Lecture 7

Recap

We are studying the spine $K_{n,s}$ of $\mathcal{O}_{n,s}$, with its action by $A_{n,s}$

We are thinking of $K_{n,s}$ as a simplicial complex. We have two ways of describing a vertex

- A marked graph $g: R_n \to G$
- A complete sphere system $S \subseteq M_{n,s}$

In general $lk(v) \simeq lk_-(v) * lk_+(v)$
we proved

- $lk_-(v) \simeq \bigvee$ spheres
- $lk_+(v) \simeq \bigvee$ spheres — but there

was a mistake in class with the induction. It's fixed in the notes, please read.

$\Rightarrow lk(v) \simeq \bigvee S^{2n-4+s}$ (exercise counting)

$A_{n,s}$ acts on $K_{n,s}$ by simplicial automorphisms

We are proving the map $A_{n,s} \longrightarrow \mathrm{Aut}(K_{n,s})$ is an isomorphism

For simplicity, take $s=0$, and don't use separating spheres (edges)

define the <u>height</u> of v =

$(\#\text{spheres in } v) - n = \#\text{vertices in } G - 1$

so roses have height 0, maximal systems have height $2n-3$

An elt of $A_{n,0} = \mathrm{Out}(F_n)$ preserves homeomorphism type, i.e. preserves height

For surjectivity: Suppose $f: K_n \to K_n$ is a simplicial automorphism. We want to show f is realized by the action of some $\varphi \in Out\, F_n$

By considering the diameter of links we have:

- f takes minimal systems to minimal systems (roses to roses)

- f takes maximal systems to maximal systems (trivalent graphs $\to$ trivalent graphs)

<u>Prop</u> f preserves the poset order

ie $S \subset S' \Rightarrow f(S) \subset f(S')$

pf Since f takes edges to edges, either $f(S) \subset f(S')$ or $f(S') \subset f(S)$

We claim S' has more roses in its link than S does:

Any minimal $R \subset S$ is also in S'

If $\sigma' \in S' \setminus \sigma$, then

there is a minimal $R' \subset S'$ containing σ' (which is non-separating!)

Since f takes roses to roses, we must have

$f(S) \subset f(S')$

~.

<u>Cor</u>: f preserves the number of spheres in a sphere system S

pf. Put S in a maximal chain

$$S_{min} \subset \cdots S \subset \cdots \subset S_{max} = S_0 \subset S_1 \subset \cdots \subset S_{2n-3}$$

The position of S in this chain determines the number of spheres in S.

(S_{min} has n spheres, S_i has $n+i$ spheres)

We know f sends this to another chain

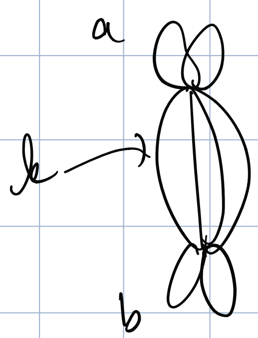
$$f(S_{min}) \subset \cdots f(S) \subset \cdots \subset f(S_{max})$$

so $f(S)$ has the same position $\Rightarrow$ same # of spheres. ✓

Prop f preserves the homeomorphism type of systems with $n+1$ spheres

($=$ 2-vertex graphs)

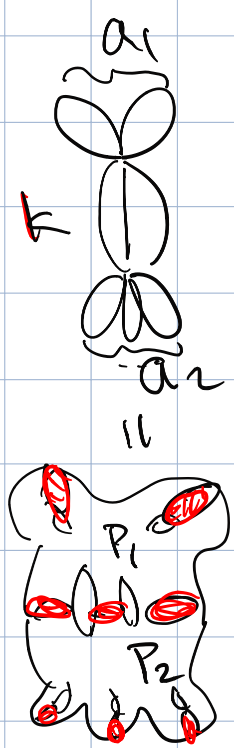
Pf

The link of 

contains exactly k roses

$n = a+b+k-1$

so can only get sent to another 2-vertex graph w/ k non-loop edges.



