n-2$

and $s_i' = (\partial_{2n} \partial_2 .. \partial_i \,|\, \partial_{i+1} \cdots \partial_{2n-1}, \partial_1)$ $i=2,\cdots 2n-2$

Then $S = \{ s_2, \cdots, s_{2n-2} \}$ and $S' = \{ s_2', \cdots, s_{2n-2}' \}$ are both maximal sphere systems

They are incompatible: in fact every sphere in S is incompatible with every sphere in S'



So to get a path from S to S', need to go first to subsystems (e.g. a single sphere):

$$S \supseteq s \qquad\qquad s' \subseteq S'$$

s and s' are not compatible, so can't find a path of length ≤ 4.
Can find a sphere s'' compatible with both

$$S \supseteq s \subseteq s \cup s'' \supseteq s'' \subseteq s' \cup s'' \supseteq s' \subseteq S'$$

$\Rightarrow$ diam ≥ 6.

If S is a <u>maximal</u> sphere system then

$$lk(S) = lk_{\mathcal{S}} S \cong |\Phi(G)|, \quad G \text{ trivalent.}$$

e = edge not a loop, not a sep. edge.

T = max tree not containing e $\quad$ (≥ 2 edges b/c rk ≥ 3)

(exists b/c e is not separating)

There is no forest $\varphi : T \supset \varphi \supset e$, so $d(T, e) \geq 3$

φ, φ' any forests st. $\varphi \cap \varphi'$, $\varphi \cup \varphi'$ not forests

Take $e' \in \varphi'$ $\quad e \in \varphi$ st. $e \cup e'$ a forest.

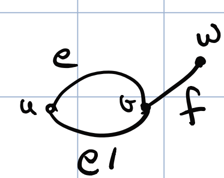
Then

$$\begin{array}{ccccc} \varphi' & & & & \varphi \\ | & & & & | \\ e' & \to & e \cup e' & \to & e \end{array} \Rightarrow d(\varphi, \varphi') \leq 4$$

works as long as $|\varphi|, |\varphi'| \geq 2$

If $\varphi = e$, $\varphi' = e'$

$n \geq 3$ $\exists$ vertex w adj to u or v, $w \neq u, v$:



Then

$$\begin{array}{ccccc} e & & & & e' \\ | & & & & | \\ e \cup f & - & f & - & e' \cup f \end{array}$$ is a path of length 4

$3 \leq \text{diam} \leq 4$. always

Proof of theorem

Let $f : K_{n,s} \to K_{n,s}$ be a simplicial automorphism.

Since it takes links of vertices to links of vertices, it must take

minimal systems = roses $\longrightarrow$ roses

maximal systems = trivalent graphs $\to$ trivalent graphs