\# 3-vertex graphs connected to this is

$=$ #spheres in P_1 + #spheres in P_2

P_1 has $2a_1+k$ ∂ components

P_2 has $2a_2+k$ ∂ components

Claim

If a punctured ball P^b has b boundary components, then it contains

$$2^{b-1}-b-1$$

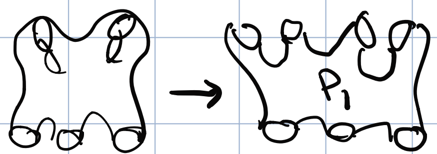
isotopy classes of non-trivial spheres

pf These are determined by the partition of the set of b boundary spheres into two parts, each with ≥ 2 spheres

$$= \frac{1}{2}(\#\text{ subsets of boundary spheres})$$

$$= \frac{1}{2}\Big(\underset{\text{all subsets}}{2^b} - \underset{\emptyset,\ \text{everything}}{2} - \underset{\text{one elt}}{b} - \underset{b-1\text{ elts}}{b}\Big)$$

$P_1 \cong P^{2a_1+k}$



contains $\binom{a_1}{1}+\binom{a_1}{2}+\cdots+\binom{a_1}{a_1}$
$= 2^{a_1}-1$ spheres that were separating in M_n

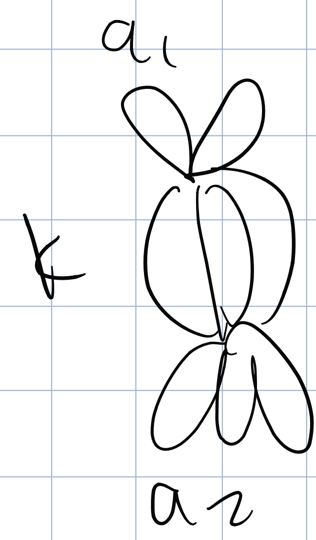
So there are $2^{2a_1+k-1}-(2a_1-k)-1$

$$+\, 2^{a_2+k-1}-(2a_2-k-1)$$

$$-(2^{a_1}-1)-(2^{a_2}-1)$$

<u>non-separating</u> spheres in $P_1 \amalg P_2$

$$= 2^{k-1}\left(2^{2a_1}+2^{2a_2}\right) - 2^{a_1}+1-2^{a_2+1} - (2a_1+2k+2a_2-2)$$



now $a_1+a_2+k-1 = n$:

<u>so</u>:

$$= 2^{k-1}\left(2^{2a_1}+2^{2a_2}\right) - \left(2^{a_1}+2^{a_2}\right) + n-2$$

The following exercise finishes the proof:

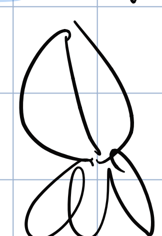
Exercise Let $a_i, a_i' \in \mathbb{N}$ and suppose

(1) $a_1+a_2 = a_1'+a_2'$ and

(2) $2^{k-1}\left(2^{2a_1}+2^{2a_2}\right)-\left(2^{a_1}+2^{a_2}\right) = 2^{k-1}\left(2^{2a_1'}+2^{2a_2'}\right)-\left(2^{a_1'}+2^{a_2'}\right)$

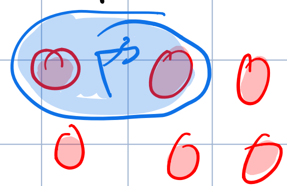
<u>Then</u> $\{a_1, a_2\} = \{a_1', a_2'\}$

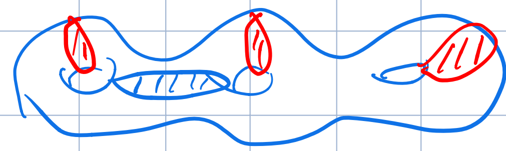
Def The **Nielsen graph** is

the 2-vertex graph 

Cor f takes marked Nielsen graphs to marked Nielsen graphs

A Nielsen system is the corresponding sphere system,





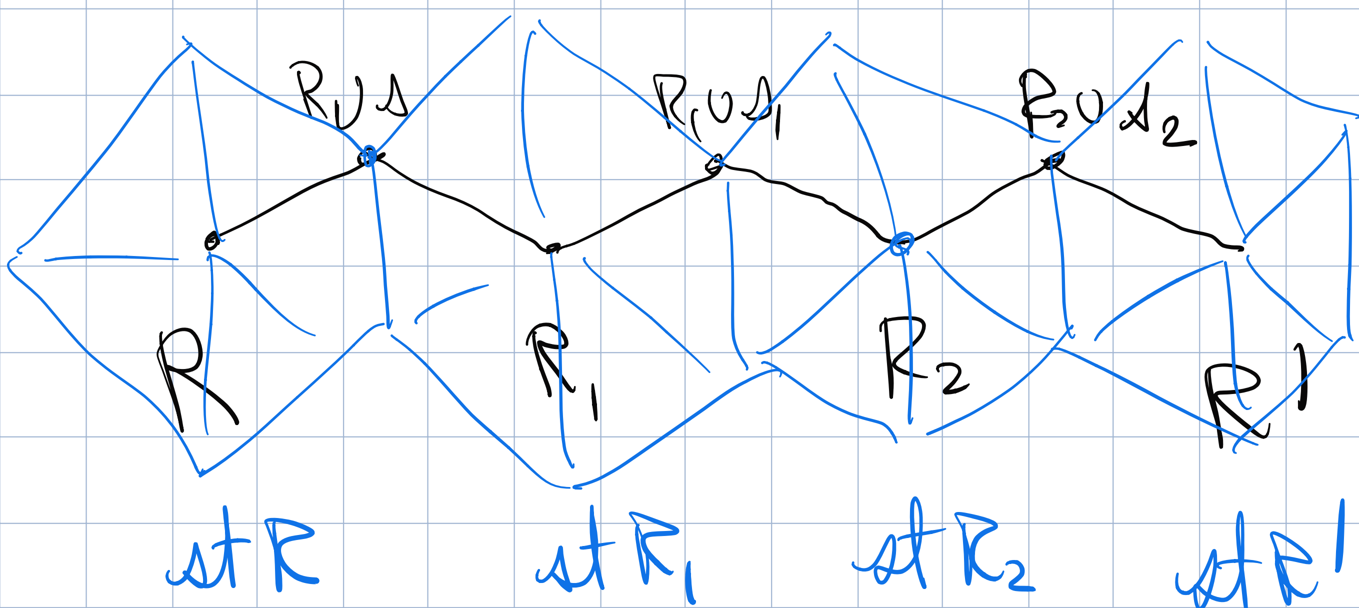
There are two components of $M-S$, one is $\cong P^3 = S^3 - \coprod_3 B^3$, the other is $\cong P^{2n+1}$.

<u>From here</u> f takes roses to roses.
$\mathrm{Out}(F_n)$ acts transitively on roses.
So after composing f with $\varphi \in \mathrm{Out}(F_n)$ we may assume f fixes a rose R.

<u>Next</u> Show that composing with some $\varphi \in \mathrm{Stab}_{\mathrm{Out}}(R)$, we may assume f fixes all Nielsen graphs ($=$ Nielsen systems $R \cup \Delta$) in the simplicial star of R

Fact: You can get from R to any other rose R' by a path in K_n that only contains roses and Nielsen systems

$$R - R\cup s - R_1 - R_1 \cup s_1 - \cdots - R'$$



Need to show:

- f fixes all Nielsen graphs in $st(R) \Rightarrow f$ fixes <u>all</u> of $st(R)$

- f fixes $\mathrm{st}(R) \cap \mathrm{st}(R_1) \Rightarrow$

  f fixes R_1

- f fixes R_1 and $\mathrm{st}\, R \cap \mathrm{st}\, R_1$

  $\Rightarrow f$ fixes $\mathrm{st}(R_1)$

Continuing along the path, we get

f fixes $\mathrm{st}(R')$

Since f fixes $\mathrm{st}\, R'$ for every R'

f is the identity

Each step is proved by either producing a homeomorphism having the desired effect ($=$ an elt of $\mathrm{Out}\, F_n$)

or counting roses or other graphs in $\mathcal{C}(S)$ to see that they have to be fixed.

Exercise (Not required!) See how far you can get with the rest of the proof. Then try it for $s > 0$.

$\Rightarrow$ all Nielsen systems in st R' are fixed by f

$\Rightarrow$ all of st R' is fixed by f.

Each step is proved by either producing a homeomorphism having the desired effect (= an elt of Out F_n)

or counting roses or other graphs in $lk(S)$ to see that they have to be fixed.

Exercise (Not required!) See how far you can get with the rest of the proof. Then try it for $s>0$.

It is clear we won't get anywhere close to most of the topics I listed in the first lecture.

Instead I would like to return to the Euler characteristic - $\bar{\chi}(A_{n,s})$

We saw to calculate $\bar{\chi}(A_{n,s})$ it's enough to calculate $\bar{\chi}(\mathrm{Out}\, F_n)$

We got the formula

$$\bar{\chi}(\mathrm{Out}\, F_n) = \sum_{G \in \mathcal{G}_n^c} \frac{\tau(G)}{|\mathrm{Aut}\, G|}$$

where $\mathcal{G}_n^c$ = finite connected graphs, all vertices at least trivalent, $\chi(G) = 1-n$

and $\tau(G) = \sum_{\text{forests } \varphi \subset G} (-1)^{e(F)}$
(including $\varnothing$)

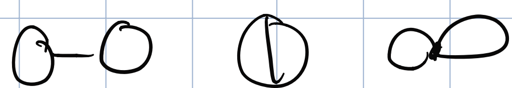
In the exercises you played a little with $\tau(G)$

Here's one more

Exercise sign $\tau(G) = (-1)^{f(G)}$ where $f(G)$ is the # of edges in a max. forest in G

So it is not even obvious what the sign of $\overline{\chi}(\mathrm{Out}\,F_n)$

But you can calculate, for small values of n

eg $n=2$ $G =$ 

$\tau = 0 \quad -2 \quad 1$

$|\mathrm{Aut}\,G| = 8 \quad 12 \quad 8$

$$\Rightarrow \overline{\chi}(\mathrm{Out}\,F_2) = \frac{0}{8} - \frac{4}{12} + \frac{1}{8}$$

$$= \frac{-4}{24} + \frac{3}{24} = \frac{-1}{24}$$

$n=3$ $\quad -\frac{1}{48}$ $\qquad n=4$ $\quad \frac{-161}{5760}$ $\qquad n=5$ $\quad \frac{-367}{5760}$

$n=12 \quad \bar{\chi} \approx -2000$

... Always <0, growing very fast!

<u>Borinsky</u> proved: • $\bar{\chi} < 0$ always
• $|\bar{\chi}|$ grows more than exponentially fast.
• $\bar{\chi}$ is closely related to the ζ-function

$=$ analytic cont of $\zeta(s) = \sum \frac{1}{n^s}$

To understand his proof, need to say a little about asymptotic expansions, Γ-functions and generating fcn's.
We already talked about Γ-functions
$\Gamma(n) = (n-1)!$ (so $\Gamma(m+1) = m\,\Gamma(m)$)

$$\Gamma(z) = \int_0^\infty t^z e^{-t}\,dt$$ converges for $Re(x) > 0$

can be analytically continued to $\mathbb{C}$, has simple poles at non-pos integers.

satisfies $\Gamma(x+1) = x\Gamma(x)$

In particular,

For fixed k, define $\varphi_k(x) = \Gamma(x-k)$

then $\displaystyle\lim_{x\to\infty} \frac{\varphi_{k+1}(x)}{\varphi_k(x)} =$

$$= \lim_{x\to\infty} \frac{\Gamma(x-k-1)}{\Gamma(x-k)} =$$

$$= \lim_{x\to\infty} \frac{\Gamma(x-k-1)}{(x-k-1)\Gamma(x-k-1)}$$

$$= \lim_{x\to\infty} \frac{1}{x-k-1} = 0$$

A sequence of functions $\varphi_k(x)$ s.t.

$$\lim_{x\to\infty} \frac{\varphi_{k+1}(x)}{\varphi_k(x)} = 0 \quad \text{is called}$$

an asymptotic scale : φ_{k+1} grows much slower than φ_k with x.

A series $\sum_{k=0}^{\infty} a_k \varphi_k(x)$ is called an asymptotic expansion for $f(x)$

if $f(x) = \sum_{k=0}^{N} a_k \varphi_k(x) + o(\varphi_N(x))$

$$\frac{f(x) - \sum^{N} a_k \varphi_k(x)}{\varphi_N(x)} \xrightarrow[x\to\infty]{} 0$$

$$a_0 = \lim_{x\to\infty} \frac{f(x)}{\varphi_0(x)}$$

$$a_1 = \lim_{x\to\infty} \frac{f(x) - a_0\varphi_0(x) - a_1\varphi_1}{\varphi_1(x)}$$

$$\lim_{x\to\infty} \frac{f(x) - a_0\varphi_0(x)}{\varphi_0(x)} = 0$$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{\varphi_0(x)} = a_0$$

$$\lim_{x\to\infty} \frac{f(x) - a_0\varphi_0(x) - a_1\varphi_1(x)}{\varphi_1(x)} = 0$$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x) - a_0\varphi_0(x)}{\varphi_1(x)} = a_1$$

etc So the coefficients in the asymptotic expansion are determined by f and $\{\varphi_k\}$

ie if an asymptotic expansion exists, it is unique.

Thm (Barinsky) $c_n = \chi(\mathrm{Out}\,F_{n+1})$

$$T(z) = c_1 z + c_2 z^2 + \cdots$$

$$\exp(T(z)) = 1 + b_1 z + b_2 z^2 + \cdots$$

$$\varphi_k(x) = \Gamma\left(x + \tfrac{1}{2} - k\right)$$

Then $\sum b_k \varphi_k(n)$ is an

asymptotic expansion for

$$f(n) = \sqrt{2\pi}\left(\frac{e}{n}\right)^n$$

There are other asymptotic expansions for f in the literature, and people have proved facts about the coefficients

This proves facts about the b_n, and $\therefore$ about the c_n!